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Stieltjes-Type Integrals for Metric Semigroup-Valued Functions Defined on Unbounded Intervals

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Abstract

We introduce the GH_k integral for functions defined on (possibly) unbounded subintervals of the extended real line and with values in metric semigroups. Basic properties and convergence theorems for this integral are deduced.

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1 Introduction.

Stieltjes-type integrals are widely studied in the literature: for example, meaningful results can be found in [8, 9, 10, 23]. In particular, in [13, 14, 15] and in a more abstract setting in [8, 9], an integral (GH_k integral) for real-valued functions defined in a compact subinterval of the real line has been investigated, which generalizes the integral studied by Š. Schwabik in [24]: the latter includes also the classical Kurzweil-Henstock and Henstock-Stieltjes integrals. Some examples of other particular cases of the GH_k integral are illustrated in [8, 9].

In this paper we extend the GH_k integral to the case of metric semigroup-valued functions, defined on (possibly) *unbounded* subintervals of the extended real line, and we prove some convergence theorems. Similar results were proved in [5] in the context of the Kurzweil-Henstock integral, for which the GH_k integral is substantially a particular case; moreover, in this paper we prove also an extension Cauchy-type theorem.

For a literature existing on the Kurzweil-Henstock integral in the context of metric semigroups, we refer to [5, 16, 26] and their bibliography, while for Riesz-space valued functions we recall [1, 2, 3, 4, 17, 18, 19, 20, 21, 22]. A particular example of metric semigroup is the set $L(\mathbb{R})$ of fuzzy numbers (see also Section 2 and [5]).

2 Metric semigroups.

Definition 2.1. A *metric semigroup* is a structure $(X, \rho, +, \cdot)$, where $\rho : X \times X \to \mathbb{R}$, $+ : X \times X \to X$, $\cdot : \mathbb{R} \times X \to X$ satisfy the following conditions:

(i) (X, ρ) is a complete metric space;

(ii) $(X, +)$ is a commutative semigroup endowed with a neutral element 0;

(iii) $\rho(w + y, z + t) \leq \rho(w, z) + \rho(y, t)$ for any $w, y, z, t \in X$;

(iv) $\rho(\alpha\, w, \alpha\, y) \leq |\alpha|\, \rho(w, y)$ for all $\alpha \in \mathbb{R}$ and $w, y \in X$;

(v) $\alpha(w + y) = \alpha w + \alpha y$ for each $\alpha \in \mathbb{R}$, $w, y \in X$;

(vi) $(\alpha + \beta)w = \alpha w + \beta w$ for every $\alpha, \beta \in \mathbb{R}_0^+$, $w \in X$, $0 \cdot w = 0$ and $1 \cdot w = w$ for each $w \in X$.

A metric semigroup $(X, \rho, +, \cdot)$ is called *invariant,* if

$$\rho(w + z, y + z) = \rho(w, y)$$

for any $w, y, z \in X$.

Observe that a consequence of invariance and of the triangular property is the following condition, which will be useful in the sequel:

(vii) $\rho(w+y,z) \leq \rho(w,t)+\rho(y+t,z)$ whenever $x,y,z,t \in X$.

An example of metric semigroup is the set of all fuzzy numbers (see also [5, 26]).

Definition 2.2. A *fuzzy number* is a function $\mu : \mathbb{R} \to [0,1]$ satisfying the following conditions:

(j) there exists $x_0 \in \mathbb{R}$ such that $\mu(x_0) = 1$;

(jj) the α-*cut* set $\mu_\alpha = \{x \in \mathbb{R} : \mu(x) \geq \alpha\}$ is convex for $\alpha \in]0,1]$;

(jjj) μ is upper semi-continuous, i. e. any α-cut μ_α is a closed subset of $\mathbb{R}$;

(jv) the support $\overline{\{x \in \mathbb{R} : \mu(x) > 0\}}$ of the function μ is a compact set.

Any real number u_0 can be identified with a fuzzy number μ_0 in the following way:

$$\mu_0(x) = \chi_{\{u_0\}}(x),$$

i. e. $\mu_0(u_0) = 1$, and $\mu_0(x) = 0$, if $x \neq u_0$.
The set of all fuzzy numbers is denoted by $L(\mathbb{R})$.

We now endow $L(\mathbb{R})$ with a metric and a linear structure (see also [5, 26]). We define the *Hausdorff distance* $\mathcal{H}$ on the set of all compact possibly degenerate intervals in $\mathbb{R}$:

$$\mathcal{H}([a,b],[c,d]) = \max(|c-a|,|d-b|).$$

Let $\mu,\nu \in L(\mathbb{R})$. It is easy to check that, for every $\alpha \in (0,1]$, there exist a, b, c, $d \in \mathbb{R}$ (depending on α) such that $\mu_\alpha = [a,b]$, $\nu_\alpha = [c,d]$ So, for $\mu,\nu \in L(\mathbb{R})$, set

$$\rho(\mu,\nu) = \sup\{\mathcal{H}(\mu_\alpha,\nu_\alpha) : \alpha \in (0,1]\}.$$

Using this definition, $(L(\mathbb{R}),\rho)$ becomes a complete metric space.

To define a linear structure on $L(\mathbb{R})$, recall that every fuzzy number is completely determined by its α-cuts. Hence, for any $\mu,\nu \in L(\mathbb{R})$, $\alpha \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}$, set

$$\begin{aligned}(\mu+\nu)_\alpha &= \mu_\alpha + \nu_\alpha,\\ (\lambda\mu)_\alpha &= \lambda\,\mu_\alpha\end{aligned}$$

(here, $V+Z = \{v+z : v \in V, z \in Z\}$; $\lambda V = \{\lambda\, v : v \in V\}$).

Finally, we note that $(L(\mathbb{R}),+)$ is not a group, but only a semigroup (see also [5]), in fact let $\mu \in L(\mathbb{R})$ be defined by the formula:

$$\mu(x) = \begin{cases} x, & \text{if } x \in [0,1];\\ 2-x, & \text{if } x \in [1,2];\\ 0, & \text{otherwise.}\end{cases}$$

Then $-\mu = (-1) \cdot \mu$ is given by

$$-\mu(x) = \begin{cases} -x, & \text{if } x \in [-1, 0]; \\ 2 + x, & \text{if } x \in [-2, -1]; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mu(x) + (-\mu(x))$ is not the zero element $0 \equiv \chi_{\{0\}}(x)$, but

$$\mu(x) + (-\mu(x)) = \begin{cases} 1 - \dfrac{x}{2}, & \text{if } x \in [0, 2]; \\ 1 + \dfrac{x}{2}, & \text{if } x \in [-2, 0]; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand the subset $R_0 \subset L(\mathbb{R})$ consisting of all functions $\chi_{\{a\}}$, $a \in \mathbb{R}$, is group isomorphic to the commutative group $(\mathbb{R}, +)$.

3 The construction of the integral.

From now on we denote by capital letters the elements of the extended real line and by small letters the real numbers. Let $[A, B]$ be a (possibly unbounded) interval of the extended real line, and $\mathcal{F}$ be the family of all closed convex subsets. By *partition* (or *k-partition*) of a set $W \in \mathcal{F}$ we denote a finite collection

$$\Pi = \{(\xi_1; F_{1,1}, \ldots, F_{1,k}), \ldots, (\xi_q; F_{q,1}, \ldots, F_{q,k})\} = \{(\xi_1; E_1), \ldots, (\xi_q; E_q)\} \quad (1)$$

such that

(i) $F_{i,j} \in \mathcal{F}$ for all $i = 1, \ldots, q$ and $j = 1, \ldots, k$;

(ii) $\bigcup_{j=1}^{k} F_{i,j} = E_i$ for all $i = 1, \ldots, q$;

(iii) $\bigcup_{i=1}^{q} E_i = W$;

(iv) $\xi_i \in E_i \ (i = 1, \ldots, q)$;

(v) the $F_{i,j}$'s are pairwise non-overlapping;

(vi) $\sup F_{i,j} = \inf F_{i,j+1}$ whenever $i = 1, \ldots, q$ and $j = 1, \ldots, k-1$.

A finite collection Π as in (1), satisfying conditions **(i), (ii), (iv), (v)** and **(vi),** but not necessarily **(iii),** is said to be a *decomposition* (or *k-decomposition*) of W.

Definitions 3.1.
- A *gauge* is a map γ defined in $[A, B]$ and taking values in the set of all open intervals in $\widetilde{\mathbb{R}}$, such that $\xi \in \gamma(\xi)$ for every $\xi \in [A, B]$ and $\gamma(\xi)$ is a bounded open interval (with respect to the topology of $[A, B]$) for every $\xi \in \mathbb{R} \cap [A, B]$.
- Given a gauge γ, a k-decomposition of $[A, B]$ of the type
$$\Pi = \{(\xi_i; E_i), i = 1, \dots, q\} \tag{2}$$
is said to be γ-*fine* if $\xi_i \in E_i \subset \gamma(\xi_i)$ for all $i = 1, \dots, q$. Observe that for any gauge γ there always exists a γ-fine k-partition (see also [8, 11]).
- Given $[a, b] \subset \mathbb{R}$ and a map $\delta : [a, b] \to \mathbb{R}^+$, a partition Π of $[a, b]$ as in (2) is said to be δ-*fine* if $\xi_i \in E_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, \dots, q$. In any case we note that, if E_i is an unbounded interval, then the element ξ_i associated with E_i is necessarily $+\infty$ or $-\infty$: otherwise $\gamma(\xi_i)$ should be a bounded interval and contain an unbounded interval, a contradiction.

From now on, we assume that X is an invariant metric semigroup. Given any k-decomposition Π as in (1) and a function $U : [A, B] \times \mathcal{F}^k \to X$, we call *Riemann sum* of U (and we write $\sum_{\Pi} U$) the expression
$$\sum_{i=1}^{q} U(\xi_i; F_{i,1}, \dots, F_{i,k}). \tag{3}$$

We now introduce the GH_k integral for X-valued functions defined on $[A, B] \times \mathcal{F}^k$. We will show that this concept can be formulated equivalently both with gauges and with positive maps δ.

Definition 3.2. We say that a function $U : [A, B] \times \mathcal{F}^k \to X$ is GH_k *integrable* on $[A, B]$ if there exists $I \in X$ such that for all $\varepsilon > 0$ there correspond a function $\delta : [A, B] \to \mathbb{R}^+$ and a positive real number P such that
$$\rho\left(I, \sum_{\Pi} U\right) \le \varepsilon \tag{4}$$
whenever Π is a δ-fine k-partition of any bounded interval $[a, b]$ with $[a, b] \supset [A, B] \cap [-P, P]$. In this case we say that I is the GH_k *integral of* U, and we denote the element I by the symbol $(GH_k)\int_A^B U$, writing usually $U \in GH_k[A, B]$.

Analogously it is possible to define the integral $(GH_k)\int_c^d U$ for each subinterval $[c, d] \subset [A, B]$.

Remark 3.3. We note that the GH_k integral is well-defined, that is there exists at most one element I, satisfying condition (4) (see also [5]).

We now give the following characterization of GH_k integrability.

Theorem 3.4. *A function $U : [A, B] \times \mathcal{F}^k \to X$ is GH_k integrable if and only if there is $J \in X$ such that for all $\varepsilon > 0$ there exists a gauge γ such that*

$$\rho\left(J, \sum_{\Pi} U\right) \leq \varepsilon \tag{5}$$

whenever Π is a γ-fine partition of $[A, B]$, and in this case we have $\int_A^B f = J$.

Proof: See also [3], Theorem 3.3., and [5]. □

4 Elementary properties of the GH_k integral

The proof of the following proposition is similar to the corresponding one in [5].

Proposition 4.1. *If $U_1, U_2 \in GH_k[A, B]$ and $c_1, c_2 \in \mathbb{R}$, then $c_1U_1 + c_2U_2 \in GH_k[A, B]$, and*

$$(GH_k) \int_A^B (c_1\, U_1 + c_2\, U_2) = c_1\, (GH_k) \int_A^B U_1 + c_2\, (GH_k) \int_A^B U_2.$$

(Here we intend by $-U$ the entity $(-1) \cdot U$)

Theorem 4.2. *A map $U : [A, B] \times \mathcal{F}^k \to X$ is GH_k integrable if and only if for all $\varepsilon > 0$ there exists a gauge $\gamma = \gamma(\varepsilon)$ on $[A, B]$ such that*

$$\rho\left(\sum_{\Pi} U, \sum_{\Pi'} U\right) \leq \varepsilon \tag{6}$$

whenever Π, Π' are γ-fine k-partitions of $[A, B]$.

Proof: We follow the lines of the proof of Proposition 3.5 of [5].
The necessary part is straightforward.
We now turn to the sufficient part. Let U satisfy (6), and set $\varepsilon = 1/n$, with $n \in \mathbb{N}$. Then for all n there exists a gauge γ_n on $[A, B]$ such that

$$\rho\left(\sum_{\Pi_1} U, \sum_{\Pi_2} U\right) \leq \frac{1}{n}$$

whenever Π_1, Π_2 are γ_n-fine partitions of $[A, B]$. Put $\eta_n = \gamma_1 \cap \gamma_2 \cap \ldots \cap \gamma_n$ for all $n \in \mathbb{N}$, and set

$$A_n = \{x \in X : \exists\, \eta_n\text{−fine partition}\, \Pi_1 : x = \sum_{\Pi_1} U\}, \quad n \in \mathbb{N}.$$

If $x, y \in A_n$, then $\rho(x,y) \le 1/n$, and hence

$$\operatorname{diam} \overline{A_n} = \operatorname{diam} A_n \le \frac{1}{n}.$$

Since $\eta_{n+1} \subset \eta_n$, we obtain $\overline{A_{n+1}} \subset \overline{A_n}$. Since X is complete, there exists exactly one element $I \in \cap_{n=1}^{\infty} \overline{A_n}$.

Pick arbitrarily $\varepsilon > 0$, and choose $n \in \mathbb{N}$ such that $\dfrac{1}{n} < \varepsilon$. If Π is any η_n-fine partition, then

$$\sum_{\Pi} U \in A_n.$$

Since $I \in \overline{A_n}$, we obtain

$$\rho\left(I, \sum_{\Pi} U\right) \le \frac{1}{n} < \varepsilon.$$

Therefore U is GH_k integrable on $[A,B]$ and $I = \displaystyle\int_A^B U$. □

We now investigate GH_k integrability on subintervals, by proceeding similarly as in [8].

Proposition 4.3. *If $U \in GH_k[A,B]$, then $U \in GH_k[c,d]$ for each $[c,d] \subset [A,B]$, and*

$$(GH_k)\int_A^B U = (GH_k)\int_A^c U + (GH_k)\int_c^B U$$

whenever $A < c < B$.

Proof: We begin with the first statement. Without loss of generality, we can assume that $[c,d] = [A,d]$, with $A < d < B$. Let γ be any gauge on $[A,B]$, pick any two γ-fine k-partitions Π_1, Π_2 of $[A,d]$, and let Π' be a γ-fine k-partition of $[d,B]$. Such a partition does exist, by virtue of the Cousin lemma. Then, for $j = 1,2$, $\Pi''_j := \Pi' \cup \Pi_j$ is a γ-fine partition of $[A,B]$. Since

$$\rho\left(\sum_{\Pi_1} U, \sum_{\Pi_2} U\right) = \rho\left(\sum_{\Pi''_1} U, \sum_{\Pi''_2} U\right),$$

then the assertion follows from the Cauchy criterion

We now turn to the last part. For every $\varepsilon > 0$ there exists a gauge γ such that for each γ-fine k-partition Π_1 of $[A,c]$ and Π_2 of $[c,B]$ we get

$$\rho\left(\sum_{\Pi_1} U, (GH_k)\int_A^c U\right) \le \varepsilon, \qquad \rho\left(\sum_{\Pi_2} U, (GH_k)\int_c^B U\right) \le \varepsilon.$$

Hence, if $\Pi = \Pi_1 \cup \Pi_2$, we have also

$$\rho\left(\sum_{\Pi} U, (GH_k)\int_A^B U\right) \le \varepsilon.$$

We obtain:

$$\begin{aligned}
0 &\le \rho\left((GH_k)\int_A^c U + (GH_k)\int_c^B U, (GH_k)\int_A^B U\right) \\
&\le \rho\left(\sum_{\Pi_1} U, (GH_k)\int_A^c U\right) + \rho\left(\sum_{\Pi_2} U, (GH_k)\int_c^B U\right) + \rho\left(\sum_{\Pi} U, (GH_k)\int_A^B U\right) \\
&\le 3\,\varepsilon.
\end{aligned}$$

By arbitrariness of $\varepsilon \in \mathbb{R}^+$ we get that

$$(GH_k)\int_A^B U = (GH_k)\int_A^c U + (GH_k)\int_c^B U.$$

This completes the proof. $\square$

In order to establish a converse of the previous result, we now introduce the following property.

Definition 4.4. Let $U : [A, B] \times \mathcal{F}^k \to X$ and fix a point $x_0 \in [A, B]$. We say that U *satisfies condition*

[H1) at x_0] if for all $\varepsilon > 0$ there exists a positive real number $\eta = \eta(\varepsilon; x_0)$ such that

$$\begin{aligned}
&\rho\Big(U(x_0; [w_0^{(0)}, w_1^{(0)}], \ldots, [w_{k-1}^{(0)}, w_k^{(0)}]), U(x_0; [w_0^{(1)}, w_1^{(1)}], \ldots, [w_{k-1}^{(1)}, w_k^{(1)}]) \\
+ \;\; &U(x_0; [w_0^{(2)}, w_1^{(2)}], \ldots, [w_{k-1}^{(2)}, w_k^{(2)}])\Big) \le \varepsilon
\end{aligned}$$

whenever $\bigcup_{l=0}^{2}\left(\bigcup_{i=1}^{k}[w_{i-1}^{(l)}, w_i^{(l)}]\right) \subset]x_0 - \eta, x_0 + \eta[$ and $w_0^{(0)} = w_0^{(1)}$, $w_k^{(0)} = w_k^{(2)}$, $x_0 = w_k^{(1)} = w_0^{(2)}$.

Note that **H1)** is a kind of "quasi-additivity" of the set function U. In many cases, when $X = \mathbb{R}$, U is defined by means of suitable "differences" (for example, $U(t; [u, v]) = V(t; v) - V(t; u)$ when $k = 1$ or

$$U(t; [w_0, w_1], \ldots, [w_{k-1}, w_k]) = V(t; w_1, \ldots, w_k) - V(t; w_0, \ldots, w_{k-1})$$

for $k \ge 2$); then, if $k = 1$, property **H1)** is automatically satisfied (see also [24], Theorem 1.11, pp. 10-12); while for $k \ge 2$ it is implied by the condition of "existence of the iterated limit J" used by A. G. Das and S. Kundu (see [8], Definition 2.9., p. 69).

We now prove the following result on additivity.

Theorem 4.5. *Let $U : [A, B] \times \mathcal{F}^k \to X$ satisfy condition* **H1)** *at $c \in]A, B[$. If $U \in GH_k[A, c]$ and $U \in GH_k[c, B]$, then $U \in GH_k[A, B]$ and*

$$(GH_k)\int_A^B U = (GH_k)\int_A^c U + (GH_k)\int_c^B U.$$

Proof: By hypothesis, for every $\varepsilon > 0$ there exist a function $\delta^* : [A, B] \to \mathbb{R}^+$ and a positive real number P (without loss of generality, greater than $|c|$) with the following property: for all δ^*-fine k-partitions Π_1 of any bounded interval $[a_1, b_1] \subset [A, c]$, $[a_1, b_1] \supset [A, c] \cap [-P, P]$ and Π_2 of every bounded interval $[a_2, b_2] \subset [c, B]$, $[a_2, b_2] \supset [c, B] \cap [-P, P]$ we get

$$\rho\left(\sum_{\Pi_1} U, (GH_k)\int_A^c U\right) \le \varepsilon, \qquad \rho\left(\sum_{\Pi_2} U, (GH_k)\int_c^B U\right) \le \varepsilon.$$

Let $\eta = \eta(\varepsilon; c)$ be related to condition **H1)** at c, and set $\delta(x) = \min\{\delta^*(x), |x - c|\}$ if $x \in [A, B] \setminus \{c\}$, $\delta(c) = \min\{\delta^*(c), \eta\}$. Pick now any bounded interval $[a, b] \subset [A, B]$, $[a, b] \supset [A, B] \cap [-P, P]$, and any δ-fine k-partition

$$\Pi = \{(\xi_i; F_{i,1}, \dots, F_{i,k}), i = 1, \dots, q\}$$

of $[a, b]$. There exists m with $1 \le m \le q$, such that $c = \xi_m$ and $\bigcup_{j=1}^{k} F_{i,j}$ contains c if and only if $i = m$ (see also [8, 24]). We get:

$$\sum_{\Pi} U = \sum_{i=1}^{m-1} U(\xi_i; F_{i,1}, \dots, F_{i,k}) + U(c; F_{m,1}, \dots, F_{m,k}) + \sum_{i=m+1}^{q} U(\xi_i; F_{i,1}, \dots, F_{i,k}).$$

Consider now the points

$$c-\delta(c) < x_{m-1,k} = y_{m,0} < \dots < y_{m,k} = c = z_{m,0} < \dots < z_{m,k} = x_{m+1,0} < c+\delta(c).$$

The parts of the partition Π for $i = 1, \dots, m-1$ ($i = m+1, \dots, q$) and the single family $\{(c; [y_{m,0}, y_{m,1}], \dots, [y_{m,k-1}, y_{m,k}])\}$ ($\{(c; [z_{m,0}, z_{m,1}], \dots, [z_{m,k-1}, z_{m,k}])\}$) form a δ^*-fine k-partition Π_1 (Π_2) of $[a, c]$ ($[c, b]$). So, we have:

$$\begin{aligned}
& \rho\left(\sum_{\Pi} U, (GH_k)\int_A^c U + (GH_k)\int_c^B U\right) \\
\le\ & \rho\left(\sum_{\Pi_1} U, (GH_k)\int_A^c U\right) + \rho\left(\sum_{\Pi_2} U, (GH_k)\int_c^B U\right) + \rho\left(\sum_{\Pi} U, \sum_{\Pi_1} U + \sum_{\Pi_2} U\right) \\
\le\ & 2\,\varepsilon + \rho(U(c; F_{m,1}, \dots, F_{m,k}), U(c; [y_{m,0}, y_{m,1}], \dots, [y_{m,k-1}, y_{m,k}]) \\
+\ & U(c; [z_{m,0}, z_{m,1}], \dots, [z_{m,k-1}, z_{m,k}])) \le 3\,\varepsilon.
\end{aligned}$$

From this it follows that $U \in GH_k[A, B]$ and

$$(GH_k)\int_A^B U = (GH_k)\int_A^c U + (GH_k)\int_c^B U.$$

This concludes the proof. $\square$

5 Convergence theorems

We begin with a version of the Saks-Henstock lemma (see also [5], Proposition 4.1). Here, the symbol $|\cdot|$ denotes the Lebesgue measure.

Lemma 5.1. *Let $U : [A,B] \times \mathcal{F}^k \to X$ be GH_k integrable on $[A,B]$. Then for every $\varepsilon > 0$ there exists a gauge γ on $[A,B]$ such that, for every γ-fine k-decomposition of $[A,B]$*

$$\Pi = \{(t_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, m\} = \{(t_i; E_i), i = 1, \ldots, m\}, \tag{7}$$

where $\bigcup_{j=1}^{k} F_{i,j} = E_i$, $i = 1, \ldots, m$; we have

$$\rho\left(\sum_{i=1,\ldots,m, |E_i|<+\infty} U(t_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^{m} (GH_k) \int_{E_i} U \right) \leq \varepsilon.$$

Proof: (see also [5]) Choose arbitrarily $\varepsilon > 0$, and let γ be a gauge on $[A,B]$ existing in correspondence with ε, according to Theorem 3.4. Fix arbitrarily any γ-fine k-decomposition Π of $[A,B]$ as in (7), and let $int\, E_i$ be the interior of E_i, $i = 1, \ldots, m$. Since the E_i's are non-overlapping, the set $[A,B] \setminus \cup_{i=1}^{m} (int\, E_i)$ is empty or is the union of non-overlapping (possibly bounded or not) intervals $B_1, \ldots, B_p$. Let $\eta > 0$. Since U is GH_k integrable on each B_j, for each $j = 1, \ldots, p$ there exists a gauge γ_j on B_j such that $\gamma_j(x) \subset \gamma(x)$ for all $x \in B_j$ and

$$\rho\left(\sum_{\Pi_j} U, (GH_k) \int_{B_j} U \right) < \frac{\eta}{p+1}$$

for every γ_j-fine partition Π_j of B_j. Let now Π_j be such a partition. We observe that

$$\Pi := \{(t_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, m\} \cup (\cup_{j=1}^{p} \Pi_j)$$

is a γ-fine partition of $[A, B]$. Then we have:

$$\begin{aligned}
& \rho\left(\sum_{i=1,\dots,m,|E_i|<+\infty} U(t_i;F_{i,1},\dots,F_{i,k}), \sum_{i=1}^{m}(GH_k)\int_{E_i} U\right) \\
= \; & \rho\left(\sum_{i=1,\dots,m,|E_i|<+\infty} U(t_i;F_{i,1},\dots,F_{i,k}) + \sum_{j=1}^{p}\sum_{\Pi_j} U, \sum_{i=1}^{m}(GH_k)\int_{E_i} U + \sum_{j=1}^{p}\sum_{\Pi_j} U\right) \\
\le \; & \rho\left(\sum_{\Pi} U, (GH_k)\int_A^B U\right) \\
+ \; & \rho\left(\sum_{i=1}^{m}(GH_k)\int_{E_i} U + \sum_{j=1}^{p}(GH_k)\int_{B_j} U, \sum_{i=1}^{m}(GH_k)\int_{E_i} U + \sum_{j=1}^{p}\sum_{\Pi_j} U\right) \\
\le \; & \varepsilon + \rho\left(\sum_{j=1}^{p}(GH_k)\int_{B_j} U, \sum_{j=1}^{p}\sum_{\Pi_j} U\right) \\
\le \; & \varepsilon + \sum_{j=1}^{p}\rho\left((GH_k)\int_{B_j} U, \sum_{\Pi_j} U\right) < \varepsilon + \sum_{j=1}^{p}\frac{\eta}{p+1} < \varepsilon + \eta.
\end{aligned}$$

Since the inequality

$$\rho\left(\sum_{i=1,\dots,m,|E_i|<+\infty} U(t_i;F_{i,1},\dots,F_{i,k}), \sum_{i=1}^{m}(GH_k)\int_{E_i} U\right) < \varepsilon + \eta$$

holds for any $\eta > 0$, then the assertion follows by arbitrariness of η. $\square$

We now prove a version of a Hake's type theorem, which is an extension of the Cauchy theorem. To do this, let $U : [A, B] \times \mathcal{F}^k \to X$ belong to $GH_k[A, c]$ for all $c \in [A, B[$, fix $I \in X$ and let us introduce the following condition:

- **H2)** for every $\varepsilon > 0$ there exists a left neighborhood $\mathcal{U}$ of B such that

  $$\rho\left(I, (GH_k)\int_A^c U + U(B; F_1, \dots, F_k)\right) \le \varepsilon$$

  whenever $F_1, \dots, F_k \in \mathcal{F}$ are pairwise non-overlapping and such that $\mathcal{U} \ni c \le \inf F_1 \le \sup F_j = \inf F_{j+1}$, $j = 1, \dots, k-1$, and $\sup F_k = B$.

In the literature several situations are considered, when, in the Riemann sums, only the terms where the involved intervals are bounded are taken: this can be done simply by postulating it or by requiring the condition

$$U(\pm\infty; \Lambda_1, \dots, \Lambda_k) = 0 \tag{8}$$

for every choice of $\Lambda_j \in \mathcal{F}$, $j = 1, \dots, k$.

Observe that, when $B = +\infty$ and we require (8), **H2)** can be automatically replaced by the simpler condition of existence in X of the limit

$$\lim_{c \to B^-} (GH_k) \int_A^c U. \tag{9}$$

Finally, we note that, when $X = \mathbb{R}$, property **H2)** is implied by the two conditions of existence in $\mathbb{R}$ of the limit as in (9) and of "existence of the iterated limit (from the left) J^-" used by A. G. Das and S. Kundu (see [8]) when $k \geq 2$. For $k = 1$, **H2)** is equivalent to the existence in $\mathbb{R}$ of the limit in [24], formula (1.11), p. 15.

Theorem 5.2. *Let $A \in \mathbb{R}^+$, $U : [A, B] \times \mathcal{F}^k \to X$ be such that $U \in GH_k[A, c]$ for every $c \in [A, B[$, and suppose that there is an element $I \in X$ such that* **H2)** *holds.*

Then $U \in GH_k[A, B]$ and $(GH_k) \int_A^B U = I$.

Moreover, if $U \in GH_k[A, B]$, then $\lim_{c \to B^-} (GH_k) \int_A^c U = (GH_k) \int_A^B U$ (this last result is independent on **H2)** *).*

Proof: Let $(c_p)_p$ be a strictly increasing sequence in $[A, b[$ with $c_p \uparrow B$ and $c_0 = A$. For every $p \in \mathbb{N}$ and $\varepsilon > 0$ there exists a gauge $\gamma_p : [A, c_p] \to \mathbb{R}^+$, such that

$$\rho\left(\sum_{\Pi_p} U, (GH_k) \int_A^{c_p} U\right) \leq \frac{\varepsilon}{2^p} \tag{10}$$

whenever Π_p is any γ_p-fine k-partition of $[A, c_p]$.

For every $\xi \in [A, B[$ there exists exactly one $p = p(\xi) \in \mathbb{N}$ such that $\xi \in [c_{p(\xi)-1}, c_{p(\xi)}[$. Given $\xi \in [A, B[$, choose $\widehat{\gamma}(\xi)$ such that $\widehat{\gamma}(\xi) \subset \gamma_{p(\xi)}(\xi)$ and $\widehat{\gamma}(\xi) \cap [A, B[\subset [A, c_{p(\xi)}(\xi))$. Let $c \in [A, B[$ and

$$\widehat{\Pi} := \{(\xi_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n\} = \{(\xi_i; E_i), i = 1, \ldots, n\},$$

with $\bigcup_{j=1}^{k} F_{i,j} = E_i$, $i = 1, \ldots, n$, be a $\widehat{\gamma}$-fine k-partition of $[A, c]$. For every $i = 1, \ldots, n$ we get:

$$E_i \subset \widehat{\gamma}(\xi_i) \subset [A, c_{p(\xi_i)}].$$

Furthermore, $E_i \subset \gamma_{p(\xi_i)}(\xi_i)$. For every $p \in \mathbb{N}$, let us indicate by

$$\sum_{i=1,\ldots,n,\, p(\xi_i)=p} \rho\left(U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{E_i} U\right)$$

the sum of those terms of

$$\sum_{i=1}^{n} \rho\left(U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{E_i} U\right)$$

for which $\xi_i \in [c_{p-1}, c_p[$. By Lemma 5.1 we obtain

$$\rho\left(\sum_{i=1,\ldots,n,\, p(\xi_i)=p} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1,\ldots,n,\, p(\xi_i)=p} (GH_k)\int_{E_i} U\right) \leq \frac{\varepsilon}{2^p}$$

for all $p \in \mathbb{N}$. Since $U \in GH_k[A, c]$ for every $c \in]A, B]$, then by Proposition 4.3 we have

$$(GH_k)\int_A^c U = \sum_{i=1}^n (GH_k)\int_{E_i} U.$$

So we get:

$$\begin{aligned}
& \rho\left(\sum_{i=1}^n U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k)\int_A^c U\right) \\
= \;& \rho\left(\sum_{i=1}^n U(\xi_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^n (GH_k)\int_{E_i} U\right) \\
\leq \;& \sum_{p=1}^\infty \left\{\rho\left(\sum_{i=1,\ldots,n,\, p(\xi_i)=p} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1,\ldots,n,\, p(\xi_i)=p} (GH_k)\int_{E_i} U\right)\right\} \\
\leq \;& \sum_{p=1}^\infty \frac{\varepsilon}{2^p} = \varepsilon.
\end{aligned}$$

Let $\mathcal{U}$ be related with condition **H2)**, and pick a gauge γ on $[A, B]$ such that $\gamma(\xi) \subset \widehat{\gamma}(\xi)$ if $\xi \in [A, B[$, and $\gamma(B) \subset \mathcal{U}$. Let

$$\Pi := \{(\xi_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n\} = \{(\xi_i; E_i), i = 1, \ldots, n\}$$

be any arbitrary γ-fine k-partition of $[A, B]$, where $\bigcup_{j=1}^k F_{i,j} = E_i$ and $E_i = [x_{i-1,k}, x_{i,k}]$, $i = 1, \ldots, n$: we get $x_{n,k} = B$ and hence $\xi_n = B$ (if not, then $E_n \subset \widehat{\gamma}(\xi_n) \subset [A, c_{p(\xi_n)}]$ and thus $x_{n,k} < B$, a contradiction). We have, thanks to the condition formulated in the hypothesis and using property **(vii)** of the function ρ,

$$\begin{aligned}
\rho\left(I, \sum_\Pi U\right) \;&\leq\; \rho\left(I, \sum_{i=1}^{n-1} U(\xi_i; F_{i,1}, \ldots, F_{i,k}) + U(B; F_{n,1}, \ldots, F_{n,k})\right) \\
&\leq\; \rho\left(\sum_{i=1}^{n-1} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k)\int_A^{x_{n-1,k}} U\right) \\
&+\; \rho\left(I, (GH_k)\int_A^{x_{n-1,k}} U + U(B; F_{n,1}, \ldots, F_{n,k})\right) \\
&\leq\; \rho\left(\sum_{i=1}^{n-1} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k)\int_A^{x_{n-1,k}} U\right) + \varepsilon.
\end{aligned}$$

As $x_{n-1,k} < B$ and $\{(\xi_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n-1\}$ is a $\widehat{\gamma}$-fine k-partition of $[A, x_{n-1,k}]$, we get

$$\rho\left(\sum_{i=1}^{n-1} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k)\int_A^{x_{n-1,k}} U\right) \leq \varepsilon,$$

and hence

$$\rho\left(I, \sum_{\Pi} U\right) \leq 2\varepsilon.$$

From this the assertion of the first part of the theorem follows

We now turn to the last part. Since, by hypothesis, $U : [A, B] \times \mathcal{F}^k \to X$ is GH_k integrable on $[A, B]$, then U is GH_k integrable on $[A, c]$ for every $A < c \leq B$. So for all $\varepsilon > 0$ and $c \in]A, B]$ there exists $\delta_1^c : [A, c] \to \mathbb{R}^+$ such that for every δ_1^c-fine k-partition Π' of $[A, c]$ we get:

$$\rho\left(\sum_{\Pi'} U, (GH_k)\int_A^c U\right) \leq \varepsilon.$$

Moreover, by GH_k integrability on $[A, B]$ (see also Definition 3.2), for any $\varepsilon > 0$ there exist $\delta : [A, B] \to \mathbb{R}^+$ and $P \in]A, B[$ such that for every bounded interval $[d_1, d_2] \subset [A, B]$ with $[d_1, d_2] \supset [-P, P]$ and for each δ-fine k-partition Π of $[d_1, d_2]$ we have

$$\rho\left(\sum_{\Pi} U, (GH_k)\int_A^B U\right) \leq \varepsilon$$

Let now $\varepsilon > 0$, $c > P$, $\delta_2^c(x) := \min\{\delta(x), \delta_1^c(x)\}$, $x \in [A, c]$, and Π be any δ_2^c-fine k-partition of $[A, c]$. Then we get:

$$\begin{aligned}\rho\left((GH_k)\int_A^c U, (GH_k)\int_A^B U\right) &\leq \rho\left(\sum_{\Pi} U, (GH_k)\int_A^c U\right) + \left(\sum_{\Pi} U, (GH_k)\int_A^B U\right) \\ &\leq 2\varepsilon.\end{aligned}$$

Thus the theorem is completely proved. □

Remark 5.3. An analogous version of Theorem 5.2 holds, if we consider, in our "limit operations" and calculus, the point A from the right instead of the point B from the left.

This concept will be useful in the sequel.

Definition 5.4. A sequence of integrable functions $(U_h : [A, B] \times \mathcal{F}^k \to X)_h$ is said to be *equiintegrable* if for any $\varepsilon > 0$ there exists a gauge γ on $[A, B]$ such that

$$\rho\left(\sum_{\Pi} U_h, (GH_k)\int_A^B U_h\right) \leq \varepsilon$$

for any γ-fine partition Π and every $h \in \mathbb{N}$.

We now prove the following convergence theorems for the GH_k integral in the context of metric semigroups.

Theorem 5.5. *Let $(U_h)_h$ be an equiintegrable sequence and let*

$$\lim_{h\to+\infty} \rho(U_h(t;\Lambda_1,\ldots,\Lambda_k), U(t;\Lambda_1,\ldots,\Lambda_k)) = 0$$

for any $t\in[A,B]$ and uniformly with respect to $\Lambda_1,\ldots,\Lambda_k\in\mathcal{F}$. Then U is GH_k integrable on $[A,B]$, and

$$\lim_{h\to+\infty} \rho\left((GH_k)\int_A^B U_h, (GH_k)\int_A^B U\right) = 0.$$

Proof: First of all, we observe that for each $\varepsilon>0$, there exist: a non-negative function $\mathcal{E}:[A,B]\times\mathcal{F}^k\to\mathbb{R}$, strictly positive on $([A,B]\cap\mathbb{R})\times\mathcal{F}^k$, GH_k integrable in $[A,B]$, with

$$(GH_k)\int_A^B \mathcal{E} \le \frac{\varepsilon}{2}$$

(for example,

$$\mathcal{E}(t;\Lambda_1,\ldots,\Lambda_k) = \sum_{j=1}^k |\Lambda_j| \frac{\varepsilon}{2\pi(1+t^2)}, \quad t\in[A,B],$$

with the convention $\mathcal{E}(\pm\infty;\Lambda_1,\ldots,\Lambda_k)=0$ for every choice of $\Lambda_j\in\mathcal{F}$, $j=1,\ldots,k$); a gauge γ_0 on $[A,B]$, such that

$$\sum_{i=1,\ldots,n,\ |I_i|<+\infty} \mathcal{E}(t_i;F_{i,1},\ldots,F_{i,k}) \le \varepsilon \tag{11}$$

for each γ_0-fine partition Π of $[A,B]$,

$$\Pi := \{(t_i;F_{i,1},\ldots,F_{i,k}), i=1,\ldots,n\} = \{(t_i;I_i), i=1,\ldots,n\},$$

with $\bigcup_{j=1}^k F_{i,j} = I_i$, $i=1,\ldots,n$.

Let now $\varepsilon>0$, γ be as in Definition 5.4, $\widehat{\gamma}=\gamma\cap\gamma_0$, and

$$\Pi := \{(t_i;F_{i,1},\ldots,F_{i,k}), i=1,\ldots,n\} = \{(t_i;I_i), i=1,\ldots,n\},$$

be any $\widehat{\gamma}$-fine k-partition of $[A,B]$, where $\bigcup_{j=1}^k F_{i,j} = I_i$, $i=1,\ldots,n$. Then for each $i=1,\ldots,n$ there exists a positive integer h_i such that

$$\rho(U_h(t_i;F_{i,1},\ldots,F_{i,k}), U(t_i;F_{i,1},\ldots,F_{i,k})) \le \mathcal{E}(t_i;F_{i,1},\ldots,F_{i,k}) \tag{12}$$

whenever $h \geq h_i$. Pick now $h \geq \max_{i=1,\dots,n} h_i$. From (11) and (12) we have:

$$\begin{aligned}
& \rho\left(\sum_{\Pi} U_h, \sum_{\Pi} U\right) \\
= & \rho\left(\sum_{i=1,\dots,n,|I_i|<+\infty} U_h(t_i; F_{i,1},\dots,F_{i,k}), \sum_{i=1,\dots,n,|I_i|<+\infty} U(t_i; F_{i,1},\dots,F_{i,k})\right) \\
\leq & \sum_{i=1,\dots,n,|I_i|<+\infty} \rho(U_h(t_i; F_{i,1},\dots,F_{i,k}), U(t_i; F_{i,1},\dots,F_{i,k})) \\
\leq & \sum_{i=1,\dots,n,|I_i|<+\infty} \mathcal{E}(t_i; F_{i,1},\dots,F_{i,k}) \leq \varepsilon.
\end{aligned}$$

It follows that

$$\lim_{h\to+\infty} \rho\left(\sum_{\Pi} U_h, \sum_{\Pi} U\right) = 0.$$

Now we get:

$$\rho\left(\sum_{\Pi} U, (GH_k)\int_A^B U_h\right) \leq \rho\left(\sum_{\Pi} U, \sum_{\Pi} U_h\right) + \rho\left(\sum_{\Pi} U_h, (GH_k)\int_A^B U_h\right) \leq 2\varepsilon.$$

Choose now arbitrarily two $\widehat{\gamma}$-fine partitions Π and Π' of $[A,B]$, and let $h^* = \max\{\max_i h_i, \max_j h'_j\}$, where the integers h_i, h'_j associated to Π and Π' respectively have the same role as the $h'_i s$ in (12). We get:

$$\begin{aligned}
& \rho\left(\sum_{\Pi} U, \sum_{\Pi'} U\right) \leq \rho\left(\sum_{\Pi} U, (GH_k)\int_A^B U_{h^*}\right) \qquad (13) \\
+ & \rho\left(\sum_{\Pi'} U, (GH_k)\int_A^B U_{h^*}\right) \leq 4\varepsilon.
\end{aligned}$$

Integrability of U on $[A,B]$ follows from (13) and the Cauchy criterion 4.2.

Finally, to every $\varepsilon > 0$ there corresponds a gauge $\overline{\gamma}$ on $[A,B]$ such that for any $\overline{\gamma}$-fine k-partition Π there exists $\overline{h} \in \mathbb{N}$ with

$$\begin{aligned}
& \rho\left((GH_k)\int_A^B U_h, (GH_k)\int_A^B U\right) \leq \rho\left((GH_k)\int_A^B U_h, \sum_{\Pi} U_h\right) \\
+ & \rho\left(\sum_{\Pi} U_h, \sum_{\Pi} U\right) + \rho\left(\sum_{\Pi} U, (GH_k)\int_A^B U\right) \leq 3\varepsilon
\end{aligned}$$

for all $h \geq \overline{h}$. This implies that

$$\lim_{h\to+\infty} \rho\left((GH_k)\int_A^B U_h, (GH_k)\int_A^B U\right) = 0. \qquad \square$$

The next step is to prove a version of the convergence theorem with respect to the "uniform convergence". To this aim we introduce the following concept.

Definition 5.6. Given a sequence of functions $(U_n : [A,B] \times \mathcal{F}^k \to X)_{n\in\mathbb{N}\cup\{0\}}$, we say that the U_n's, $n \geq 1$, *variationally uniformly converge to* U_0 if to every $\varepsilon > 0$ an integer n_0 can be found, such that

$$\rho\left(\sum_{i=1}^{q} U_n(t_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^{q} U_0(t_i; F_{i,1}, \ldots, F_{i,k})\right) \leq \varepsilon$$

for every $n \geq n_0$ and any k-partition $\Pi = \{(t_i, F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, q\} = \{(t_i, I_i), i = 1, \ldots, q\}$ of $[A,B]$, where $\bigcup_{j=1}^{k} F_{i,j} = I_i$, $i = 1, \ldots, q$.

Observe that, if $k = 1$ and

$$U_n(t; [u,v]) = [g(v) - g(u)] \cdot f_n(t), \quad n \in \mathbb{N} \cup \{0\},$$

where $g : [A,B] \to \mathbb{R}$ is of bounded variation and the sequence $(f_n : [A,B] \to X)_n$ is uniformly convergent to f_0 on $[A,B]$, then the U_n's variationally uniformly converge to U_0. In this case, under the hypothesis of uniform convergence of $(f_n)_n$ to f_0, if the f_n's, $n \geq 1$, are Henstock-Stieltjes integrable with respect to g, then f_0 is too, and we get the exchange of limits under the sign of integral.

An example in which this happens is when we take $X = L(\mathbb{R})$ (i. e. the set of all fuzzy numbers), and define $f_n : [0,1] \to X$ by setting $f_n(x) = \chi_{[0,1]\cap[x-1/n,x+1/n]}$, $n \in \mathbb{N}$, then the sequence $(f_n)_n$ is uniformly convergent to the "identity" function (in the sense that the generic element $x \in [0,1]$ is identified with the element $\chi_{\{x\}}$).

Theorem 5.7. *Let* $(U_n : [A,B] \times \mathcal{F}^k \to X)_n$ *be a sequence of functions,* GH_k *integrable on* $[A,B]$ *and variationally uniformly convergent to a map* U.

Then U *is* GH_k *integrable on* $[A,B]$ *and*

$$\lim_{n\to+\infty} \rho\left((GH_k)\int_A^B U_n, (GH_k)\int_A^B U\right) = 0.$$

Proof: Let $\varepsilon > 0$, and take $n_0 = n_0(\varepsilon)$ according to variationally uniform convergence. Then

$$\begin{aligned}
& \rho\left(\sum_{\Pi_1} U, \sum_{\Pi_2} U\right) \leq \rho\left(\sum_{\Pi_1} U, \sum_{\Pi_1} U_{n_0}\right) \\
+ \quad & \rho\left(\sum_{\Pi_1} U_{n_0}, \sum_{\Pi_2} U_{n_0}\right) + \rho\left(\sum_{\Pi_2} U_{n_0}, \sum_{\Pi_2} U\right) \\
\leq \quad & 2\varepsilon + \rho\left(\sum_{\Pi_1} U_{n_0}, \sum_{\Pi_2} U_{n_0}\right)
\end{aligned}$$

for any two partitions Π_1, Π_2 of $[A, B]$. Since U_{n_0} is GH_k integrable on $[A, B]$, then there is a map $\delta = \delta_{n_0} : [A, B] \to \mathbb{R}^+$, such that, for any two δ-fine k-partitions Π_1, Π_2 of $[A, B]$,

$$\rho\left(\sum_{\Pi_1} U_{n_0}, \sum_{\Pi_2} U_{n_0}\right) \leq \varepsilon,$$

and hence

$$\rho\left(\sum_{\Pi_1} U, \sum_{\Pi_2} U\right) \leq 3\varepsilon.$$

Thus U is GH_k integrable on $[A, B]$, by virtue of the Cauchy criterion 4.2 So there exists a map $\delta' : [A, B] \to \mathbb{R}^+$ such that

$$\rho\left(\sum_{\Pi} U, (GH_k)\int_A^B U\right) \leq \varepsilon$$

for each δ'-fine partition Π of $[A, B]$. Fix $n \geq n_0$ and choose $\kappa_n : [A, B] \to \mathbb{R}^+$ such that

$$\rho\left(\sum_{\Pi} U_n, (GH_k)\int_A^B U_n\right) \leq \varepsilon$$

whenever Π is a κ_n-fine partition of $[A, B]$. Put $\overline{\delta}_n = \min\{\delta', \kappa_n\}$: for any $\overline{\delta}_n$-fine k-partition Π of $[A, B]$ we obtain

$$\rho\left((GH_k)\int_A^B U_n, (GH_k)\int_A^B U\right) \leq \rho\left((GH_k)\int_A^B U, \sum_{\Pi} U\right)$$
$$+ \quad \rho\left(\sum_{\Pi} U, \sum_{\Pi} U_n\right) + \rho\left(\sum_{\Pi} U_n, (GH_k)\int_A^B U_n\right) \leq 3\varepsilon,$$

and thus the last part of the assertion. $\square$

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