Directional autocorrelation function of the polarization-mode dispersion vector

Saeed Hadjifaradji, Liang Chen, David S. Waddy and Xiaoyi Bao[a]

[a]University of Ottawa, Department of Physics, Ottawa, ON Canada K1N 6N5

ABSTRACT

The autocorrelation function (ACF) for the direction of the principal state of polarization (PSP) vector is reported. The analytical results are compared with the simulation results. It is shown that the magnitude and the directional ACF of the PSP vector are not independent. The directional correlation bandwidth for the PSP vector is verified to be narrower than that of the PSP magnitude.

Keywords: polarization mode dispersion, autocorrelation function, correlation bandwidth

1. INTRODUCTION

Polarization mode dispersion (PMD) in the single mode fibers (SMF) severely limits a communication system at a speed of more than 10Gb/s. PMD of a SMF can be described by its (PSP) vector[1] $\mathbf{\Omega}(\omega)$ in the Stokes space whose magnitude, $|\Omega(\omega)|$ describes the differential group delay (DGD). For a light launched into a SMF having PMD the output state of polarization (SOP) varies as a function of optical frequency as $\partial \mathbf{s}(\omega)/\partial\omega = \mathbf{\Omega}(\omega) \times \mathbf{s}(\omega)$.[2,3] The frequency dependence of the PSP vector would cause the reduction of the degree of polarization (DOP) for an optical pulse and the distortion of its pulse shape. Many properties of the PSP vector $\mathbf{\Omega}(\omega)$ have been investigated,[4,5] for example its magnitude, i.e. DGD, satisfies the Maxwellian probability density function for a highly mode coupled SMF.[5,6] Its ACFs have also been studied.[7] However, to the authors' knowledge, there is no report on the exact calculation of the PSP directional ACF. In this paper, we present an analytical result confirming that the de-alignment of PSP vector direction is more significant than the mismatch of the PSP magnitude (i.e. DGD) between neighboring frequencies. We also show that the direction and magnitude of PSP vector ACF are not independent.

2. ANALYTICAL MODEL

We follow the same mathematical procedure introduced by Karlsson and Brentel.[6,7] Here we treat a SMF as having n segments of polarization maintaining fibers (PMF), and each segment has a constant birefringence axis, described in Stokes space by unity vector $\hat{\mathbf{e}}_n$, and a retardation $\gamma_n(\omega)$. The Muller matrix of a single section can be written by the use of the matrix exponential as: $M_n = \exp[\gamma_n(\omega)\,\hat{\mathbf{e}}_n\times] = I + \sin(\gamma_n(\omega))\,\hat{\mathbf{e}}_n \times + [1-\cos(\gamma_n(\omega))]\,\hat{\mathbf{e}}_n \times \hat{\mathbf{e}}_n\times$. The PSP vector $\mathbf{\Omega}_n(\omega)$ of the first n fiber segments can be obtained from the Gisin-Pellaux[8] recursion relation:

$$\mathbf{\Omega}_n = \frac{\mathrm{d}\gamma_n}{\mathrm{d}\omega_n}\hat{\mathbf{e}}_n + M_n\mathbf{\Omega}_{n-1}. \tag{1}$$

Using the above recursion relation, Karlsson and Brentel have calculated the ACF of the PSP vector, i.e., $E(\mathbf{\Omega}(\omega_1)\cdot\mathbf{\Omega}(\omega_2))$ where the average is done over $\hat{\mathbf{e}}_n$. In the limit of highly mode coupled SMF they obtained the following analytical result:

$$E\left[\mathbf{\Omega}(\omega_1)\cdot\mathbf{\Omega}(\omega_2)\right] = \frac{3\left[1-\exp\left(-\frac{E(\Delta\tau^2)(\Delta\omega)^2}{3}\right)\right]}{(\Delta\omega)^2}, \tag{2}$$

Further author information: (Send correspondence to S.H.)
S.H.: E-mail: saeed.hadjifaradji@science.uottawa.ca, Telephone: 1 613 562-5800 ext. 6917

where $\Delta\omega = \omega_1 - \omega_2$. However, this result does not separate the ACF for PSP direction $\hat{\mathbf{\Omega}}$ and its magnitude $\Delta\tau$. Our strategy to get the PSP directional ACF involves calculating two quantities: $E[(\mathbf{\Omega}(\omega_1)\cdot\mathbf{\Omega}(\omega_2))^2]$ and $E[|\Omega(\omega_1)|^2|\Omega(\omega_2)|^2]$. The second autocorrelation has been also reported,[6] and it is:

$$E\left[|\Omega(\omega_1)|^2|\Omega(\omega_2)|^2\right] = \left(E\left[\Delta\tau^2\right]\right)^2 + \frac{4E\left[\Delta\tau^2\right]}{(\Delta\omega)^2} - \frac{12\left[1-\exp\left(-\frac{E\left[\Delta\tau^2\right](\Delta\omega)^2}{3}\right)\right]}{(\Delta\omega)^4}. \tag{3}$$

Following Karlsson and Brentel, we assume each fiber segment having the same DGD value t except their direction can be pointing anywhere on the Poincaré sphere, then we can have the following:

$$\begin{aligned}\mathbf{\Omega}_n(\omega_1)\cdot\mathbf{\Omega}_n(\omega_2) = \; & t^2 + t\hat{\mathbf{e}}_n\cdot\left[\mathbf{\Omega}_{n-1}(\omega_1)+\mathbf{\Omega}_{n-1}(\omega_2)\right] + \cos(t\Delta\omega)\,\mathbf{\Omega}_{n-1}(\omega_1)\cdot\mathbf{\Omega}_{n-1}(\omega_2) \\ & + \sin(t\Delta\omega)\,\hat{\mathbf{e}}_n\cdot\left[\mathbf{\Omega}_{n-1}(\omega_1)\times\mathbf{\Omega}_{n-1}(\omega_2)\right] \\ & + \left[1-\cos(t\Delta\omega)\right]\left[\hat{\mathbf{e}}_n\cdot\mathbf{\Omega}_{n-1}(\omega_1)\right]\left[\hat{\mathbf{e}}_n\cdot\mathbf{\Omega}_{n-1}(\omega_2)\right].\end{aligned} \tag{4}$$

Now averaging over the solid angle $\hat{\mathbf{e}}_n$ for the square of the last expression, we get the following expression:

$$\begin{aligned}E\left[(\mathbf{\Omega}_n(\omega_1)\cdot\mathbf{\Omega}_n(\omega_2))^2\right] = \; & t^4 + \tfrac{t^2}{3}\left[E\left(|\Omega_{n-1}(\omega_1)|^2\right)+E\left(|\Omega_{n-1}(\omega_2)|^2\right)\right] \\ & + t^2 bE\left[\mathbf{\Omega}_{n-1}(\omega_1)\cdot\mathbf{\Omega}_{n-1}(\omega_2)\right] \\ & + cE\left[|\Omega_{n-1}(\omega_1)|^2|\Omega_{n-1}(\omega_2)|^2\right] \\ & + dE\left[(\mathbf{\Omega}_{n-1}(\omega_1)\cdot\mathbf{\Omega}_{n-1}(\omega_2))^2\right],\end{aligned} \tag{5}$$

where $b = 4(1+\cos(t\Delta\omega))/3$, $c = 2(3-\cos(t\Delta\omega)-2\cos^2(t\Delta\omega))/15$ and $d = (4\cos^2(t\Delta\omega)+2\cos(t\Delta\omega)-1)/5$. In arriving the last equation, we have used the following $\hat{\mathbf{e}}_n$ solid angle average formula:

$$\begin{aligned}E\left[(\hat{\mathbf{e}}_n\cdot\mathbf{A})(\hat{\mathbf{e}}_n\cdot\mathbf{B})\right] = \; & \tfrac{1}{3}(\mathbf{A}\cdot\mathbf{B}) \\ E\left[(\hat{\mathbf{e}}_n\cdot\mathbf{A})(\hat{\mathbf{e}}_n\cdot\mathbf{B})(\hat{\mathbf{e}}_n\cdot\mathbf{C})(\hat{\mathbf{e}}_n\cdot\mathbf{D})\right] = \; & \tfrac{1}{15}\left[(\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\cdot\mathbf{D})+(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})\right],\end{aligned} \tag{6}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are constant vectors. To proceed, we use the recursion relation derived by Karlsson and Brentel:

$$E[\mathbf{\Omega}_n(\omega_1)\cdot\mathbf{\Omega}_n(\omega_2)] = t^2 + \left[1+2\cos(t\Delta\omega)\right]E[\mathbf{\Omega}_{n-1}(\omega_1)\cdot\mathbf{\Omega}_{n-1}(\omega_2)]/3. \tag{7}$$

Furthermore, one can derive the following two recursion relations:

$$\begin{aligned}E\left[|\Omega_n(\omega_1)|^2|\Omega_n(\omega_2)|^2\right] = \; & t^4 + t^2\left(E\left[|\Omega_{n-1}(\omega_1)|^2\right]+E\left[|\Omega_{n-1}(\omega_2)|^2\right]\right) \\ & + \tfrac{4t^2}{3}E\left[\mathbf{\Omega}_{n-1}(\omega_1)\cdot\mathbf{\Omega}_{n-1}(\omega_2)\right] \\ & + E\left[|\Omega_{n-1}(\omega_1)|^2|\Omega_{n-1}(\omega_2)|^2\right] \\ E\left[|\Omega_n(\omega)|^2\right] = \; & t^2 + E\left[|\Omega_{n-1}(\omega)|^2\right].\end{aligned} \tag{8}$$

After some algebra, one can get the following:

$$\begin{aligned}E\left[|\Omega_n(\omega)|^2\right] = \; & nt^2, \\ E\left[\mathbf{\Omega}_n(\omega_1)\cdot\mathbf{\Omega}_n(\omega_2)\right] = \; & t^2\tfrac{1-a^n}{1-a}, \\ E\left[|\Omega_n(\omega_1)|^2|\Omega_n(\omega_2)|^2\right] = \; & t^4\left[n^2 + \frac{4\left(n-\frac{1-a^n}{1-a}\right)}{3(1-a)}\right],\end{aligned} \tag{9}$$

where $a = (1+2\cos(t\Delta\omega))/3$. By plugging the above results into (5), we have the following recursion relation:

$$\begin{aligned}E\left[(\mathbf{\Omega}_n(\omega_1)\cdot\mathbf{\Omega}_n(\omega_2))^2\right] = \; & t^4 + \tfrac{2}{3}t^4(n-1) + t^4 b\tfrac{1-a^{n-1}}{1-a} \\ & + \tfrac{4}{3}\tfrac{t^4 c}{1-a}\left((n-1)-\tfrac{1-a^{n-1}}{1-a}\right) + t^4 c(n-1)^2 \\ & + dE\left[(\mathbf{\Omega}_{n-1}(\omega_1)\cdot\mathbf{\Omega}_{n-1}(\omega_2))^2\right].\end{aligned} \tag{10}$$

By formally introducing the following function:

$$f(n) = t^4 + \frac{2}{3}t^4 n + t^4 b \frac{1-a^n}{1-a} + t^4 c \left[n^2 + \frac{4}{3}\frac{1}{1-a}\left(n - \frac{1-a^n}{1-a} \right) \right]. \tag{11}$$

Eq. 10 can be written as:

$$E\left[\left(\mathbf{\Omega}_n(\omega_1) \cdot \mathbf{\Omega}_n(\omega_2) \right)^2 \right] = \sum_{i=1}^{n} f(n-i) d^{i-1}. \tag{12}$$

The series sum can be done; however, we neglect the final expression because of its length. In the last step, similar to the work of Karlsson and Brentel, we take the limit $n \to \infty$ while keeping $n^2 t = E[\Delta\tau^2] =$ constant. We finally find the $E[(\mathbf{\Omega}(\omega_1) \cdot \mathbf{\Omega}(\omega_2))^2]$ in the highly mode coupling limit:

$$\begin{aligned} E\left[\left(\mathbf{\Omega}(\omega_1) \cdot \mathbf{\Omega}(\omega_2) \right)^2 \right] = & \frac{(E[\Delta\tau^2])^2}{3} + \frac{4}{3}\frac{E[\Delta\tau^2]}{(\Delta\omega)^2} + \frac{8}{3(\Delta\omega)^4} \\ & - \frac{6}{(\Delta\omega)^4} \exp\left[-\frac{E[\Delta\tau^2](\Delta\omega)^2}{3} \right] \\ & + \frac{10}{3(\Delta\omega)^4} \exp\left[-E\left[\Delta\tau^2\right] (\Delta\omega)^2 \right]. \end{aligned} \tag{13}$$

If we use the conjecture[7] that the two properties of the PSP vector i.e. its length $\Delta\tau$ (the DGD) and its direction $\hat{\mathbf{\Omega}}$ (the PSP) are independent, we should easily derive the PSP directional ACF:

$$E\left[\left(\hat{\mathbf{\Omega}}(\omega_1) \cdot \hat{\mathbf{\Omega}}(\omega_2) \right)^2 \right] = \frac{E\left[\left(\mathbf{\Omega}(\omega_1) \cdot \mathbf{\Omega}(\omega_2) \right)^2 \right]}{E\left[|\Omega(\omega_1)|^2 |\Omega(\omega_2)|^2 \right]}. \tag{14}$$

However, numerical simulation shows that the two properties are not independent, therefore, Eq. (14) is an approximation for the PSP directional ACF. Furthermore, we will next show that it is a good approximation.

3. SIMULATION RESULTS

To verify the analytical results on the ACFs, we perform Monte Carlo simulations of the directional ACF given in Eq. (14) as well as ACFs given in Eqs. (3) and (13). We consider 20000 SMFs each having 500 segments. Each segment is considered to have a uniformed random DGD such that over all DGD in fiber obeys the Maxwellian distribution. Figure 1 (left) compares the ACF given in Eq. (3) with its simulation result for average DGD value of 10 ps which corresponds to $\langle \Delta\tau^2 \rangle = 117.8$ ps^2. Figure 1 (right) compares the ACF given in Eq. (13) with its simulation result for the same DGD value. Note that we scaled the ACFs by $\langle \Delta\tau^2 \rangle^2$. That makes the graphs to be depended on $E\left[\Delta\tau^2\right](\Delta\omega)^2$ rather than $\langle \Delta\tau \rangle$; therefore, the graphs remain unchanged as one changes $\langle \Delta\tau \rangle$. The dash-dotted line with a few error bars represents the simulation and the solid line represents the analytical ACF; the agreement is quite perfect.

Figure 2 shows the simulation and analytical results of the second order directional ACF for the same parameters as Fig. 1. Since we have a perfect match between the analytical results and the simulations in Figure 1, we expect the same perfect agreement for the analytical expression given in Eq. (14). However, Figure 2 shows a discriminable gap between them. This clearly proves that the ACF of the two properties of the PSP vector i.e. its length $\Delta\tau$ and its direction $\hat{\mathbf{\Omega}}$ are not independent. In fact, it shows that they are independent only for small $\Delta\omega$ and for large $\Delta\omega$, but there is a dependency (at most 10%) elsewhere. Note again that both simulation and analytic directional ACF depend only on the product of $E\left[\Delta\tau^2\right]$ and $(\Delta\omega)^2$.

To verify our conclusion for the dependency of the $\Delta\tau$ and $\hat{\mathbf{\Omega}}$ ACF, we perform Monte Carlo simulations of the following ACFs: $E[|\Omega(\omega_1)||\Omega(\omega_2)|]$, $E[\hat{\mathbf{\Omega}}(\omega_1) \cdot \hat{\mathbf{\Omega}}(\omega_2)]$ and $E[\mathbf{\Omega}(\omega_1) \cdot \mathbf{\Omega}(\omega_2)]$. Figure 3 shows the numerical results for the first order directional and magnitude ACF for PSP. Also, in Figure 3 (left), the numerical results for the directional ACF has been compared with the ratio of $E[\mathbf{\Omega}(\omega_1) \cdot \mathbf{\Omega}(\omega_2)]/E[|\Omega(\omega_1)||\Omega(\omega_2)|]$ (solid line). Similar as Fig. 2, it shows a gap between them. Figure 4 compares the first order analytical ACF for PSP given in Eq. (2) with its simulation as well as with the numerical product of the directional and magnitude ACFs from

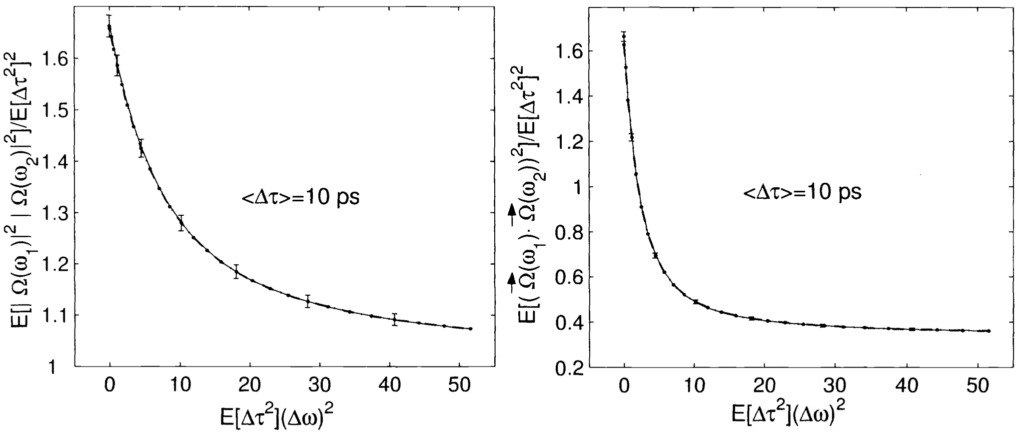

Figure 1. The magnitude ACF given by Eq. (3) (left) and the ACF given by Eq. (13) (right) are compared with their simulation results for average DGD values of $\langle \Delta\tau \rangle = 10$ ps over 20000 SMFs each having 500 segments.

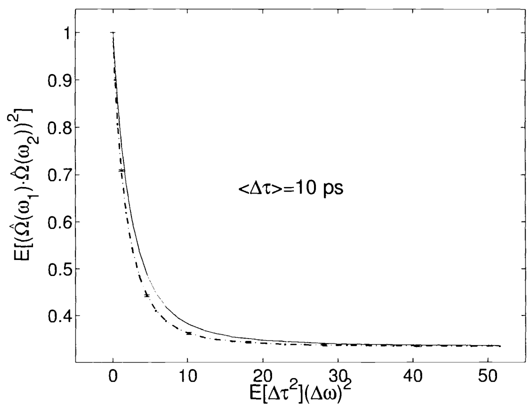

Figure 2. The second order directional ACF for the PSP vector for $E[\Delta\tau] = 10$ ps. Dash-dotted line with the error bars is the simulation result and the solid line is the analytical result given in Eq. (14).

simulation. The first order analytical ACF for PSP has been discussed and compared with the experimental result by Karlsson and Brentel.[7] They conjectured the independence between the magnitude and directional ACF for PSP. However, note that the numerical product of the PSP directional and magnitude ACFs from simulation clearly deviates from the analytical ACF given by Eq. (2). Figure 4 again shows the dependency between the magnitude and the direction ACF of the PSP vector.

Furthermore, we report the correlation bandwidth for the first and second order directional ACF for PSP as well as the correlation bandwidth for the first and second order magnitude ACF for PSP. We define the correlation bandwidths to be the half width of the variation of the corresponding ACFs. For example, one can show that Eq. (14) approaches to 1 as $E\left[\Delta\tau^2\right](\Delta\omega)^2$ goes to zero, and to 1/3 as $E\left[\Delta\tau^2\right](\Delta\omega)^2$ goes to infinity. Hence, placing Eq. (14) equal to (1+1/3)/2=2/3, we can find the correlation bandwidth of the second order directional ACF for PSP from: $E\left[\Delta\tau^2\right](\Delta\omega_B[\hat{\mathbf{\Omega}}])^2 = 1.8953$. From its simulation result, we can find the correlation bandwidth from: $E\left[\Delta\tau^2\right](\Delta\omega_B[\hat{\mathbf{\Omega}}])^2 \approx 1.425$. Similarly, we can find the correlation bandwidth for the second order magnitude ACF for PSP given in Eq. (3) from $E\left[\Delta\tau^2\right](\Delta\omega_B[|\Omega|])^2 = 7.670787257$. Clearly $\Delta\omega_B[\hat{\mathbf{\Omega}}] < \Delta\omega_B[|\Omega|]$. Table 1 summarizes all correlation bandwidths to the corresponding ACF. Again $\Delta\omega_B[\hat{\mathbf{\Omega}}] < \Delta\omega_B[|\Omega|]$ for the first order

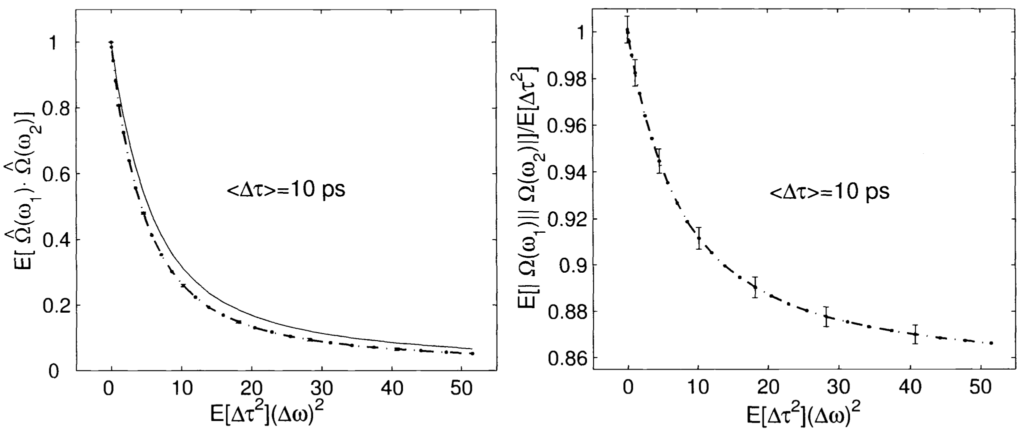

Figure 3. Simulation results for the first order directional (left) and the magnitude (right) ACF. Directional ACF (left) is also compared with the ratio of $E[\mathbf{\Omega}(\omega_1) \cdot \mathbf{\Omega}(\omega_2)]/E[|\Omega(\omega_1)||\Omega(\omega_2)|]$ (solid line).

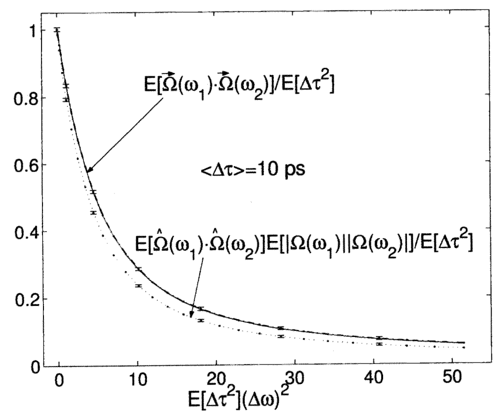

Figure 4. Analytical ACF (solid line) for the PSP vector given by Eq. (2) is compared with its simulation (dash-dotted line). The numerical product of the PSP directional and magnitude ACFs from simulation is the dotted line. $E[\Delta\tau] = 10$ ps.

ACFs. Results obtained from simulations are marked by (*).

4. CONCLUSION

We have derived an approximate analytical expression for the second order directional ACF of the PSP vector. We have shown that our results agree with the simulation within 10%. We have shown that the two correlation properties of the PSP vector i.e. its length $\Delta\tau$ (the DGD) and its direction $\hat{\mathbf{\Omega}}$ (the PSP) are not independent.

Table 1. Correlation bandwidths

$E[\Delta\tau^2](\Delta\omega_B)^2$	Directional ACF	Magnitude ACF	ACF
First order	4.269*	7.397*	4.781
Second order	1.425*	7.671	2.047

Our results show that the ACF frequency bandwidths are inversely proportional to the mean DGD. However, the frequency bandwidth for the PSP directional ACF is narrower than that for the PSP magnitude ACF for a given DGD for both first and second order ACFs. (First order about $1 - \sqrt{4.269/7.397} \approx 24\%$ narrower and second order about $1 - \sqrt{1.425/7.6707} \approx 57\%$ narrower.) In other words, the PSP directional ACF de-correlates faster than the PSP magnitude ACF. Moreover, one can obtain ACFs of the higher order frequency derivative of PSP vector (and therefore correlation bandwidths) simply by differentiations with respect to ω_1 and/or ω_2.

REFERENCES

1. C. D. Poole and R. E. Wagner, "Phenomenological Approach to Polarization Mode Dispersion in Long Single-Mode Fibers," *Electronics Letters*, vol. 22 no. 19, pp. 1029-1030, 1986.
2. C. D. Poole, N. S. Bergano, R. E. Wagner and H.J. Schulte, "Polarization dispersion and principal states in a 147-km undersea lightwave cable," *Lightwave Technology, Journal of*, vol. 6 no. 7, pp. 1185-1190, 1988.
3. P. Lu, L. Chen and X. Bao, "Polarization mode dispersion and polarization dependent loss for a pulse in single-mode fibers," *Lightwave Technology, Journal of*, vol. 19 no. 6, pp. 856-860, 2001.
4. Kaminov, I. P. and T. Li, Eds., *Optical Fiber Telecommunications IVB: Systems and Impairments*, Academic Press, Incorporated, 2002.
5. J. P. Gordon and H. Kogelnik, "PMD fundamentals: Polarization mode dispersion in optical fibers," *Proc. Natl. Acad. Sci. USA*, vol. 97 no. 9, pp. 4541-4550, 2000.
6. M. Shtaif and A. Mecozzi, "Study of the frequency autocorrelation of the differential group delay in fiber with polarization mode dispersion," *Optics Letters*, vol. 25 no. 10, pp. 707-709, 2000.
7. M. Karlsson and J. Brentel, "Autocorrelation Function of the Polarization-Mode Dispersion Vector," *Optics Letters*, vol. 24 no. 14, pp. 939-941, 1999.
8. H. Gisin and J. P. Pellaux, "Polarization Mode Dispersion: Time Versus Frequency Domains," *Optics Communications*, vol. 89 no. 2,3,4, pp. 316-323, May 1992.