Near-field Response in Lossy Media with Exponential Conductivity Inhomogeneity

R. J. Riddolls
Defence R&D Canada–Ottawa, Canada

Abstract—This paper examines the near-field response to source currents in lossy media with exponential conductivity inhomogeneity. The motivation for this work is to understand the modification of the polar ionosphere D region (50–90 km altitude) by powerful high frequency transmitters. The transmitted waves heat the D region plasma, causing a localized conductivity perturbation. In the presence of the DC electric field of the polar electrojet, the conductivity perturbation produces a current perturbation referred to as "antenna current" that can drive extremely/very low frequency radiation. Here we seek to understand the production of antenna current in a strongly inhomogeneous plasma. In the lower D region, the static approximation is valid, and we solve using a scalar potential description. In the upper D region, we use the magnetoquasistatic approximation and solve using a vector potential approach.

1. Introduction

We begin the formulation by defining standard scalar and vector potentials for the electric and magnetic field perturbations introduced by the conductivity perturbation. In time-harmonic form, we have

$$\mathbf{E} = i\omega\mathbf{A} - \nabla\Phi \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A}, \tag{1}$$

where i is the imaginary unit and ω is frequency. Let us suppose that the charge relaxation time and electromagnetic transit time are both small compared to the time scale of interest. This assumption allows us to ignore the effect of displacement current, so that current consists of only the imposed antenna current $\mathbf{J}_s$ due to the conductivity perturbation, and a self-consistent conduction current $\sigma\mathbf{E}$, where σ is the conductivity of the medium. Adopting a Coulomb gauge, the wave equation is given by

$$\nabla^2\mathbf{A} + i\omega\mu_0\sigma\mathbf{A} = -\mu_0\mathbf{J_s} + \mu_0\sigma\nabla\Phi, \tag{2}$$

where μ_0 is the permeability of the medium, assumed the same as free space. The two terms on the right side can be viewed as source terms for the vector potential. We will proceed as follows. In the lower ionosphere D region, the conductivity is small such that the magnetic relaxation time is fast compared to the time scale of interest, and thus we ignore effects of vector potential. In the upper D region, the conductivity is large such that the magnetic relaxation time is slower than the time scale of interest. In this case, magnetic diffusion dominates the behaviour of the system, and we ignore the effects of space charge and its associated scalar potential. We will analyze each of the two limits.

The above statements assume a simple scalar conductivity. In practice, the plasma conductivity is anisotropic and requires a matrix representation. In the northern polar region the direction z (altitude) is antiparallel the earth's magnetic field. The appropriate conductivity tensor is given by

$$\boldsymbol{\sigma} = e^{hz} \left[\begin{array}{ccc} \sigma_P & \sigma_H & 0 \\ -\sigma_H & \sigma_P & 0 \\ 0 & 0 & \sigma_0 \end{array} \right], \tag{3}$$

where $1/h$ is the scale height of the conductivity. Here, the exponential factor models the variability in the plasma conductivity due to the plasma density inhomogeneity, and the matrix entries are constants pertaining to the anisotropic plasma conductivity tensor. The quantity σ_P is the Pedersen conductivity, σ_H is the Hall conductivity, and σ_0 is the specific conductivity. We are assuming that all conductivities vary in altitude at the same rate. Strictly speaking this is not the case as the specific conductivity increases with altitude somewhat more rapidly than the Pedersen or Hall conductivities. However, for the purposes of a simple treatment, we ignore the fine details of the altitude dependence of the individual conductivity elements.

2. Static Solution

We now turn to the problem of determining the scalar potential Φ in the static limit. If we incorporate the tensor definition for $\boldsymbol{\sigma}$ into Equation (2), ignore the vector potential, and take the divergence of both sides, we find that

$$\nabla^2 \Phi + \left(\frac{\sigma_0}{\sigma_P} - 1\right)\frac{\partial^2 \Phi}{\partial z^2} + \frac{h\sigma_0}{\sigma_P}\frac{\partial \Phi}{\partial z} = \frac{e^{-hz}}{\sigma_P}\nabla \cdot \mathbf{J}_s \equiv S(\mathbf{r}), \tag{4}$$

where $S(\mathbf{r})$ is the source distribution. Let us expand the right and left sides of Equation (4):

$$S(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mathbf{r}_0 S(\mathbf{r}_0)\delta(\mathbf{r} - \mathbf{r}_0) \tag{5}$$

$$\Phi(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mathbf{r}_0 S(\mathbf{r}_0)G_\Phi(\mathbf{r}, \mathbf{r}_0). \tag{6}$$

Inserting these expansions into Equation (4) yields an expression for the Green's function $G_\Phi(\mathbf{r}, \mathbf{r}_0)$:

$$\nabla^2 G_\Phi(\mathbf{r}, \mathbf{r}_0) + \left(\frac{\sigma_0}{\sigma_P} - 1\right)\frac{\partial^2 G_\Phi(\mathbf{r}, \mathbf{r}_0)}{\partial z^2} + \frac{h\sigma_0}{\sigma_P}\frac{\partial G_\Phi(\mathbf{r}, \mathbf{r}_0)}{\partial z} = \delta(\mathbf{r} - \mathbf{r}_0). \tag{7}$$

This is a constant coefficient equation, and therefore $G_\Phi(\mathbf{r}, \mathbf{r}_0)$ is the same as $G_\Phi(\mathbf{r} - \mathbf{r}_0)$. We can write

$$\nabla^2 G_\Phi(\mathbf{r}) + \left(\frac{\sigma_0}{\sigma_P} - 1\right)\frac{\partial^2 G_\Phi(\mathbf{r})}{\partial z^2} + \frac{h\sigma_0}{\sigma_P}\frac{\partial G_\Phi(\mathbf{r})}{\partial z} = \delta(\mathbf{r}). \tag{8}$$

This equation solves easily using the method of Fourier transforms. Taking the Fourier transform of Equation (8), solving for $G_\Phi(\mathbf{k})$, and then inverse transforming, results in the following solution for $G_\Phi(\mathbf{r})$:

$$G_\Phi(\mathbf{r}) = -\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mathbf{k}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k_x^2 + k_y^2 + (\sigma_0/\sigma_P)k_z^2 - ih(\sigma_0/\sigma_P)k_z}. \tag{9}$$

We can now convert Equation (9) to cylindrical co-ordinates (ρ, ϕ, z) and (k_ρ, α, k_z) and perform the integrals:

$$G_\Phi(\mathbf{r}) = -\frac{1}{8\pi^3}\int_0^{\infty} dk_\rho k_\rho \int_{-\infty}^{\infty} dk_z \frac{e^{ik_z z}}{k_\rho^2 + (\sigma_0/\sigma_P)k_z^2 - ih(\sigma_0/\sigma_P)k_z}\int_0^{2\pi} d\alpha e^{ik_\rho \rho \cos(\phi - \alpha)} \tag{10}$$

$$= -\frac{1}{4\pi^2}\int_0^{\infty} dk_\rho k_\rho J_0(k_\rho \rho)\int_{-\infty}^{\infty} dk_z \frac{e^{ik_z z}}{k_\rho^2 + (\sigma_0/\sigma_P)k_z^2 - ih(\sigma_0/\sigma_P)k_z} \tag{11}$$

$$= -\frac{e^{-hz/2}}{2\pi\sigma_0/\sigma_P}\int_0^{\infty} dk_\rho \frac{k_\rho J_0(k_\rho \rho)e^{-\sqrt{(h/2)^2 + (\sigma_P/\sigma_0)k_\rho^2}|z|}}{\sqrt{(h/2)^2 + (\sigma_P/\sigma_0)k_\rho^2}} \tag{12}$$

$$= -\frac{e^{-hz/2 - h\sqrt{(\sigma_0/\sigma_P)\rho^2 + z^2}/2}}{4\pi\sqrt{(\sigma_0/\sigma_P)\rho^2 + z^2}}. \tag{13}$$

The integral over k_z above is facilitated by the residue theorem, and the integral over k_ρ uses the following identity

$$\int_1^{\infty} du e^{-\alpha u} J_0(\beta\sqrt{u^2 - 1}) = \frac{e^{-\sqrt{\alpha^2 + \beta^2}}}{\sqrt{\alpha^2 + \beta^2}}, \tag{14}$$

which can be found in standard tables. The scalar potential for a given source distribution can then be found by integrating this Green's function over the source distribution. The basic form of the scalar potential is similar to that of sources in homogeneous isotropic media, except there is exponential decay in the upward direction, and the potential is squeezed in the ρ direction compared to the z direction by a factor corresponding to the degree of anisotropy σ_0/σ_P. We also note that the Hall conductivity σ_H does not play a factor in the static scalar potential.

3. Static Solution Example

In this section, we provide an example of the static solution. Let us consider a current source $\mathbf{J}_s$ that consists of a horizontal cylinder-like structure modelled by

$$\mathbf{J}_s = \hat{\mathbf{x}} I \delta(y) \delta(z) [\mu(x + L/2) - \mu(x - L/2)], \tag{15}$$

where I is the current and L is the cylinder length. The source distribution is given by

$$S(\mathbf{r}_0) = (e^{-hz}/\sigma_P) \nabla \cdot \mathbf{J}_s \tag{16}$$
$$= (I/\sigma_P)[\delta(\mathbf{r}_0 + \hat{\mathbf{x}} L/2) - \delta(\mathbf{r}_0 - \hat{\mathbf{x}} L/2)]. \tag{17}$$

The potential is given by

$$\Phi(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\mathbf{r}_0 S(\mathbf{r}_0) G_\Phi(\mathbf{r} - \mathbf{r}_0) \tag{18}$$

$$= \frac{I e^{-hz/2}}{4\pi\sigma_P} \left\{ \frac{e^{-h\sqrt{(\sigma_0/\sigma_P)[(x-L/2)^2+y^2]+z^2}/2}}{\sqrt{(\sigma_0/\sigma_P)[(x-L/2)^2+y^2]+z^2}} - \frac{e^{-h\sqrt{(\sigma_0/\sigma_P)[(x+L/2)^2+y^2]+z^2}/2}}{\sqrt{(\sigma_0/\sigma_P)[(x+L/2)^2+y^2]+z^2}} \right\}. \tag{19}$$

The total current $\mathbf{J} = \mathbf{J}_s - \boldsymbol{\sigma} \cdot \nabla\Phi$, near the z axis, is given by:

$$\mathbf{J}_{(x,y)\approx 0} = (\mathbf{J}_s - \boldsymbol{\sigma} \cdot \nabla\Phi)_{(x,y)\approx 0} \tag{20}$$
$$= \hat{\mathbf{x}} I \delta(y) \delta(z) - (\hat{\mathbf{x}}\sigma_P - \hat{\mathbf{y}}\sigma_H) I L \sigma_0 (1 + hw/2) \exp[h(z - w)/2]/(4\pi\sigma_P^2 w^3), \tag{21}$$

where $w = \sqrt{(\sigma_0/\sigma_P)(L/2)^2 + z^2}$. The conduction current $-\boldsymbol{\sigma} \cdot \nabla\Phi$ flows largely above the origin, opposite the source current, effectively forming a vertical current loop. The conduction current distributions are shown for $L = 15\,\text{km}$ and the cases of homogeneous isotropic, inhomogeneous isotropic, and inhomogeneous anisotropic media.
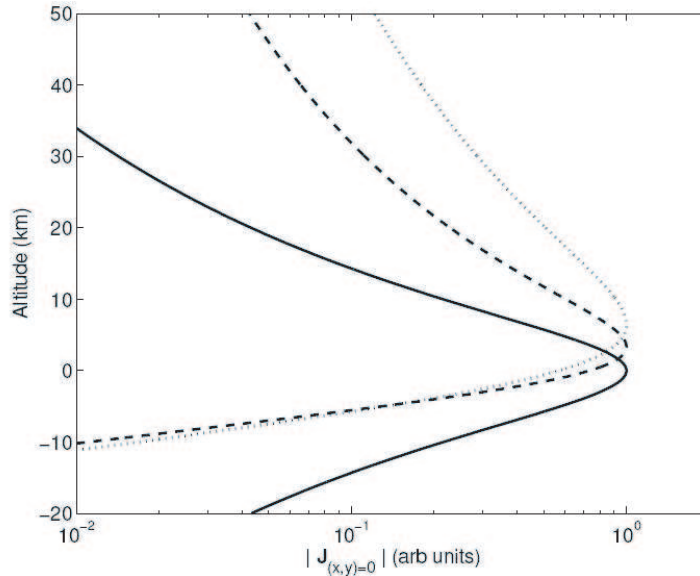

Figure 1: Static conduction current distributions. Solid line: homogeneous isotropic media. Dashed line: inhomogeneous isotropic media ($1/h = 2.5\,\text{km}$). Dotted line: inhomogeneous anisotropic media ($1/h = 2.5\,\text{km}$, $\sigma_0/\sigma_P = 2$).

4. Magnetoquasistatic Solution

Let us now consider the problem of determining the vector potential relevant to the magnetoquasistatic limit. Returning to Equation (2), we ignore the scalar potential so that we have

$$\nabla^2 \mathbf{A} + i\omega\mu_0 \boldsymbol{\sigma} \cdot \mathbf{A} = -\mu_0 \mathbf{J}_s. \tag{22}$$

By Equation (4), the z component is decoupled from the x and y components. Since the current perturbation $\mathbf{J}_s$ is generally horizontally directed in practical situations, A_z is not driven, and we assume it is zero. The x and y components are decoupled by transforming to a basis of eigenvectors of the conductivity tensor:

$$\begin{bmatrix} \hat{A}_x \\ \hat{A}_y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix}. \tag{23}$$

After the transformation the equations for the vector potential components $\hat{A}_x$ and $\hat{A}_y$ can be written as

$$\nabla^2 \begin{bmatrix} \hat{A}_x \\ \hat{A}_y \end{bmatrix} + i\omega\mu_0 e^{hz} \begin{bmatrix} \sigma_P + i\sigma_H & 0 \\ 0 & \sigma_P - i\sigma_H \end{bmatrix} \begin{bmatrix} \hat{A}_x \\ \hat{A}_y \end{bmatrix} = -\frac{\mu_0}{\sqrt{2}} \begin{bmatrix} J_{sx} - iJ_{sy} \\ J_{sx} + iJ_{sy} \end{bmatrix} \equiv -\mu_0 \hat{\mathbf{J}}_s. \tag{24}$$

The Green's function for a component of $\hat{\mathbf{A}}$ is given by

$$\left[\nabla^2 + i\omega\mu_0 e^{hz}(\sigma_P \pm i\sigma_H) \right] G_{\hat{A}}(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0). \tag{25}$$

In view of the e^{hz} factor, $G_{\hat{A}}(\mathbf{r}, \mathbf{r}_0) = G_{\hat{A}}(x - x_0, y - y_0, z, z_0) \neq G_{\hat{A}}(\mathbf{r} - \mathbf{r}_0)$. Thus we write

$$\left[\nabla^2 + i\omega\mu_0 e^{h(z+z_0)}(\sigma_P \pm i\sigma_H) \right] G_{\hat{A}}(x, y, z + z_0, z_0) = \delta(\mathbf{r}). \tag{26}$$

A solution by the method of Fourier transforms is confounded by the e^{hz} factor. Thus we transform in the x and y directions only, which converts the partial differential Equation (25) into an ordinary differential equation:

$$\left[\frac{\partial^2}{\partial z^2} - k_\rho^2 + i\omega\mu_0 e^{h(z+z_0)}(\sigma_P \pm i\sigma_H) \right] G_{\hat{A}}(k_x, k_y, z + z_0, z_0) = \delta(z). \tag{27}$$

The solutions are the Bessel functions $J_\nu[\lambda e^{h(z+z_0)/2}]$ and $Y_\nu[\lambda e^{h(z+z_0)/2}]$, with $\lambda = 2\sqrt{i\omega\mu_0(\sigma_P \pm i\sigma_H)}/h$ and $\nu = 2k_\rho/h$. In the $z \to \infty$ limit, the only bounded linear combination of solutions for $0 < \arg(\lambda) < \pi$ is a Hankel function of the form $C_1 H_\nu^{(1)}[\lambda e^{h(z+z_0)/2}]$. Similarly, in the $z \to -\infty$ limit, the only bounded solution for all complex λ is a Bessel function of the form $C_2 J_\nu[\lambda e^{h(z+z_0)/2}]$. To determine the constants C_1 and C_2 we impose that the solutions in the regions $z > 0$ and $z < 0$ are continuous at $z = 0$:

$$C_1 H_\nu^{(1)}\left(\lambda e^{hz_0/2}\right) - C_2 J_\nu\left(\lambda e^{hz_0/2}\right) = 0, \tag{28}$$

and that inhomogeneous Equation (27) is satisfied, which is done by integrating over a small interval at $z = 0$:

$$C_1 H_\nu^{(1)\prime}\left(\lambda e^{hz_0/2}\right) - C_2 J_\nu'\left(\lambda e^{hz_0/2}\right) = 2/\left(h\lambda e^{hz_0/2}\right). \tag{29}$$

Recalling the Wronskian relationship $W_z[J_\nu(z), H_\nu^{(1)}(z)] = 2i/(\pi z)$, the solution for C_1 and C_2 is

$$C_1 = -i\pi J_\nu\left(\lambda e^{hz_0/2}\right)/h \qquad C_2 = -i\pi H_\nu^{(1)}\left(\lambda e^{hz_0/2}\right)/h. \tag{30}$$

$G_{\hat{A}}(x, y, z + z_0, z_0)$ is found by performing the inverse Fourier transforms, which in cylindrical coordinates are

$$\begin{aligned} G_{\hat{A}}(x, y, z + z_0, z_0) &= \frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} dk_\rho d\alpha k_\rho e^{ik_\rho\rho\cos(\phi-\alpha)} G_{\hat{A}}(k_\rho, \alpha, z + z_0, z_0) \tag{31} \\ &= -\frac{i}{2h} \int_0^\infty dk_\rho k_\rho J_0(k_\rho\rho) J_\nu\left[\lambda e^{h(z+z_0)/2}\mu(-z)\right. \\ &\quad \left. + \lambda e^{hz_0/2}\mu(z)\right] H_\nu^{(1)}\left[\lambda e^{h(z+z_0)/2}\mu(z) + \lambda e^{hz_0/2}\mu(-z)\right], \tag{32} \end{aligned}$$

where $\mu(z)$ is the Heaviside step function. Therefore $G_{\hat{A}}(\mathbf{r}, z_0)$ is given by

$$\begin{aligned} G_{\hat{A}}(\mathbf{r}, z_0) &= -\frac{i}{2h} \int_0^\infty dk_\rho k_\rho J_0(k_\rho\rho) J_\nu\left[\lambda e^{hz/2}\mu(z_0 - z)\right. \\ &\quad \left. + \lambda e^{hz_0/2}\mu(z - z_0)\right] H_\nu^{(1)}\left[\lambda e^{hz/2}\mu(z - z_0) + \lambda e^{hz_0/2}\mu(z_0 - z)\right]. \tag{33} \end{aligned}$$

We find $\mathbf{A}$ by integrating $G_{\hat{A}}(\mathbf{r}, z_0)$ over the source $-\mu_0 \hat{\mathbf{J}}_s$ and transforming $\hat{\mathbf{A}}$ to $\mathbf{A}$ using Equation (23).

5. Magnetoquasistatic Solution Example

We consider, as an analytically tractable example, the response to a current sheet

$$\mathbf{J}_s = \hat{\mathbf{x}} K \delta(z), \tag{34}$$

where K is a surface current density. The response for a component of $\hat{\mathbf{A}}$ is found as follows

$$\hat{A} = -\mu_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\mathbf{r}_0 K \delta(z_0) G_{A'}(\mathbf{r}, z_0) \tag{35}$$

$$= \frac{iK\mu_0}{2h} \int_0^{\infty} \int_0^{2\pi} d\rho \, d\phi \rho \int_0^{\infty} dk_\rho k_\rho J_0(k_\rho \rho) J_\nu \left[\lambda e^{hz/2} \mu(-z) + \lambda\mu(z) \right] H_\nu^{(1)} \left[\lambda e^{hz/2} \mu(z) + \lambda\mu(-z) \right] \tag{36}$$

$$= \frac{i\pi K\mu_0}{h} \int_0^{\infty} dk_\rho \delta(k_\rho) J_\nu \left[\lambda e^{hz/2} \mu(-z) + \lambda\mu(z) \right] H_\nu^{(1)} \left[\lambda e^{hz/2} \mu(z) + \lambda\mu(-z) \right] \tag{37}$$

$$= \frac{i\pi K\mu_0}{h} J_0 \left[\lambda e^{hz/2} \mu(-z) + \lambda\mu(z) \right] H_0^{(1)} \left[\lambda e^{hz/2} \mu(z) + \lambda\mu(-z) \right]. \tag{38}$$

The x component of the conduction current $i\omega\boldsymbol{\sigma}\cdot\mathbf{A}$ is shown in Fig. 2. The upper cutoff of the conduction current distribution results from the exponential increase in magnetic diffusion time with altitude, and the lower cutoff arises from the exponential decrease in conductivity.
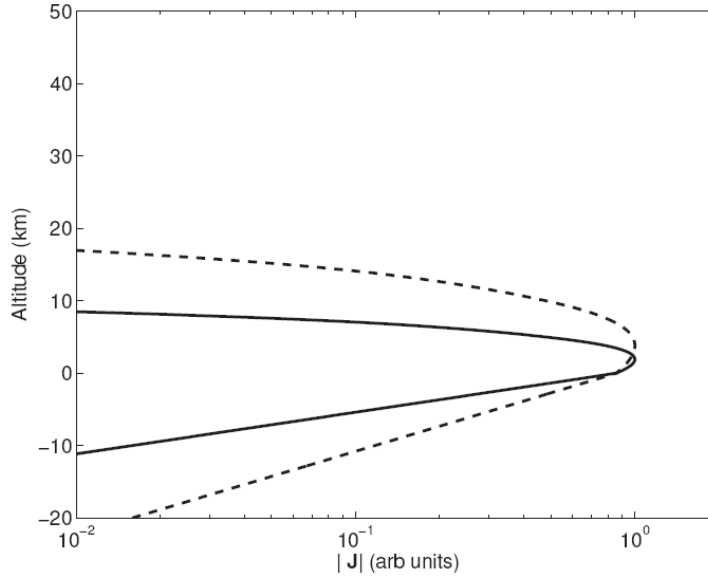

Figure 2: Magnetoquasistatic conduction current distributions. Solid line: $1/h = 2.5\,\mathrm{km}$, $1/\sqrt{\omega\mu_0\sigma_p} = 100\,\mathrm{km}$, $\sigma_P = \sigma_H$. Dashed line: $1/h = 5.0\,\mathrm{km}$, $1/\sqrt{\omega\mu_0\sigma_p} = 100\,\mathrm{km}$, $\sigma_P = \sigma_H$.

6. Conclusion

This work has determined the response of inhomogeneous, anisotropic media to conductivity perturbations in the static and magnetoquasistatic limits. The responses have been characterized as Green's functions, which can provide the response current distribution if the source currents are known a priori. Some simple source currents have been considered here. More discussion of ionospheric source currents can be found in [1].

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