

An Extension Theorem for Non-transitive Preferences

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Abstract

The paper shows that any σ -transitive preference can be extended to a complete preference preserving σ -transitivity. The result has potential applications to the theory of choice and specifically to revealed preference theory.

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1 Introduction

In his classical contribution Szpilrajn (1930) shows that any quasi-ordering (a reflexive and transitive binary relation) can be extended to an ordering (a complete quasi-ordering).¹ The purpose of this paper is to provide an analogous extension result for non-transitive preferences. The case of quasi-transitivity (transitivity of the strict preference) is trivial since it suffices to ‘complete’ the original preference by putting indifference for all pairs of alternatives that are non-comparable. The focus of our analysis is a different

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¹Strictly speaking Szpilrajn proved a slightly different result.

notion of non-transitivity called σ -transitivity and put forward by Sen (1970). Our main result is an extension theorem which shows that any σ -transitive preference can be extended to a complete preference preserving σ -transitivity. We also provide the dual result in terms of a strict preference approach. Finally, we show an application to the theory of choice by providing a new characterization of the Weak Axiom of Revealed Preference.

2 Definitions and notation

Let X be an arbitrary set of alternatives. By S we denote a binary relation in X , i.e. $S \subseteq X^2$; we say that $x \in X$ is related to $y \in X$ and write $(x, y) \in S$ or, equivalently, xSy . Some of the most frequently used properties of binary relations are stated below. A binary relation S in X is

Reflexive if xSx for all $x \in X$.

Irreflexive if $\neg xSx$ for all $x \in X$.

Symmetric if xSy implies ySx .

Asymmetric if xSy implies $\neg ySx$.

Complete if for all $x, y \in X$ either xSy or ySx or both.

Transitive if, for all $x, y, z \in X$, xSy and ySz imply xSz .

An irreflexive binary relation in X is called a *strict preference* and is denoted by $Q \subseteq X^2$. A strict preference can be seen as the union of two disjoint components. By $Q^\alpha \subseteq X^2$ we denote the asymmetric component of Q , i.e. $xQ^\alpha y$ iff xQy and $\neg yQx$. The symmetric component of Q is denoted by $E \subseteq X^2$, thus xEy iff xQy and yQx . Clearly, $Q = Q^\alpha \cup E$ and $Q^\alpha \cap E = \emptyset$. When the strict preference Q is asymmetric the component E is empty.

From Q we derive the *non-comparability* relation $N \subseteq X^2$ which is defined by xNy iff $\neg xQy$ and $\neg yQx$. Therefore, N is symmetric and non empty, since by irreflexivity it contains the set $\Delta := \{(x, x) \mid x \in X\}$.

By $Q(x)$ we denote the upper contour set of Q , thus the set of strictly preferred alternatives, i.e. $Q(x) = \{z \in X \mid zQx\}$. On the other hand, $Q^{-1}(x) = \{z \in X \mid xQz\}$ is the lower contour set of Q , i.e. the set of alternatives ‘dominated’ by x .

Let us introduce a few mild properties of transitivity.

Definition 1. Let Q be a strict preference.

Q is *Superiorly Regular* (SR) if, for all $x, y \in X$, xNy implies $Q(x) = Q(y)$ or, equivalently, if, for all $x, y, z \in X$, zQx and xNy imply zQy .

Q is *Inferiorly Regular* (IR) if, for all $x, y \in X$, xNy implies $Q^{-1}(x) = Q^{-1}(y)$ or, equivalently, if, for all $x, y, z \in X$, zNx and xQy imply zQy .

We notice that either SR or IR imply transitivity of the non comparability relation N , i.e. if xNy and yNz then xNz . However, SR and IR do not imply transitivity of Q .

A non irreflexive binary relation in X is a *weak* preference or, simply, a preference, and will be denoted by $R \subseteq X^2$. The asymmetric and the symmetric components of a preference R are respectively denoted by R^α and R^σ , i.e. $xR^\alpha y$ iff xRy and $\neg yRx$, and $xR^\sigma y$ iff xRy and yRx . Clearly, $R = R^\alpha \cup R^\sigma$ and $R^\alpha \cap R^\sigma = \emptyset$. The following definitions have been introduced by Sen (1970)

Definition 2. The preference R is

- i) $\alpha\sigma$ -transitive if $zR^\alpha x$ and $xR^\sigma y$ imply $zR^\alpha y$
- ii) $\sigma\alpha$ -transitive if $zR^\sigma x$ and $xR^\alpha y$ imply $zR^\alpha y$
- iii) $\sigma\sigma$ -transitive if $zR^\sigma x$ and $xR^\sigma y$ imply $zR^\sigma y$
- iv) σ -transitive if (i), (ii) and (iii) hold.

Notice that σ -transitivity does not imply transitivity of R .

In this paper we use both the strict and the weak preference approach. As it is well known,² they are intimately related by the following relationship. A given strict preference Q generates the weak preference R as follows: xRy iff $\neg yQx$. Equivalently, R can be defined by $R = X^2 - Q^{-1}$ where $Q^{-1} := \{(x, y) \in X^2 \mid (y, x) \in Q\}$ is the *inverse* relation of Q . Starting from a reflexive weak preference R , we can generate in a similar way a strict preference Q . We say that R and Q are *conjugate* preferences.

A *subrelation* of a given strict preference S is a strict preference Q satisfying $Q \subseteq S$ and $S^\alpha \subseteq Q^\alpha$. Similarly, an *extension* of a given preference S is a preference R satisfying $S^\alpha \subseteq R^\alpha$ and $S^\sigma \subseteq R^\sigma$.

3 A preliminary result

Given a SR strict preference Q we propose a general method for constructing a subrelation $P \subseteq Q$ which is asymmetric and preserves the SR property. As we will see, this result also applies to the case of weak preferences and allows us to obtain a complete σ -transitive extension.

The construction of the desired subrelation P may seem a trivial matter. In fact, if we select ‘one half’ of the symmetric component E , i.e. an asymmetric set E^* such that $E^* \cup E^{*-1} = E$, we immediately obtain the asymmetric subrelation $P = Q^\alpha \cup E^*$. However, P need not preserve the SR property. Take the following simple example.

Example. Let $X = \{x, y, z\}$ and $Q = \{(x, z), (z, x), (y, z), (z, y)\}$. Then $Q^\alpha = \emptyset$, $E = Q$ and $N = \{(x, y), (y, x)\} \cup \Delta$. Clearly, Q has the SR property and is not asymmetric. Let us consider the asymmetric relation $E^* = \{(x, z), (z, y)\}$. Then $E^* \cup E^{*-1} = E$ and the subrelation $P = E^*$ is asymmetric. However, P has not the SR property since xNy , $z \in P(y)$ but $z \notin P(x)$. In this trivial example it is easy to find a subrelation P which fulfills the desired requirements; take, for instance, $E^* = \{(z, x), (z, y)\}$. In general, however, things are not so simple especially when the number of alternatives is not finite.

²See Section 6 in Kim and Richter (1986).

All over the present section, we assume that $Q = Q^\alpha \cup E$ is a strict preference satisfying the SR property. The aim of the present paper is to devise a general method for splitting the symmetric component E of the strict preference by preserving the SR property. A few useful facts deriving from SR and concerning the symmetric component of the strict preference are stated below.

Fact 1. $(z, y) \in E$ if and only if there exists $(x, w) \in X^2$ such that (z, y) and (x, w) satisfy the following conditions:

$$\begin{aligned} zQx, & \quad xNy \\ yQw, & \quad wNz \end{aligned} \tag{1}$$

Indeed, (1) and SR imply $(z, y) \in E$. Conversely, (1) trivially holds for $x = y$ and $w = z$.

The relationships between x and w satisfying (1) are characterized by the following fact.

Fact 2. If $(z, y) \in E$ and $(x, w) \in X^2$ satisfy (1) there are only three possible mutually exclusive cases:

- i) $wQ^\alpha x$ iff $zQ^\alpha x$
- ii) $xQ^\alpha w$ iff $yQ^\alpha w$
- iii) wEx .

Proof. Obviously, the three cases are mutually exclusive. To prove that they exhaust all the possibilities we must show that xNw is excluded. Indeed, by transitivity of N , wNz would imply xNz which is impossible since zQx . Next, let us consider the equivalence in (i) and suppose that xQz . Then from zNw and SR we have xQw which contradicts $wQ^\alpha x$, thus $zQ^\alpha x$. The converse and the equivalence in (ii) are proved in a similar way. \square

As we mentioned before, the construction of an asymmetric subrelation of Q requires a careful analysis of the set E . In particular, we are looking for an asymmetric subset of E . The first step is given by the following definition.

Definition 3. The set $E_1^* \subseteq X^2$ is defined as follows: $(z, y) \in E_1^*$ if $(z, y) \in E$ and for *some* $(x, w) \in X^2$ satisfying (1) it holds $zQ^\alpha x$.

As required the relation E_1^* is asymmetric. To check this claim let us suppose that both $(z, y) \in E_1^*$ and $(y, z) \in E_1^*$, then there exist $(x, w), (x', w') \in X^2$ such that (i) (z, y) and (x, w) satisfy (1) with $zQ^\alpha x$ and (ii) (z, y) and (x', w') satisfy (1) with $yQ^\alpha w'$. But, since then (z, y) and (x, w') would satisfy (1), by Fact 2, $zQ^\alpha x$ and $yQ^\alpha w'$ cannot hold together and this contradiction establish our claim.

Other potential elements of the symmetric component E are collected in the set $E_2 \subseteq X^2$ defined as follows:

$(z, y) \in E_2$ if $(z, y) \in E$ and for *all* $(x, w) \in X^2$ satisfying (1) it holds xEw .

Clearly E_2 is symmetric and its intersection with E_1^* is empty. The next step in the construction of the desired subrelation consists in selecting an appropriate subset of E_2 . To this aim we introduce a new family of subsets of E_2 :

Let $(z, y) \in E_2$; The set $G(z, y)$ consists of all $(x, w) \in X^2$ satisfying (1).

It is easily verified that $(y, z) \in G(z, y)$ so that the sets $G(z, y)$ are not empty. Other useful properties are stated below.

Lemma 1. The set $G(z, y)$ has the following properties:

- i) $G(z, y) \subseteq E_2$.
- ii) $G(z, y)$ is asymmetric.
- iii) $G(y, z) = G(z, y)^{-1} := \{(w, x) \in X^2 \mid (x, w) \in G(z, y)\}$.
- iv) If $(x, w) \in G(z, y)$ then $G(z, y) = G(z, x)$.

The proof is the Appendix.

The sets $G(z, y)$ allow us to obtain a partition of E_2 .

Definition 4. The binary relation \mathcal{R} in E_2 is defined as follows: for all (z, y) and (z', y') in E_2 ,

$$(z, y)\mathcal{R}(z', y') \quad \text{iff} \quad G(z, y) = G(z', y')$$

It is easily seen that \mathcal{R} is an equivalence relation, i.e. it is reflexive, symmetric and transitive, therefore E_2 can be partitioned into equivalence classes. We denote by \mathcal{C} the family of equivalence classes and by C a generic element of \mathcal{C} , thus $E_2 = \cup_{C \in \mathcal{C}} C$ and $C \cap C' = \emptyset$ for all $C \neq C'$.

Lemma 2. The set E_2 is partitioned into equivalence classes by the binary relation \mathcal{R} . Any equivalence class $C \in \mathcal{C}$ has the following properties: (i) C is an asymmetric subset of E_2 and (ii) the set $C^{-1} = \{(z, y) \in E_2 \mid (y, z) \in C\}$ is an equivalence class in E_2 , i.e. if $C \in \mathcal{C}$ then $C^{-1} \in \mathcal{C}$.

The proof is in the Appendix.

The above construction allows us to split in a suitable way the set E_2 . Indeed, according to Lemma 2, \mathcal{C} can be partitioned by the family of sets $\{C, C^{-1}\}$. By the Axiom of Choice, then there exists a subset $\mathcal{D} \subseteq \mathcal{C}$ with the following property: if $C \in \mathcal{C}$ then either $C \in \mathcal{D}$ or $C^{-1} \in \mathcal{D}$, but not both.

Lemma 3. Let $E_2^* := \cup_{C \in \mathcal{D}} C$. The set E_2^* is an asymmetric subset of E_2 .

The proof of Lemma 3 follows easily from the definition of \mathcal{D} and Lemma 2.

We are ready for the final step of our construction.

Proposition 1. Let $E \subseteq X^2$ be the symmetric component of a strict preference Q satisfying the SR property and let

$$E^* := E_1^* \cup E_2^* \tag{2}$$

where E_1^* is as in Definition 3 and E_2^* as in Lemma 3. The set E^* is asymmetric and satisfies the condition $E^* \cup E^{*-1} = E$.

The proof is in the Appendix.

It may be useful to illustrate our construction by means of the example discussed at the beginning of this section.

Example (continued). Since $Q^\alpha = \emptyset$, by Definition 3, also E_1^* is empty. On the other hand, it is easy to check that (z, y) and (z, x) are in E_2 , so that

by symmetry of E_2 we have $E_2 = E = Q$. With the help of Lemma 1 we can also compute the sets G .

$$\begin{aligned} G(z, y) &= \{(x, z), (y, z)\} & G(y, z) &= \{(z, x), (z, y)\} \\ G(z, x) &= \{(x, z), (y, z)\} & G(x, z) &= \{(z, x), (z, y)\} \end{aligned}$$

Thus, we have $(z, y)\mathcal{R}(z, x)$ and $(y, z)\mathcal{R}(x, z)$, and the equivalence classes are $\mathcal{C} = \{C, C'\}$ where

$$C = \{(z, x), (z, y)\} \quad \text{and} \quad C' = \{(x, z), (y, z)\}$$

We can set $\mathcal{D} = \{C\}$, therefore, $E_2^* = C$ and, finally, $E^* = E_1^* \cup E_2^* = \emptyset \cup C = \{(z, x), (z, y)\}$, which is the solution proposed in the example.

We notice that the construction proposed in Proposition 1 works when the set of alternative X is finite as well as when it is infinite. However, only in the latter case we are forced to resort to the Axiom of Choice.

4 The main result

In this section we show that the set E^* actually serves the purpose of finding the desired asymmetric subrelation of Q .

Theorem 1. Let Q be a strict preference satisfying the SR property. Then there exists a strict preference P with the following properties:

- i) P is an asymmetric subrelation of Q , i.e. $P \cap P^{-1} = \emptyset$, $P \subseteq Q$ and $Q^\alpha \subseteq P$.
- ii) P has the SR and IR properties.

Proof. Set $P = Q^\alpha \cup E^*$ where E^* is given by (2). Then (i) follows trivially from Proposition 1.

As for (ii), let us suppose that P does not satisfy the SR property, i.e. there exist x, y and z such that xNy , $z \in P(x)$ and $z \notin P(y)$. Since $P \subseteq Q$ and $z \in P(x)$ we have $(z, x) \in Q$ and by SR of Q it must be $(z, y) \in Q$.

Thus, since $(z, y) \notin P$ it must be $(z, y) \in E$ and specifically $(z, y) \in E^{*-1}$ or equivalently $(y, z) \in E^*$. We will show that this leads to a contradiction.

Let us suppose that $(y, z) \in E_1^*$, so that there exists $w \in X$ such that $yQ^\alpha w$ and wNz . Thus, we can write

$$\begin{array}{ll} yQ^\alpha w, & wNz \\ zQx, & xNy \end{array}$$

By Fact 2(ii) we have $xQ^\alpha w$ and it is easily seen that $(x, z) \in E_1^*$. Thus $x \in P(z)$, which is impossible since $z \in P(x)$ and P is asymmetric. Therefore, we conclude that $(y, z) \notin E_1^*$.

Next, let us suppose that $(y, z) \in E_2^*$ so that (y, z) belongs to an equivalence class, i.e. $(y, z) \in C$ for some $C \in \mathcal{C}$. Moreover, since $(y, z) \in E_2$ and (z, x) satisfy (1), by Lemma A (in the Appendix), we also have $(x, z) \in E_2$. We show that $(x, z) \in C$, i.e. $G(y, z) = G(x, z)$.

By symmetry of E_2 , $(y, z) \in E_2$ implies $(z, y) \in E_2$, thus there exists $w \in X$ such that $(x, w) \in G(z, y)$. Then by Lemma 1(iv), $G(z, y) = G(z, x)$, so that $G(z, y)^{-1} = G(z, x)^{-1}$ and by Lemma 1(iii) $G(y, z) = G(x, z)$. Since (y, z) and (x, z) belong to the same equivalence class C and $(y, z) \in E_2^*$, we also have $(x, z) \in E_2^*$. But that means $x \in P(z)$, which is impossible since P is asymmetric and $z \in P(x)$ by assumption. Hence $(y, z) \notin E_2^*$ and we conclude that P has the SR property.

To complete point (ii) we show that an asymmetric and SR preference is also IR. Indeed, let us suppose that IR is violated, i.e. there exist xNy , xPz and $\neg yPz$. We have two cases: (a) zPy , then SR yields zPx which is impossible since P is asymmetric. (b) $\neg zPy$, so that zNy and since xPz , SR yields xPy which is impossible since xNy . Thus we conclude that P is also IR. □

The Extension Theorem is the dual of Theorem 1.

Theorem 2.(Extension Theorem) Let S be a reflexive and σ -transitive binary relation. Then there exists a reflexive preference R with the following properties:

- i) R is complete, i.e. $R \cup R^{-1} = X^2$.

ii) R is an extension of S , i.e. $S^\alpha \subseteq R^\alpha$ and $S^\sigma \subseteq R^\sigma$.

ii) R is σ -transitive.

Proof. Let us set $Q = X^2 - S^{-1}$ or equivalently define Q by xQy iff $x \neq y$ and $\neg ySx$. By reflexivity of S , the relation Q is irreflexive. Let us suppose that Q is not SR, i.e. there exist x, y and z such that xNy , zQx and $\neg zQy$. By definition of Q and by $z \neq y$ we have yRz . In addition, xNy means $xR^\sigma y$ so that by σ -transitivity we have xRz which contradicts zQx . Hence Q is SR.

By Theorem 1 there exists an asymmetric SR subrelation $P = Q^\alpha \cup E^* = S^\alpha \cup E^*$. Taking the complement of P^{-1} we obtain the reflexive preference $R = X^2 - P^{-1}$, equivalently defined by xRy iff $\neg yPx$. By asymmetry of P the preference R is complete. Moreover, $R^\alpha = P = S^\alpha \cup E^* \supseteq S^\alpha$ and $R^\sigma = S^\sigma$. Thus R is an extension of S . It is not difficult to check that R has the desired property of σ -transitivity. For instance, let us consider $\sigma\alpha$ -transitivity and suppose $xR^\sigma y$, $yR^\alpha z$ but $\neg xR^\alpha z$. In terms of the strict preference P we have xNy , yPz and $\neg xPz$, which violate IR of P . □

Theorem 1 and 2 do not place any requirement on the algebraic or topological structure of the set of alternatives, therefore they provide quite abstract and general results. An interesting application is discussed in the next section.

5 An application

In this final section of the paper we illustrate our result with an application to the theory of choice and specifically to revealed preference theory. To begin with, let us introduce some more notation.

Let \mathcal{A} be the set of all the subsets of X and \mathcal{B} any nonempty subset of \mathcal{A} which does not contain the empty set. A function $\Phi: \mathcal{B} \rightarrow \mathcal{A}$ is a choice if $\Phi(B) \subseteq B$ for all $B \in \mathcal{B}$. Φ is single-valued if $\Phi(B)$ is a singleton for all

$B \in \mathcal{B}$. Following Richter (1966), a preference R is said to *rationalize* the choice function Φ if for all $B \in \mathcal{B}$

$$\Phi(B) = \{x \in B \mid xRy \text{ for all } y \in B\}.$$

A choice function Φ is rational if there exists some preference R satisfying the above condition. A strict preference Q *motivates* the choice function Φ if for all $B \in \mathcal{B}$

$$\Phi(B) = \{x \in B \mid Q(x) \cap B = \emptyset\}.$$

Φ is motivated if there exists some strict preference Q satisfying the above condition.

A choice function Φ generates a *revealed preference* relation, that is a binary relation $V \subseteq X^2$, which is defined by xVy iff there exists $B \in \mathcal{B}$ such that $x \in \Phi(B)$ and $y \in B$, in words, x is revealed preferred to y if y is available when x is chosen. The revealed *strict* preference V^* is defined by xV^*y iff there exists $B \in \mathcal{B}$ such that $x \in \Phi(B)$ and $y \in [B - \Phi(B)]$ (i.e. y is rejected when x is chosen). Samuelson (1938) introduced the Weak Axiom of Revealed Preference (WARP) as a criterion of consistency of choice.

The choice function $\Phi: \mathcal{B} \rightarrow \mathcal{A}$ satisfies WARP iff xV^*y implies $\neg yVx$. As an application of our Extension Theorem we obtain a new characterization of WARP.

Proposition 2. Let Φ be a single-valued choice. The following statements are equivalent:

- i) Φ satisfies the WA
- ii) Φ is rationalized by a reflexive and σ -transitive preference S
- iii) Φ is motivated by a SR and IR strict preference Q
- iv) Φ is motivated by an asymmetric SR and IR strict preference P
- v) Φ is rationalized by a complete, σ -transitive preference R

Proof. (i) to (ii). It is well known that a choice satisfying the WARP is rationalized by the revealed preference V . Moreover, since Φ is single-valued

V is antisymmetric, i.e. xVy and yVx imply $x = y$. Thus $V^\sigma \subseteq \Delta = \{(x, y) \in X^2 \mid x = y\}$ and V is trivially σ -transitive. Hence $S = V \cup \Delta$ is reflexive, σ -transitive and rationalizes Φ .

(ii) to (iii). Set $Q = X^2 - S^{-1}$, i.e. xQy iff $x \neq y$ and $\neg ySx$. Clearly, Q is irreflexive and using a similar argument as that used in the proof of Theorem 1.(ii) one easily shows that Q is also SR and IR.

Next, we show that Q motivates Φ . It is easily seen that $\Phi(B) \subseteq \Psi(B) = \{x \in B \mid Q(x) \cap B = \emptyset\}$ for all $B \in \mathcal{B}$. To show the converse let us suppose that $x \in \Psi(B)$ and $x \notin \Phi(B)$ for some B . By definition of Ψ we have $\neg yQx$ for all $y \in B$ and, for $y \neq x$ we have xSy . Since $x \notin \Phi(B)$ it must be $\neg xSx$, which contradicts reflexivity of S .

(iii) to (iv). By Theorem 1 the desired strict preference P can be obtained from Q . Next we show that P motivates Φ . Let $\Psi(B) = \{x \in B; P(x) \cap B = \emptyset\}$. Since Φ is motivated by Q and $P(x) \subseteq Q(x)$, it is easily seen that $\Phi(B) \subseteq \Psi(B)$. To show the converse let $x \in \Psi(B)$ so that $P(x) \cap B = \emptyset$. Arguing by contradiction, let us suppose that $x \notin \Phi(B)$, which means that there exists z such that $z \in Q(x) \cap B$. Since $\Phi(B)$ is not empty, there exists $y \neq x$ such that $y \in \Phi(B)$, thus $y \in \Psi(B)$ since $P(x) \subseteq Q(x)$. Clearly, x and y are non comparable under P and, by (ii), also under Q , i.e. xNy . However, since Q has the SR property and zQx , it must be zQy so that $z \in Q(y) \cap B$ and y cannot be in $\Phi(B)$. Therefore, we conclude that P motivates Φ .

(iv) to (v). By the same argument used in the proof of the Extension Theorem we obtain the required preference R . Moreover, since R is the conjugate preference of P it rationalizes the same choice function motivated by P (see Kim and Richter (1986), Lemma 5).

(v) to (i). Let us suppose that Φ violates WARP, i.e. there exist x and y such that xV^*y and yVx . From xV^*y , there exists B such that $x \in \Phi(B)$ and $y \in B - \Phi(B)$. Since R rationalizes Φ we must have xRy and for some $z \in B$, $zR^\alpha y$. Moreover, from yVx we have yRx , so that $yR^\sigma x$. By σ -transitivity of R we have $zR^\alpha x$, but then R cannot rationalize Φ .

□

The characterization of WARP by (ii) and (iii) are respectively due to Clark (1988) and Scapparone (2000). The last two characterizations, (iv)

and (v), are new and are obtained as applications of the extension results.

Appendix

Lemma A. If $(z, y) \in E_2$ and (x, w) satisfies (1) then (x, w) , (z, x) and (y, w) are in E_2 .

Proof. Let us first show that $(z, x) \in E_2$. By (1) we have zQx and by $(z, y) \in E_2$ we have xEw . Thus, by SR, xQw and wNz yield xQz . Thus (z, x) is in E .

Let (z, x) and (x', w') $\in X^2$ satisfy conditions (1), i.e.

$$\begin{aligned} zQx', & \quad x'Nx \\ xQw', & \quad w'Nz \end{aligned}$$

If $w'Q^\alpha x'$ then, by Fact 2, $zQ^\alpha x'$. Noting that, by SR, $x'Nx$ and xNy yield $x'Ny$ we see that (z, y) and (x', w') satisfy (1) so that $(z, y) \in E_1^*$ which contradicts $(z, y) \in E_2$.

Next, let us suppose that $x'Q^\alpha w'$. Then, by Fact 2, $xQ^\alpha w'$ and it is easily seen that (x, z) and (w', y) satisfy (1). By Fact 2 and $xQ^\alpha w'$ we then have $yQ^\alpha w'$, therefore (y, z) and (w', x) satisfy (1) and $(y, z) \in E_1^*$. But this contradicts $(z, y) \in E_2$ since E_2 is symmetric and E_1^* and E_2 are disjoint.

Thus, by Fact 2, we conclude that for all (x', w') it must be $x'Ew'$ so that $(z, x) \in E_2$.

The remaining two cases are shown similarly. For example, notice that $(x, z) \in E_2$ and (w, y) satisfy (1) so that, as shown above, $(x, w) \in E_2$. \square

Proof of Lemma 1.

- (i) $(x, w) \in G(z, y)$ implies, by Lemma A, that $(x, w) \in E_2$.
- (ii) Let $(x, w) \in G(z, y)$ then, by Definition 5, wNz . If (w, x) is in $G(z, y)$ then zQw which contradicts wNz .
- (iii) We have to show that $(x, w) \in G(z, y)$ iff $(w, x) \in G(y, z)$. This follows trivially from (1).
- (iv) By Lemma A, $(z, x) \in E_2$ thus $G(z, x)$ is well defined. First, we show that $G(z, y) \subseteq G(z, x)$.

Let $(x', w') \in G(z, y)$, i.e. (z, y) and (x', w') satisfy conditions (1) so that we have zQx' . Moreover, $x'Nx$ follows from SR, $x'Ny$ and yNx . Next, it is easily seen that (z, y) and (x, w') satisfy (1), therefore we have $w'Nz$ and, by Lemma A, xQw' . Hence (z, x) and (x', w') satisfy (1) and we conclude that $(x', w') \in G(z, x)$.

The inclusion $G(z, x) \subseteq G(z, y)$ is shown in a similar way. \square

Proof of Lemma 2.

(i) If (z, y) and (y, z) are in the same equivalence class we have $G(z, y) = G(y, z)$. Then, by Lemma 1(iii), $G(z, y) = G(z, y)^{-1}$, which is impossible since $G(z, y)$ is asymmetric and non empty.

(ii) Let $(z, y) \in C$. By (i), C is asymmetric thus $(y, z) \notin C$. Therefore $(y, z) \in C'$ for some $C' \in \mathcal{C}$ with $C' \neq C$. We have to show that $C' = C^{-1}$. Now, let $(y', z') \in C'$. Thus $G(y, z) = G(y', z')$ so that $G(y, z)^{-1} = G(y', z')^{-1}$ and, by Lemma 1(iii), $G(z, y) = G(z', y')$. Hence $(z', y') \in C$ and $(y', z') \in C^{-1}$. Therefore, we have shown that $C' \subseteq C^{-1}$. The converse is also true and is proved similarly. \square

Proof of Proposition 1.

Let $(z, y) \in E^*$ and $(y, z) \in E^*$. As we know, E_1^* and E_2^* are asymmetric thus $(z, y) \in E_1^*$ and $(y, z) \in E_2^*$ (or vice versa). Since $(y, z) \in E_2^*$ we also have $(z, y) \in E_2$, which is impossible since E_1^* and E_2 are disjoint.

Next, we show that $E \subseteq E^* \cup E^{*-1}$. Let $(z, y) \in E$, then by Fact 1 and 2 we have the following cases:

(a) For some $(x, w) \in X^2$ satisfying (1) it holds $xQ^\alpha w$. Then, by Definition 4, $(z, y) \in E_1^* \subseteq E^* \cup E^{*-1}$.

(b) For some $(x, w) \in X^2$ satisfying (1) it holds $wQ^\alpha x$. Then, by Definition 4, $(y, z) \in E_1^*$ thus $(z, y) \in E_1^{*-1} \subseteq E^* \cup E^{*-1}$.

(c) Finally, for all $(x, w) \in X^2$ satisfying (1) it holds xEw . Then, by Definition 4, $(z, y) \in E_2$ and, by Remark 3, $E_2 \subseteq E^* \cup E^{*-1}$.

This completes the proof, since the inclusion $E^* \cup E^{*-1} \subseteq E$ trivially holds. \square

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