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Relative orders and slowly changing functions oriented growth analysis of composite entire functions

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Abstract

In the paper we establish some new results depending on the comparative growth properties of composition of entire functions using relative L^*-order (relative L^*-lower order) as compared to their corresponding left and right factors where $L \equiv L(r)$ is a slowly changing function.

Keywords: Entire function; maximum modulus; maximum term; composition; growth; relative L*-order (relative L*-lower order); slowly changing function.
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1. Introduction, Definitions and Notations.

Let $\mathbb{C}$ denote the set of all finite complex numbers. Also let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r, f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu_f(r) = \max_{n\geq 0}(|a_n| r^n)$. On the other hand, maximum modulus $M(r, f)$ of f on $|z| = r$ is defined as $M(r, f) = \max_{|z|=r} |f(z)|$. The following notation:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, ... \text{ and } \log^{[0]} x = x$$

is frequently used in the paper.

To start our paper we just recall the following definition :

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Definition 1.1. The *order* ρ_f and *lower order* λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} .$$

Using the inequalities $\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f)$ (cf. [**15**]) , for $0 \leq r < R$ one may verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r} .$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [12] defined it in the following way:

Definition 1.2. [12] A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon\,(>0)$,

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \; for \; r \geq r(\varepsilon) \text{ and}$$

uniformly for $k\,(\geq 1)$.

Somasundaram and Thamizharasi [13] introduced the notions of *L-order* and *L-lower order* for entire functions. The more generalised concept for *L-order* and *L-lower order* for entire function are L^**-order* and L^**-lower order* respectively. Their definitions are as follows:

Definition 1.3. [13] The L^**-order* $\rho_f^{L^*}$ *and the* L^**-lower order* $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} .$$

If an entire function g is non-constant then $M_g(r)$ is strictly increasing and continuous. Its inverse $M_g^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_g^{-1}(s) = \infty$.

Bernal [1] introduced the definition of *relative order* of an entire function f with respect to an entire function g , denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r} . \end{aligned}$$

For $g(z) = \exp z$, the above definition coincides with the classical one (cf. [16]).

Similarly, one can define the *relative lower order* of an entire function f with respect to another entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r} .$$

Datta and Maji [5] gave an alternative definition of *relative order* and *relative lower order* of an entire function with respect to another entire function in the following way:

Definition 1.4. [5] The *relative order* $\rho_g(f)$ and *relative lower order* $\lambda_g(f)$ of an entire function f with respect to another entire function g are defined as follows:

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r} \quad \text{and} \quad \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r} .$$

In the line of Somasundaram and Thamizharasi (cf. [13]), one can define the *relative L^*-order* of an entire function in the following manner:

Definition 1.5. ([3],[4]) The *relative L^*-order* of an entire function f with respect to another entire function g , denoted by $\rho_g^{L^*}(f)$ is defined in the following way

$$\begin{aligned} \rho_g^{L^*}(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g\left\{re^{L(r)}\right\}^{\mu} \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1}M_f(r)}{\log\left[re^{L(r)}\right]} . \end{aligned}$$

Analogusly, one may define the *relative L^*-lower order* of an entire function f with respect to another entire function g denoted by $\lambda_g^{L^*}(f)$ as follows:

$$\lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1}M_f(r)}{\log\left[re^{L(r)}\right]} .$$

Datta, Biswas and Ali [6] also gave an alternative definition of *relative L^*-order* and *relative L^*-lower order* of an entire function which are as follows:

Definition 1.6. [6] The *relative L^*-order* $\rho_g^{L^*}(f)$ and *the relative L^*-lower order* $\lambda_g^{L^*}(f)$ of an entire function f with respect to g are as follows:

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log\left[re^{L(r)}\right]} .$$

For entire functions, the notions of thier growth indicators such as *order* (*lower order), L-order* (*L-lower order*) and *L^*-order* (*L^*-lower order)* are classical in complex analysis and their respective generalized concepts are *relative order* (*relative lower order), relative L-order* (*relative L-lower order*) and *relative L^*-order* (*relative L^*-lower order*). During the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of relative growth indicators of entire functions and as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of composite entire functions using their relative growth indicators are the prime concern of this paper. In fact, some light has already been thrown on such type of works in [7], [8], [9], [10] and [11]. In the paper we study some growth properties of maximum term and maximum modulus of composition of entire functions with respect to another entire function and compare the growth of their corresponding left and right factors on the basis of *relative L^*-order* and *relative L^*-lower order*. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [17].

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [14] Let f and g be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,

$$\mu_{f\circ g}(r) \leq \frac{\alpha}{\alpha-1}\mu_f\left(\frac{\alpha R}{R-r}\mu_g(R)\right) .$$

Lemma 2.2. [14] Let f and g be any two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right)\right) .$$

Lemma 2.3. [2] If f and g are any two entire functions then for all sufficiently large values of r,

$$M_{f\circ g}(r) \leq M_f(M_g(r)) .$$

Lemma 2.4. [2] If f and g are any two entire functions with $g(0) = 0$ then for all sufficiently large values of r,

$$M_{f\circ g}(r) \geq M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right) .$$

Lemma 2.5. [5] If f be entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,

$$\mu_f(\alpha r) \geq \beta\mu_f(r) .$$

3. Theorems.

In this section we present the main results of the paper.

Theorem 3.1. *Let f and h be any two entire functions with $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$. Also let g be an entire function with $\lambda_g^{L^*} > 0$ and $g(0) = 0$. Then for every positive constant A and real number x,*

$$\lim_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\left\{\log\mu_h^{-1}\mu_f\left(r^A\right)\right\}^{1+x}} = \infty .$$

Proof . If x is such that $1 + x \leq 0$, then the theorem is obvious. So we suppose that $1 + x > 0$. Now in view of Lemma 2.2 and Lemma 2.5, we have for all sufficiently large values of r that

$$\mu_{f\circ g}(r) \geq \mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right) . \tag{3.1}$$

Since μ_h^{-1} is an increasing function, it follows from (3.1) for all sufficiently large values of r that

$$\begin{aligned} \mu_h^{-1}\mu_{f\circ g}(r) &\geq \mu_h^{-1}\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right) \\ \textit{i.e.,}\quad \log\mu_h^{-1}\mu_{f\circ g}(r) &\geq \log\mu_h^{-1}\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right) \end{aligned}$$

$$i.e., \quad \log \mu_h^{-1}\mu_{f\circ g}(r) \geq O(1) + \left(\lambda_h^{L^*}(f) - \varepsilon\right)\left[\log\left\{\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right\} + L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right]$$

$$i.e., \quad \log \mu_h^{-1}\mu_{f\circ g}(r) \geq O(1) + \left(\lambda_h^{L^*}(f) - \varepsilon\right)\left[\log \mu_g\left(\frac{r}{4}\right) + O(1) + L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right]$$

$$\begin{aligned} & i.e., \ \log \mu_h^{-1}\mu_{f\circ g}(r) \\ \geq \ & O(1) + \left(\lambda_h^{L^*}(f) - \varepsilon\right)\left\{\left(\frac{r}{4}\right)e^{L(r)}\right\}^{\lambda_g^{L^*}-\varepsilon} + L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right) . \end{aligned} \tag{3.2}$$

where we choose $0 < \varepsilon < \min\left\{\lambda_h^{L^*}(f), \lambda_g^{L^*}\right\}$.
Also for all sufficiently large values of r we get that

$$\begin{aligned} \log \mu_h^{-1}\mu_f\left(r^A\right) &\leq \left(\rho_h^{L^*}(f) + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\} \\ i.e., \ \log \mu_h^{-1}\mu_f\left(r^A\right) &\leq \left(\rho_h^{L^*}(f) + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\} \\ i.e., \ \left\{\log \mu_h^{-1}\mu_f\left(r^A\right)\right\}^{1+x} &\leq \left(\rho_h^{L^*}(f) + \varepsilon\right)^{1+x}\left(A\log r + L\left(r^A\right)\right)^{1+x} . \end{aligned} \tag{3.3}$$

Therefore from (3.2) and (3.3), it follows for all sufficiently large values of r that

$$\begin{aligned} & \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\left\{\log \mu_h^{-1}\mu_f\left(r^A\right)\right\}^{1+x}} \\ & \geq \frac{O(1) + \left(\lambda_h^{L^*}(f) - \varepsilon\right)\left\{\left(\frac{r}{4}\right)e^{L(r)}\right\}^{\lambda_g^{L^*}-\varepsilon} + O(1) + L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)}{\left(\rho_h^{L^*}(f) + \varepsilon\right)^{1+x}\left(A\log r + L\left(r^A\right)\right)^{1+x}} . \end{aligned} \tag{3.4}$$

Since $\frac{r^{\lambda_g^{L^*}-\varepsilon}}{\log r^{1+x}} \to \infty$ as $r \to \infty$, the theorem follows from (3.4). □

In the line of Theorem 3.1, one may state the following theorem without proof :

Theorem 3.2. Let f and h be any two entire functions with $0 < \lambda_h^{L^*}(f) < \infty$ or $0 < \rho_h^{L^*}(f) < \infty$. Also let g be an entire function with $\lambda_g^{L^*} > 0$ and $g(0) = 0$.Then for every positive constant A and real number x,

$$\limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\left\{\log \mu_h^{-1}\mu_g\left(r^A\right)\right\}^{1+x}} = \infty .$$

Using the same technique of Theorem 3.1 and Theorem 3.2 and using Lemma 2.4 one may easily verify the following two theorems:

Theorem 3.3. Let f and h be any two entire functions with $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$. Alos let g be an entire function with $\lambda_g^{L^*} > 0$ and $g(0) = 0$. Then for every positive constant A and real number x,

$$\lim_{r\to\infty} \frac{\log M_h^{-1}M_{f\circ g}(r)}{\left\{\log \mu_h^{-1}\mu_f\left(r^A\right)\right\}^{1+x}} = \infty .$$

Theorem 3.4. Let f, g and h be any three entire functions where g is with non zero L^*-lower order, $g(0)=0$ and either $0<\lambda_h^{L^*}(f)<\infty$ or $0<\rho_h^{L^*}(f)<\infty$.Then for every positive constant A and real number x,

$$\limsup_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{\left\{\log\mu_h^{-1}\mu_g\left(r^A\right)\right\}^{1+x}}=\infty\ .$$

Theorem 3.5. *Let f and h be any two entire functions with $0<\lambda_h^{L^*}(f)\le\rho_h^{L^*}(f)<\infty$ and g be an entire function with non zero L^*-lower order and $g(0)=0$. Then for any positive integer α and β,*

$$\lim_{r\to\infty}\frac{\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(\exp(\exp(r^\alpha)))}{\log\mu_h^{-1}\mu_f\left(\exp\left(r^\beta\right)\right)+K(r,\alpha;L)}=\infty\ ,$$

where $K(r,\alpha;L)=\begin{cases}0\ \textit{if}\ r^\beta=o\{L(\exp(\exp(r^\alpha)))\} \\ \qquad\qquad\qquad\textit{as}\ r\to\infty \\ L(\exp(\exp(r^\alpha)))\ \ \textit{otherwise}\ .\end{cases}$

Proof . Taking $x=0$ and $A=1$ in Theorem 3.1 we obtain for all sufficiently large values of r and for $K>1$

$$\begin{aligned}\log\mu_h^{-1}\mu_{f\circ g}(r) &> K\log\mu_h^{-1}\mu_f(r)\\ i.e.,\ \mu_h^{-1}\mu_{f\circ g}(r) &> \left\{\mu_h^{-1}\mu_f(r)\right\}^K\\ i.e.,\ \mu_h^{-1}\mu_{f\circ g}(r) &> \mu_h^{-1}\mu_f(r)\ .\end{aligned}\tag{3.5}$$

Therefore from (3.5), we get for all sufficiently large values of r that

$$\log\mu_h^{-1}\mu_{f\circ g}(\exp(\exp(r^\alpha)))>\log\mu_h^{-1}\mu_f(\exp(\exp(r^\alpha)))$$

$$\begin{aligned}&i.e.,\ \ \log\mu_h^{-1}\mu_{f\circ g}(\exp(\exp(r^\alpha)))\\ >\ &\left(\lambda_h^{L^*}(f)-\varepsilon\right).\log\{\exp(\exp(r^\alpha)).\exp L(\exp(\exp(r^\alpha)))\}\end{aligned}$$

$$\begin{aligned}&i.e.,\ \ \log\mu_h^{-1}\mu_{f\circ g}(\exp(\exp(r^\alpha)))\\ >\ &\left(\lambda_h^{L^*}(f)-\varepsilon\right).\{(\exp(r^\alpha))+L(\exp(\exp(r^\alpha)))\}\end{aligned}$$

$$\begin{aligned}&i.e.,\ \ \log\mu_h^{-1}\mu_{f\circ g}(\exp(\exp(r^\alpha)))\\ >\ &\left(\lambda_h^{L^*}(f)-\varepsilon\right).\left\{(\exp(r^\alpha))\left(1+\frac{L(\exp(\exp(r^\alpha)))}{(\exp(r^\alpha))}\right)\right\}\end{aligned}$$

$$\begin{aligned}i.e.,\ \ \log^{[2]}\mu_h^{-1}\mu_{f\circ g}(\exp(\exp(r^\alpha))) &> O(1)+\log\exp(r^\alpha)\\ &\quad+\log\left\{1+\frac{L(\exp(\exp(r^\alpha)))}{(\exp(r^\alpha))}\right\}\end{aligned}$$

$$i.e.,\qquad \log^{[2]}\mu_h^{-1}\mu_{f\circ g}(\exp(\exp(r^\alpha)))\quad>\quad O(1)\ +\ r^\alpha\ +\ \log\left\{1+\frac{L(\exp(\exp(r^\alpha)))}{(\exp(r^\alpha))}\right\}$$

$$i.e., \ \log^{[2]}\mu_h^{-1}\mu_{f\circ g}\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) > O\left(1\right) + r^{\alpha} + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) - \log\left[\exp\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}\right] + \log\left[1 + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\exp\left(\mu r^{\alpha}\right)}\right]$$

$$i.e., \ \log^{[2]}\mu_h^{-1}\mu_{f\circ g}\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) > O\left(1\right) + r^{\alpha} + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) + \log\left[\frac{1}{\exp\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}} + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\exp\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}.\exp\left(r^{\alpha}\right)}\right]$$

$$i.e., \ \log^{[2]}\mu_h^{-1}\mu_{f\circ g}\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) > O\left(1\right) + r^{(\alpha-\beta)}.r^{\beta} + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right). \tag{3.6}$$

Again we have for all sufficiently large values of r that

$$\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right) \le \left(\rho_h^{L^*}(f)+\varepsilon\right)\log\left\{\exp\left(r^{\beta}\right)e^{L\left(\exp\left(r^{\beta}\right)\right)}\right\}$$

$$\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right) \le \left(\rho_h^{L^*}(f)+\varepsilon\right)\log\left\{\exp\left(r^{\beta}\right)e^{L\left(\exp\left(r^{\beta}\right)\right)}\right\}$$

$$i.e., \ \log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right) \le \left(\rho_h^{L^*}(f)+\varepsilon\right)\left\{\log\exp\left(r^{\beta}\right) + L\left(\exp\left(r^{\beta}\right)\right)\right\}$$

$$i.e., \ \log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right) \le \left(\rho_h^{L^*}(f)+\varepsilon\right)\left\{r^{\beta} + L\left(\exp\left(r^{\beta}\right)\right)\right\}$$

$$i.e., \ \frac{\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right) - \left(\rho_h^{L^*}(f)+\varepsilon\right)L\left(\exp\left(r^{\beta}\right)\right)}{\left(\rho_h^{L^*}(f)+\varepsilon\right)} \le r^{\beta}. \tag{3.7}$$

Now from (3.6) and (3.7), it follows for all sufficiently large values of r that

$$\log^{[2]}\mu_h^{-1}\mu_{f\circ g}\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \ge O\left(1\right) + \left(\frac{r^{(\alpha-\beta)}}{\rho_h^{L^*}(f)+\varepsilon}\right)\left[\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right) - \left(\rho_h^{L^*}(f)+\varepsilon\right)L\left(\exp\left(r^{\beta}\right)\right)\right] + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \tag{3.8}$$

$$i.e., \ \frac{\log^{[2]}\mu_h^{-1}\mu_{f\circ g}\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right)} \ge \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) + O\left(1\right)}{\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right)} + \frac{r^{(\alpha-\beta)}}{\rho_h^{L^*}(f)+\varepsilon}\left\{1 - \frac{\left(\rho_h^{L^*}(f)+\varepsilon\right)L\left(\exp\left(r^{\beta}\right)\right)}{\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right)}\right\}. \tag{3.9}$$

Again from (3.8) we get for all sufficiently large values of r that

$$\frac{\log^{[2]}\mu_h^{-1}\mu_{f\circ g}\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} \ge \frac{O\left(1\right) + r^{(\alpha-\beta)}L\left(\exp\left(r^{\beta}\right)\right)}{\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} + \frac{\left(\frac{r^{(\alpha-\beta)}}{\rho_h^{L^*}(f)+\varepsilon}\right)\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right)}{\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\log\mu_h^{-1}\mu_f\left(\exp\left(r^{\beta}\right)\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}$$

$$i.e., \ \frac{\log^{[2]} \mu_h^{-1} \mu_{f \circ g} (\exp (\exp (r^{\alpha})))}{\log \mu_h^{-1} \mu_f (\exp (r^{\beta})) + L (\exp (\exp (r^{\alpha})))} \geq \frac{\frac{O(1)+r^{(\alpha-\beta)} L\left(\exp\left(r^{\beta}\right)\right)}{L(\exp(\exp(r^{\alpha})))}}{\frac{\log \mu_h^{-1} \mu_f\left(\exp\left(r^{\beta}\right)\right)}{L(\exp(\exp(r^{\alpha})))} + 1}$$

$$+ \frac{\left(\frac{r^{(\alpha-\beta)}}{\rho_h^{L^*}(f)+\varepsilon}\right)}{1 + \frac{L(\exp(\exp(r^{\alpha})))}{\log \mu_h^{-1} \mu_f\left(\exp\left(r^{\beta}\right)\right)}} + \frac{1}{1 + \frac{\log \mu_h^{-1} \mu_f\left(\exp\left(r^{\beta}\right)\right)}{L(\exp(\exp(r^{\alpha})))}} \ . \tag{3.10}$$

Case I. If $r^{\beta} = o\{L(\exp(\exp(r^{\alpha})))\}$ then it follows from (3.9) that

$$\lim_{r\to\infty} \frac{\log^{[2]} \mu_h^{-1} \mu_{f \circ g} (\exp (\exp (r^{\alpha})))}{\log \mu_h^{-1} \mu_f (\exp (r^{\beta}))} = \infty \ .$$

Case II. $r^{\beta} \neq o\{L(\exp(\exp(r^{\alpha})))\}$ then the following two sub cases may arise:
Sub case (a). If $L(\exp(\exp(r^{\alpha}))) = o\left\{\log \mu_h^{-1} \mu_f\left(\exp\left(r^{\beta}\right)\right)\right\}$, then we get from (3.10) that

$$\lim_{r\to\infty} \frac{\log^{[2]} \mu_h^{-1} \mu_{f \circ g} (\exp (\exp (r^{\alpha})))}{\log \mu_h^{-1} \mu_f (\exp (r^{\beta})) + L (\exp (\exp (r^{\alpha})))} = \infty \ .$$

Sub case (b). If $L(\exp(\exp(r^{\alpha}))) \sim \log \mu_h^{-1} \mu_f\left(\exp\left(r^{\beta}\right)\right)$ then

$$\lim_{r\to\infty} \frac{L (\exp (\exp (r^{\alpha})))}{\log \mu_h^{-1} \mu_f (\exp (r^{\beta}))} = 1$$

and we obtain from (3.10) that

$$\liminf_{r\to\infty} \frac{\log^{[2]} \mu_h^{-1} \mu_{f \circ g} (\exp (\exp (r^{\alpha})))}{\log \mu_h^{-1} \mu_f (\exp (r^{\beta})) + L (\exp (\exp (r^{\alpha})))} = \infty \ .$$

Combining Case I and Case II we obtain that

$$\lim_{r\to\infty} \frac{\log^{[2]} \mu_h^{-1} \mu_{f \circ g} (\exp (\exp (r^{\alpha})))}{\log \mu_h^{-1} \mu_f (\exp (r^{\beta})) + L (\exp (\exp (r^{\alpha})))} = \infty \ ,$$

where $K(r,\alpha;L) = \begin{cases} 0 \text{ if } r^{\mu} = o\{L(\exp(\exp(r^{\alpha})))\} \\ \qquad\qquad\qquad\qquad \text{as } r \to \infty \\ L(\exp(\exp(r^{\alpha}))) \ \text{otherwise} \ . \end{cases}$
This proves the theorem. □

Theorem 3.6. Let f, g and h be any three entire functions where g is an entire function with non zero L^*-lower order, $g(0) = 0$, $\lambda_h^{L^*}(f) > 0$ and $\rho_h^{L^*}(g) < \infty$. Then for any positive integer α and β,

$$\lim_{r\to\infty} \frac{\log^{[2]} \mu_h^{-1} \mu_{f \circ g} (\exp (\exp (r^{\alpha})))}{\log \mu_h^{-1} \mu_g (\exp (r^{\beta})) + K(r,\alpha;L)} = \infty \ ,$$

where $K(r,\alpha;L) = \begin{cases} 0 \text{ if } r^{\beta} = o\{L(\exp(\exp(r^{\alpha})))\} \\ \qquad\qquad\qquad\qquad \text{as } r \to \infty \\ L(\exp(\exp(r^{\alpha}))) \ \text{otherwise} \ . \end{cases}$

The proof is omitted because it can be carried out in the line of Theorem 3.5.

In the line of Theorem 3.5 and Theorem 3.6 and with the help of Theorem 3.3 one may easily establish the following two theorems:

Theorem 3.7. Let f and h be any two entire functions with $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ and g be an entire function with non zero L^*-lower order and $g(0) = 0$. Then for any positive integer α and β,

$$\lim_{r\to\infty} \frac{\log^{[2]} M_h^{-1} M_{f\circ g}(\exp(\exp(r^\alpha)))}{\log M_h^{-1} M_f(\exp(r^\beta)) + K(r,\alpha;L)} = \infty ,$$

where $K(r,\alpha;L) = \begin{cases} 0 \text{ if } r^\beta = o\{L(\exp(\exp(r^\alpha)))\} \\ \qquad\qquad \text{as } r\to\infty \\ L(\exp(\exp(r^\alpha))) \text{ otherwise .} \end{cases}$

Theorem 3.8. Let f, g and h be any three entire functions where g is an entire function with non zero L^*-lower order, $g(0) = 0$, $\lambda_h^{L^*}(f) > 0$ and $\rho_h^{L^*}(g) < \infty$. Then for any positive integer α and β,

$$\lim_{r\to\infty} \frac{\log^{[2]} M_h^{-1} M_{f\circ g}(\exp(\exp(r^\alpha)))}{\log M_h^{-1} M_g(\exp(r^\beta)) + K(r,\alpha;L)} = \infty ,$$

where $K(r,\alpha;L) = \begin{cases} 0 \text{ if } r^\beta = o\{L(\exp(\exp(r^\alpha)))\} \\ \qquad\qquad \text{as } r\to\infty \\ L(\exp(\exp(r^\alpha))) \text{ otherwise .} \end{cases}$

Theorem 3.9. *Let f, g and h be any three entire functions such that $\rho_g^{L^*} < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$. Then for any $\beta > 1$,*

$$\lim_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)\cdot K(r,g;L)} = 0 ,$$

where $K(r,g;L) = \begin{cases} 1 \textit{ if } L(\mu_g(\beta r)) = o\left\{r^\alpha e^{\alpha L(r)}\right\} \textit{ as } r\to\infty \\ \qquad\qquad \textit{and for some } \alpha < \lambda_h^{L^*}(f) \\ L(\mu_g(\beta r)) \textit{ otherwise.} \end{cases}$

Proof . Taking $R = \beta r$ in Lemma 2.1 and in view of Lemma 2.5 we have for all sufficiently large values of r that

$$\begin{aligned} \mu_{f\circ g}(r) &\leq \left(\frac{\alpha}{\alpha-1}\right)\mu_f\left(\frac{\alpha\beta}{(\beta-1)}\mu_g(\beta r)\right) \\ \textit{i.e., } \mu_{f\circ g}(r) &\leq \mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right). \end{aligned}$$

Since μ_h^{-1} is an increasing function, it follows from above for all sufficiently large values of r that

$$\mu_h^{-1}\mu_{f\circ g}(r) \leq \mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)$$

$$\textit{i.e., } \log \mu_h^{-1}\mu_{f\circ g}(r) \leq \log \mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)$$

$$\textit{i.e., } \log \mu_h^{-1}\mu_{f\circ g}(r) \leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\left[\log \mu_g(\beta r)\cdot e^{L\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)} + O(1)\right]$$

$$i.e.,\ \log \mu_h^{-1}\mu_{f\circ g}(r) \le \left(\rho_h^{L^*}(f)+\varepsilon\right)\left[\log \mu_g(\beta r) + L(\mu_g(\beta r)) + O(1)\right] \tag{3.11}$$

$$i.e.,\ \log \mu_h^{-1}\mu_{f\circ g}(r) \le \left(\rho_h^{L^*}(f)+\varepsilon\right)\left[\left\{\beta r e^{L(\beta r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)} + L(\mu_g(\beta r)) + O(1)\right]$$

$$i.e.,\ \log \mu_h^{-1}\mu_{f\circ g}(r) \le \left(\rho_h^{L^*}(f)+\varepsilon\right)\left[\left\{\beta r e^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)} + L(\mu_g(\beta r))\right]. \tag{3.12}$$

Also we obtain for all sufficiently large values of r that

$$\log \mu_h^{-1}\mu_f(r) \ge \left(\lambda_h^{L^*}(f)-\varepsilon\right)\log\left[re^{L(r)}\right]$$
$$i.e.,\ \log \mu_h^{-1}\mu_f(r) \ge \left(\lambda_h^{L^*}(f)-\varepsilon\right)\log\left[re^{L(r)}\right]$$
$$i.e.,\ \mu_h^{-1}\mu_f(r) \ge \left[re^{L(r)}\right]^{\left(\lambda_h^{L^*}(f)-\varepsilon\right)}\ . \tag{3.13}$$

Now from (3.12) and (3.13) we get for all sufficiently large values of r that

$$\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \le \frac{\left(\rho_h^{L^*}(f)+\varepsilon\right)\left[\left\{\beta r e^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)} + L(\mu_g(\beta r))\right]}{\left[re^{L(r)}\right]^{\left(\lambda_h^{L^*}(f)-\varepsilon\right)}}\ . \tag{3.14}$$

Since $\rho_g^{L^*} < \lambda_h^{L^*}(f)$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\rho_g^{L^*}+\varepsilon < \lambda_h^{L^*}(f)-\varepsilon\ . \tag{3.15}$$

Case I. Let $L(\mu_g(\beta r)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r\to\infty$ and for some $\alpha < \lambda_h^{L^*}(f)$.
As $\alpha < \lambda_h^{L^*}(f)$ we can choose $\varepsilon\,(>0)$ in such a way that

$$\alpha < \lambda_h^{L^*}(f)-\varepsilon\ . \tag{3.16}$$

Since $L(\mu_g(\beta r)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r\to\infty$ we get on using (3.16) that

$$\frac{L(\mu_g(\beta r))}{r^\alpha e^{\alpha L(r)}} \to 0 \text{ as } r\to\infty$$
$$i.e.,\ \frac{L(\mu_g(\beta r))}{\left[re^{L(r)}\right]^{\left(\lambda_h^{L^*}(f)-\varepsilon\right)}} \to 0 \text{ as } r\to\infty\ . \tag{3.17}$$

Now in view of (3.14), (3.15) and (3.17) we obtain that

$$\lim_{r\to\infty}\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} = 0\ . \tag{3.18}$$

Case II. If $L(\mu_g(\beta r)) \ne o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r\to\infty$ and for some $\alpha < \lambda_h^{L^*}(f)$ then we get from (3.14) that for a sequence of values of r tending to infinity that

$$\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r).L(\mu_g(\beta r))} \le \frac{\left(\rho_h^{L^*}(f)+\varepsilon\right)\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)}}{\left[re^{L(r)}\right]^{\left(\lambda_h^{L^*}(f)-\varepsilon\right)}.L(\mu(\beta r,g))} + \frac{\left(\rho_h^{L^*}(f)+\varepsilon\right)}{\left[re^{L(r)}\right]^{\left(\lambda_h^{L^*}(f)-\varepsilon\right)}}\ . \tag{3.19}$$

Now using (3.15) it follows from (3.19) that

$$\lim_{r\to\infty}\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)\cdot L(\mu_g(\beta r))} = 0\ . \tag{3.20}$$

Combining (3.18) and (3.20) we obtain that

$$\lim_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)\cdot L(\mu_g(\beta r))}=0\ ,$$

where $K(r,g;L)=\begin{cases}1 \text{ if } L(\mu_g(\beta r))=o\left\{r^\alpha e^{\alpha L(r)}\right\} \text{ as } r\to\infty \\ \qquad \text{and for some } \alpha<\lambda_h^{L^*}(f) \\ L(\mu_g(\beta r)) \text{ otherwise.}\end{cases}$

Thus the theorem is established. □

Theorem 3.10. Let f, g and h be any three entire functions with $\rho_g^{L^*}<\rho_h^{L^*}(f)<\infty$. Then for any $\beta>1$,

$$\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)\cdot K(r,g;L)}=0\ ,$$

where $K(r,g;L)=\begin{cases}1 \text{ if } L(\mu_g(\beta r))=o\left\{r^\alpha e^{\alpha L(r)}\right\} \text{ as } r\to\infty \\ \qquad \text{and for some } \alpha<\rho_h^{L^*}(f) \\ L(\mu_g(\beta r)) \text{ otherwise.}\end{cases}$

The proof of Theorem 3.10 is omitted because it can be carried out in the line of Theorem 3.9.

Theorem 3.11. Let f, g and h be any three entire functions such that $\rho_g^{L^*}<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$. Then

$$\lim_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_f(r)\cdot K(r,g;L)}=0\ ,$$

where $K(r,g;L)=\begin{cases}1 \text{ if } L(M_g(r))=o\left\{r^\alpha e^{\alpha L(r)}\right\} \text{ as } r\to\infty \\ \qquad \text{and for some } \alpha<\lambda_h^{L^*}(f) \\ L(M_g(r)) \text{ otherwise.}\end{cases}$

Theorem 3.12. Let f, g and h be any three entire functions with $\rho_g^{L^*}<\rho_h^{L^*}(f)<\infty$. Then

$$\liminf_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_f(r)\cdot K(r,g;L)}=0\ ,$$

where $K(r,g;L)=\begin{cases}1 \text{ if } L(M_g(r))=o\left\{r^\alpha e^{\alpha L(r)}\right\} \text{ as } r\to\infty \\ \qquad \text{and for some } \alpha<\rho_h^{L^*}(f) \\ L(M_g(r)) \text{ otherwise.}\end{cases}$

We omit the proof of Theorem 3.11 and Theorem 3.12 as those can be carried out with the help of Lemma 2.3 and in the line of Theorem 3.9 and Theorem 3.10.

Theorem 3.13. *Let f, g and h be any three entire functions such that $\rho_h^{L^*}(f)<\infty$, $\lambda_h^{L^*}(g)>0$ and $\rho_g^{L^*}<\infty$. Then for any $\beta>1$,*
(a) if $L(\mu_g(\beta r))=o\left\{\log\mu_h^{-1}\mu_g(r)\right\}$ then

$$\limsup_{r\to\infty}\frac{\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\mu_h^{-1}\mu_g(r)+L(\mu_g(\beta r))}\leq\frac{\rho_g^{L^*}}{\lambda_h^{L^*}(g)}$$

and (b) if $\log\mu_h^{-1}\mu_g(r)=o\{L(\mu_g(\beta r))\}$ then

$$\lim_{r\to\infty}\frac{\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\mu_h^{-1}\mu_g(r)+L(\mu_g(\beta r))}=0\ .$$

Proof . Taking $\log\left\{1+\frac{O(1)+L(\mu_g(\beta r))}{\log\mu_g(\beta r)}\right\}\sim\frac{O(1)+L(\mu_g(\beta r))}{\log\mu_g(\beta r)}$ we have from (3.11) for all sufficiently large values of r that

$$\log\mu_h^{-1}\mu_{f\circ g}(r)\le\left(\rho_h^{L^*}(f)+\varepsilon\right).\log\mu_g(\beta r)\left[1+\frac{O(1)+L(\mu_g(\beta r))}{\log\mu_g(\beta r)}\right]$$

$$i.e.,\quad\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)\le\log\left(\rho_h^{L^*}(f)+\varepsilon\right)+\log^{[2]}\mu_g(\beta r)+\log\left\{1+\frac{O(1)+L(\mu_g(\beta r))}{\log\mu_g(\beta r)}\right\}$$

$$i.e.,\quad\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)\le\log\left(\rho_h^{L^*}(f)+\varepsilon\right)+\left(\rho_g^{L^*}+\varepsilon\right)\log\left\{\beta re^{L(\beta r)}\right\}+\log\left\{1+\frac{O(1)+L(\mu_g(\beta r))}{\log\mu_g(\beta r)}\right\}$$

$$i.e.,\quad\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)\le\log\left(\rho_h^{L^*}(f)+\varepsilon\right)+\left(\rho_g^{L^*}+\varepsilon\right)\log\left\{\beta re^{L(r)}\right\}+\log\left\{1+\frac{O(1)+L(\mu_g(\beta r))}{\log\mu_g(\beta r)}\right\}$$

$$i.e.,\quad\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)\le O(1)+\left(\rho_g^{L^*}+\varepsilon\right)\{\log\beta r+L(r)\}+\frac{O(1)+L(\mu_g(\beta r))}{\log\mu_g(\beta r)}$$

$$i.e.,\quad\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)\le O(1)+\left(\rho_g^{L^*}+\varepsilon\right)\{\log r+L(r)\}+\left(\rho_g^{L^*}+\varepsilon\right)\log\beta+\frac{O(1)+L(\mu_g(\beta r))}{\log\mu_g(\beta r)}\ . \tag{3.21}$$

Again from the definition of relative L^*-lower order of an entire function with respect to another entire function in term of their maximum terms we have for all sufficiently large values of r that

$$\log\mu_h^{-1}\mu_g(r)\ge\left(\lambda_h^{L^*}(g)-\varepsilon\right)\log\left[re^{L(r)}\right]$$
$$i.e.,\quad\log\mu_h^{-1}\mu_g(r)\ge\left(\lambda_h^{L^*}(g)-\varepsilon\right)\log\left[re^{L(r)}\right]$$
$$i.e.,\quad\log\mu_h^{-1}\mu_g(r)\ge\left(\lambda_h^{L^*}(g)-\varepsilon\right)[\log r+L(r)]$$
$$i.e.,\quad\log r+L(r)\le\frac{\log\mu_h^{-1}\mu_g(r)}{\left(\lambda_h^{L^*}(g)-\varepsilon\right)}\ . \tag{3.22}$$

Hence from (3.21) and (3.22) , it follows for all sufficiently large values of r that

$$\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)\le O(1)+\left(\frac{\rho_g^{L^*}+\varepsilon}{\lambda_h^{L^*}(g)-\varepsilon}\right)\cdot\log\mu_h^{-1}\mu_g(r)+\left(\rho_g^{L^*}+\varepsilon\right)\log\beta+\frac{O(1)+L(\mu_g(\beta r))}{\log\mu_g(\beta r)}$$

$$i.e.,\quad\frac{\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\mu_h^{-1}\mu_g(r)+L(\mu_g(\beta r))}\le\frac{O(1)+\left(\rho_g^{L^*}+\varepsilon\right)\log\beta}{\log\mu_h^{-1}\mu_g(r)+L(\mu_g(\beta r))}+\left(\frac{\rho_g^{L^*}+\varepsilon}{\lambda_h^{L^*}(g)-\varepsilon}\right)\cdot\frac{\log\mu_h^{-1}\mu_g(r)}{\log\mu_h^{-1}\mu_g(r)+L(\mu_g(\beta r))}+\frac{O(1)+L(\mu_g(\beta r))}{\left[\log\mu_h^{-1}\mu_g(r)+L(\mu_g(\beta r))\right]\log\mu_g(\beta r)}$$

i.e.,

$$\frac{\log^{[2]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_g(r) + L(\mu_g(\beta r))} \leq \frac{\frac{O(1)+\left(\rho_g^{L^*}+\varepsilon\right)\log\beta}{L(\mu_g(\beta r))}}{\frac{\log \mu_h^{-1}\mu_g(r)}{L(\mu_g(\beta r))}+1} + \frac{\left(\frac{\rho_g^{L^*}+\varepsilon}{\lambda_h^{L^*}(g)-\varepsilon}\right)}{1+\frac{L(\mu_g(\beta r))}{\log \mu_h^{-1}\mu_g(r)}}$$
$$+\frac{1}{\left[1+\frac{\log \mu_h^{-1}\mu_g(r)}{L(\mu_g(\beta r))}\right]\log \mu_g(\beta r)}. \tag{3.23}$$

Since $L(\mu_g(\beta r)) = o\left\{\log \mu_h^{-1}\mu_g(r)\right\}$ as $r \to \infty$ and $\varepsilon (> 0)$ is arbitrary we obtain from (3.23) that

$$\limsup_{r\to\infty} \frac{\log^{[2]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_g(r) + L(\mu_g(\beta r))} \leq \frac{\rho_g^{L^*}}{\lambda_h^{L^*}(g)} . \tag{3.24}$$

Again if $\log \mu_h^{-1}\mu_g(r) = o\{L(\mu_g(\beta r))\}$ then from (3.23) we get that

$$\lim_{r\to\infty} \frac{\log^{[2]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_g(r) + L(\mu_g(\beta r))} = 0 . \tag{3.25}$$

Thus from (3.24) and (3.25), the theorem is established. □

Corollary 3.14. Let f, g and h be any three entire functions with $\rho_h^{L^*}(f) < \infty$, $\rho_h^{L^*}(g) > 0$ and $\rho_g^{L^*} < \infty$. Then for any $\beta > 1$,
(a) if $L(\mu_g(\beta r)) = o\left\{\log \mu_h^{-1}\mu_g(r)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[2]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_g(r) + L(\mu_g(\beta r))} \leq \frac{\rho_g^{L^*}}{\rho_h^{L^*}(g)}$$

and (b) if $\log \mu_h^{-1}\mu_g(r) = o\{L(\mu_g(\beta r))\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[2]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_g(r) + L(\mu_g(\beta r))} = 0 .$$

We omit the proof of Corollary 3.14 because it can be carried out in the line of Theorem 3.13.

Theorem 3.15. Let f, g and h be any three entire functions such that $\rho_h^{L^*}(f) < \infty$, $\lambda_h^{L^*}(g) > 0$ and $\rho_g^{L^*} < \infty$. Then
(a) if $L(M_g(r)) = o\left\{\log M_h^{-1}M_g(r)\right\}$ then

$$\limsup_{r\to\infty} \frac{\log^{[2]} M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_g(r) + L(M_g(r))} \leq \frac{\rho_g^{L^*}}{\lambda_h^{L^*}(g)}$$

and (b) if $\log M_h^{-1}M_g(r) = o\{L(M_g(r))\}$ then

$$\lim_{r\to\infty} \frac{\log^{[2]} M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_g(r) + L(M_g(r))} = 0 .$$

Corollary 3.16. Let f, g and h be any three entire functions with $\rho_h^{L^*}(f) < \infty$, $\rho_h^{L^*}(g) > 0$ and $\rho_g^{L^*} < \infty$.
(a) If $L(M_g(r)) = o\left\{\log M_h^{-1} M_g(r)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[2]} M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_g(r) + L(M_g(r))} \leq \frac{\rho_g^{L^*}}{\rho_h^{L^*}(g)}$$

and (b) if $\log M_h^{-1} M_g(r) = o\{L(M_g(r))\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[2]} M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_g(r) + L(M_g(r))} = 0\ .$$

We omit the proof of Theorem 3.15 and Corollary 3.16 because those can be respectively carried out in the line of Theorem 3.13 and Corollary 3.14 and with the help of Lemma 2.3.

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