Separation in Stability Analysis of Piecewise Linear Systems in Discrete Time

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Abstract. Stability analysis of piecewise linear systems, without affine terms, consists of the problem of finding maximal stabilizing sets of switching paths among possible system coefficients and that of obtaining a sequence of state-space partitions in the order of increasing refinement. Exploiting the fact that these two problems can be solved separately, one can find subsets of the state space such that the piecewise linear system restricted to these sets is uniformly exponentially stable.

1 Introduction

Successful analysis of the stability of a piecewise linear system hinges on one's ability to construct an appropriate Lyapunov function. Common approaches involve piecewise quadratic Lyapunov functions [1–4] and piecewise higher-order polynomial Lyapunov functions [5, 6]. However, these approaches are conservative because only a subset of all asymptotically stable piecewise linear systems admits these types of Lyapunov functions.

We focus on discrete-time piecewise linear systems under a polyhedral partition of the state space but without affine terms, and propose that the problem of determining the asymptotic stability of such a system be divided into two separate problems. The first problem draws on the recent characterization of all uniformly stabilizing sets of switching sequences [7]. To obtain stabilizing switching sequences, it suffices to obtain so-called maximal admissible sets of switching paths of length L over $L = 0, 1, \ldots$. These sets are independent of the switching structure imposed by the underlying state-dependent switching among different system coefficients, and associated with each of them is a switching-path-dependent quadratic Lyapunov function. On the other hand, the second problem is to obtain all admissible polyhedral partitions of depth L over $L = 0, 1, \ldots$. The task here is to explore the underlying switching structure of the system by obtaining an increasing family of state-space partitions. This task can be done regardless of how each state-space partition affects the form of the Lyapunov function. Combining these two problems leads to a novel stability analysis method for piecewise linear systems.

2 Problem Formulation

Let $\mathcal{A} = \{\mathbf{A}_1, \ldots, \mathbf{A}_N\}$ with $\mathbf{A}_1, \ldots, \mathbf{A}_N \in \mathbb{R}^{n\times n}$. Let $\mathcal{D} = \{D_1, \ldots, D_N\}$ be a partition of $\mathbb{R}^n$ (i.e., $\bigcup_{i=1}^N D_i = \mathbb{R}^n$ and $D_i \cap D_j = \varnothing$ whenever $i \neq j$). Then the pair $(\mathcal{A}, \mathcal{D})$ defines the discrete-time piecewise linear system represented by

$$\mathbf{x}(t+1) = \mathbf{A}_{\theta(t)}\mathbf{x}(t) \tag{1}$$

with $\theta(t) = \{i \colon \mathbf{x}(t) \in D_i\}$ for $t = 0, 1, \ldots$.

Definition 1. *Let $C \subset \mathbb{R}^n$. The pair $(\mathcal{A}, \mathcal{D})$ is said to be* C-uniformly exponentially stable *if there exist $c \geq 1$ and $\lambda \in (0,1)$ such that*

$$\|\mathbf{x}(t)\| \leq c\lambda^{t-t_0}\|\mathbf{x}(t_0)\| \tag{2}$$

for all t_0, $t \in \{0, 1, \ldots\}$ with $t \geq t_0$ and for all $\mathbf{x}(t_0) \in C$.

Given a pair $(\mathcal{A}, \mathcal{D})$, our stability analysis problem is *to determine a (maximal) set $C \subset \mathbb{R}^n$ such that the pair $(\mathcal{A}, \mathcal{D})$ is C-uniformly exponentially stable.*

3 Two Separate Problems

The first problem is to find maximal admissible sets of switching paths. Let $\Theta \subset \{1, \ldots, N\}^\infty$ be nonempty. The pair $(\mathcal{A}, \Theta)$ defines the discrete-time switched linear system represented by (1) over all $(\theta(0), \theta(1), \ldots) \in \Theta$. Searching for all Θ such that the pair $(\mathcal{A}, \Theta)$ is uniformly exponentially stable amounts to finding (the countable family of) all $\mathcal{A}$-maximal sets [7]. We shall write $\mathbf{X} < \mathbf{0}$ to mean that $\mathbf{X}$ is symmetric and negative definite. To simplify notation, set $(i_j, \ldots, i_k) = 0$ if $j > k$, and set $\{1, \ldots, N\}^0 = \{0\}$.

Definition 2. *The pair $(\mathcal{A}, \Theta)$ is said to be* uniformly exponentially stable *if there exist $c \geq 1$ and $\lambda \in (0,1)$ such that (2) holds for all t_0, $t \in \{0, 1, \ldots\}$ with $t \geq t_0$, for all $\mathbf{x}(t_0) \in \mathbb{R}^n$, and for all $(\theta(0), \theta(1), \ldots) \in \Theta$.*

Definition 3. *Let L be a nonnegative integer. Elements of $\{1, \ldots, N\}^{L+1}$ are called* L-paths. *A nonempty set $\mathcal{N}$ of L-paths is said to be* $\mathcal{A}$-admissible *if, for each $(i_0, \ldots, i_L) \in \mathcal{N}$, there exist an integer $M > L$ and a $(i_{L+1}, \ldots, i_M) \in \{1, \ldots, N\}^{M-L}$ such that $(i_{M-L}, \ldots, i_M) = (i_0, \ldots, i_L)$ and $(i_t, \ldots, i_{t+L}) \in \mathcal{N}$ for $0 \leq t \leq M-L$, and if there exist matrices $\mathbf{X}_{(j_1,\ldots,j_L)} > \mathbf{0}$ such that*

$$\mathbf{A}_{i_L}^{\mathrm{T}}\mathbf{X}_{(i_1,\ldots,i_L)}\mathbf{A}_{i_L} - \mathbf{X}_{(i_0,\ldots,i_{L-1})} < \mathbf{0} \tag{3}$$

for all L-paths $(i_0, \ldots, i_L) \in \mathcal{N}$. Moreover, if the only $\mathcal{A}$-admissible $\widetilde{\mathcal{N}}$ with $\widetilde{\mathcal{N}} \subset \mathcal{N}$ (resp. $\mathcal{N} \subset \widetilde{\mathcal{N}}$) is $\mathcal{N}$ itself, then $\mathcal{N}$ is called $\mathcal{A}$-minimal *(resp.* $\mathcal{A}$-maximal*).*

Lemma 4. [7] *There exists a nonempty $\Theta \subset \Omega$ such that the pair $(\mathcal{A}, \Theta)$ is uniformly exponentially stable if and only if there exist an integer $L \geq 0$ and an $\mathcal{A}$-admissible $\mathcal{N} \subset \{1, \ldots, N\}^{L+1}$. Associated with each $\mathcal{A}$-minimal $\mathcal{N}$ is a periodic $\theta = (\theta(0), \theta(1), \ldots)$ such that $(\mathcal{A}, \{\theta\})$ is uniformly exponentially stable.*

The second problem is to generate a countable family of partitions of the state space in the order of increasing refinement. Each of these partitions are made according to the switching structure that the underlying state-dependence of the switching sequence dictates. Define sets $D_{(i_0,\dots,i_L)} \subset \mathbb{R}^n$ recursively by

$$D_{(i_0,\dots,i_{L+1})} = \{\mathbf{x} \in D_{(i_0,\dots,i_L)} \colon \mathbf{A}_{i_0}\mathbf{x} \in D_{(i_1,\dots,i_{L+1})}\}$$

for $L = 0,\ 1,\ \dots$ and for $(i_0,\dots,i_L) \in \{1,\dots,N\}^{L+1}$. Then, for each L, the indexed family $\{D_{(i_0,\dots,i_L)} \colon (i_0,\dots,i_L) \in \{1,\dots,N\}^{L+1}\}$ defines a partition of $\mathbb{R}^n$, which we shall call an *L-path partition* of $\mathbb{R}^n$.

4 Proposed Algorithm for Stability Analysis

We propose that the stability analysis formulated in Section 2 be tackled by combining the two decoupled problems described in Section 3. Suppose we have solved the two problems described above. Let us fix a nonnegative integer L, and suppose that $\mathcal{D}_L$ and $\mathcal{D}_{L+1}$ are the L-path partition and $(L+1)$-path partition of the state space. Partition $\mathcal{D}_{L+1}$ is finer than $\mathcal{D}_L$ and enables one to construct a switching sequence as follows: given a nonempty $D_{(i_0,\dots,i_L)} \in \mathcal{D}_L$, let $\theta(0) = i_0$, $\dots$, $\theta(L) = i_L$; if there exists a nonempty $D_{(i_0,\dots,i_L,i_{L+1})} \in \mathcal{D}_{L+1}$, then let $\theta(L+1) = i_{L+1}$; if there exists a nonempty $D_{(i_1,\dots,i_{L+1},i_{L+2})} \in \mathcal{D}_{L+1}$, then let $\theta(L+2) = i_{L+2}$; and so on. Any switching sequence that can be constructed this way generates an infinite chain of L-paths, which we shall call a *chain of L-paths generated by $D_{(i_0,\dots,i_L)}$ and $\mathcal{D}_{L+1}$*. The following is immediate by construction:

Lemma 5. *Let $D_{(i_0,\dots,i_L)} \in \mathcal{D}_L$. If each chain of L-paths generated by $D_{(i_0,\dots,i_L)}$ and $\mathcal{D}_{L+1}$ has a limit set that is contained in an $\mathcal{A}$-maximal set of L-paths, then the piecewise linear system $(\mathcal{A},\mathcal{D})$ is $D_{(i_0,\dots,i_L)}$-uniformly exponentially stable.*

This lemma suggests an algorithm to generate a nested sequence $C_0 \subset C_1 \subset \cdots$ such that the pair $(\mathcal{A},\mathcal{D})$ is C_i-uniformly exponentially stable for each i:

Step 0. Set $C_{-1} = \varnothing$; set $L = 0$.
Step 1. Obtain the partition $\mathcal{D}_{L+1}$ of the state space.
Step 2. Obtain $\mathcal{A}$-maximal sets of L-paths.
Step 3. Let C_L be the union of C_{L-1} and all $D_{(i_0,\dots,i_L)}$ such that each chain of L-paths generated by $D_{(i_0,\dots,i_L)}$ and $\mathcal{D}_{L+1}$ has a limit set that is contained in an $\mathcal{A}$-maximal set of L-paths.
Step 4. Increment L to $L+1$; go to Step 1.

For example, if $N = 2$ and if $\mathcal{A}$ and $\mathcal{D}$ have

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ -1/2 & 3/2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1/2 & 1 \\ -1 & 1/2 \end{bmatrix}; \quad \begin{cases} \mathbf{D}_1 = \left\{[x_1\ x_2]^{\mathrm{T}} \in \mathbb{R}^2 \colon x_1 \geq x_2\right\}, \\ \mathbf{D}_2 = \left\{[x_1\ x_2]^{\mathrm{T}} \in \mathbb{R}^2 \colon x_1 < x_2\right\}, \end{cases}$$

then the algorithm gives us $C_0 = C_1 = C_2 = \varnothing$, $C_3 = D_{1212} \cup D_{2121} \cup D_{2212}$, and $C_4 = C_5 = \cdots = D_{1212} \cup D_{2121} \cup D_{2212} \cup D_{22212}$. In this particular example, the

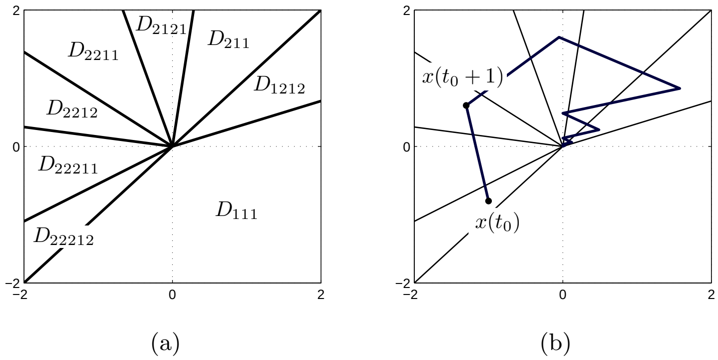

Fig. 1. Illustrative example. (a) the four-path partition of the state space. (b) a typical state trajectory converging to the origin.

process of iteratively partitioning the state space terminates at the path length of $L = 4$ since none of the states in $\mathbb{R}^2 \setminus C_4$ converges to the origin. The stability of the pair $(\mathcal{A}, \mathcal{D})$ can be completely assessed using the four-path partition given by Fig. 1(a); a typical state trajectory that starts in C_4 is depicted in Fig. 1(b).

5 Conclusion

A novel stability analysis method was proposed based on the fact that the task of characterizing all stabilizing sets of switching sequences can be done independently of that of successively refining the partition of the state space. Questions to be answered regarding the algorithm presented in Section 4 are as follows: (a) Is $C_\infty = \lim_{i \to \infty} C_i$ maximal? (b) Under what condition, do we have $C_\infty = \mathbb{R}^n$? (c) Under what condition, do we have $C_\infty = C_L$ for some finite L?

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