Degree theory: old and new

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1. Introduction

The theory of degree has a long history which is a cascade of successive generalizations. Presumably, the oldest notion is the degree of a (smooth) map u from S^1 into S^1 (S^1 =the unit circle). The degree of u, also called winding number, counts "how many times u covers its range taking into account the algebraic multiplicity." More generally, a smooth (say C^1) map u from S^1 into $\mathbb{C}$, such that $u \neq 0$ on S^1 has a degree which may be computed through the very classical integral formula

$$\deg u = \frac{1}{2\pi i} \int_{S^1} \frac{\dot{u}}{u} \tag{1.1}$$

which measures the "algebraic change of phase" of u as the variable goes around S^1 once. Similarly, if Γ is a simple curve in $\mathbb{R}^2$ and u is a smooth map from Γ into S^1, then its degree can be computed as

$$\deg(u, \Gamma) = \frac{1}{2\pi} \int_\Gamma u \times u_\tau, \tag{1.2}$$

where $\times$ denotes the cross product of vectors in $\mathbb{R}^2$ (here S^1 is viewed as a subset of $\mathbb{R}^2$, not $\mathbb{C}$) and u_τ denotes the tangential derivative of u along Γ.

Starting at the end of the 19th century people realized that the notion of degree also makes sense in higher dimensions. To simplify the presentation I will consider only (smooth) maps u from S^n into S^n ($= n$ dimensional unit sphere in $\mathbb{R}^{n+1}$), but the theory extends to maps $u : X \to Y$ where X and Y are smooth n-dimensional oriented manifolds without boundary.

Here is the precise definition of degree. Fix any $y \in S^n$ which is a regular value of u, i.e.,

$$\varphi^{-1}(y) = \{x_1, x_2, \dots, x_p\}$$

is a finite set and, for each i, the Jacobian determinant $\det J_u(x_i) \neq 0$. (Recall that, by Sard's theorem, almost every y is a regular value.) The degree of u is by definition the number of solutions of the equation $u(x) = y$ taking into account their algebraic multiplicity:

$$\deg u = \sum_{i=1}^{p} \operatorname{sign} \det J_u(x_i). \tag{1.3}$$

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In principle this number depends on the choice of the regular value y. A **remarkable property** is that $\deg u$ defined above is **independent of** y, so that one may talk about $\deg u$ without specifying y.

A very important representation formula, which is the n-dimensional analogue of (1.1) or (1.2) allows to compute the degree as an integral of the Jacobian determinant:

$$\deg u = \frac{1}{|S^n|} \int_{S^n} \det J_u \tag{1.4}$$

where the Jacobian determinant is computed using geodesic normal coordinates (both in the domain and the range) and $|S^n|$ denotes the measure (length, area, volume, etc...) of S^n. For the proof of (1.4) see e.g. L. Nirenberg [1] or H. Brezis and L. Nirenberg [3]. It is sometimes convenient to observe that

$$\det J_u = \det(u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}) \tag{1.5}$$

(recall that u takes its values in $\mathbb{R}^{n+1}$ and det on the righthand side of (1.5) refers to the determinant of an $(n+1) \times (n+1)$ matrix); this follows easily from the fact that $|u|^2 \equiv 1$ and thus $u \cdot u_{x_i} \equiv 0$ for every i. (For example (1.2) corresponds to the form (1.5)).

There is a "cousin" of formula (1.4) where the "surface" integral in (1.4) is replaced by a "**volume**" integral in the unit ball B^{n+1} of $\mathbb{R}^{n+1}$. Let $\widetilde{u}$ be any **extension** of u to B^{n+1} with values into $\mathbb{R}^{n+1}$, then

$$\deg u = \frac{1}{|B^{n+1}|} \int_{B^{n+1}} \det J_{\widetilde{u}}. \tag{1.6}$$

(Note that, in (1.6), det refers to the determinant of an $(n+1) \times (n+1)$ matrix). It is easy to pass from (1.5) to (1.6) by writing $\det J_{\widetilde{u}}$ in a divergence form and then integrating by parts (see e.g. H. Brezis and L. Nirenberg [3]). For example, when $n = 1$ write

$$\det J_{\widetilde{u}} = \widetilde{u}_x \times \widetilde{u}_y = \frac{1}{2}\big[(\widetilde{u} \times \widetilde{u}_y)_x + (\widetilde{u}_x \times \widetilde{u})_y\big].$$

Green's formula allows us to replace the integral over B^2 by an integral over S^1 which coincides with (1.2).

The next major step in degree theory came at the beginning of the 20th century, especially through the work of Brouwer. It was then realized that the C^1 assumption about u used either in (1.3) or (1.4) is **not** necessary to define a degree. Continuity suffices. The key observation is the following

Lemma 1. *Assume $u, v \in C^1(S^n, S^n)$ satisfy*

$$\|u - v\|_{L^\infty} < 1,$$

then

$$\deg u = \deg v.$$

Using Lemma 1 we may now define $\deg u$ for a general map $u \in C^0(S^n, S^n)$. Clearly, there is a sequence (u_j) of C^1 maps from S^n to S^n such that $u_j \to u$ uniformly. Hence $\|u_j - u_k\|_{L^\infty} < 1 \quad \forall j, k \geq N$. By definition we let

$$\deg u = \deg u_j \quad \text{for } j \geq N.$$

In this manner every map $u \in C^0(S^n, S^n)$ has a well defined degree which belongs to $\mathbb{Z}$. Moreover the degree is **stable** (i.e., unchanged) under small C^0 perturbation:

Property 1. If $u, v \in C^0(S^n, S^n)$ are such that

$$\|u - v\|_{L^\infty} < 1$$

then

$$\deg u = \deg v.$$

As a consequence the degree is constant under homotopy, i.e., if $H(x,t) \in C^0(S^n \times [0,1], S^n)$ then

$$\deg(H(\cdot\,,0)) = \deg(H(\cdot\,,1)).$$

Let us summarize the main properties of the degree (they were discovered during the first part of this century):

Property 2. If $u \in C^0(S^n, S^n)$ is such that

$$\deg u \neq 0$$

then

$$u \quad \text{maps} \quad S^n \quad \text{onto} \quad S^n.$$

Property 3 (Borsuk). If $u \in C^0(S^n, S^n)$ is odd, then

$$\deg u \qquad \text{is odd.}$$

Property 4 (Hopf). If $u, v \in C^0(S^n, S^n)$ are such that

$$\deg u = \deg v$$

then there is a homotopy $H(x,t)$ (as in Property 1) connecting u and v.

A variant of the above degree theory has also been developed for maps $u : \overline{\Omega} \to \mathbb{R}^n$ where Ω is a bounded domain in $\mathbb{R}^n$. Given a point $y \in \mathbb{R}^n$ such that

$$y \notin u(\partial\Omega) \tag{1.7}$$

one defines $\deg(u, \Omega, y)$ provided $u \in C^0(\overline{\Omega}, \mathbb{R}^n)$.

The strategy is the same as above; namely one starts with a **smooth map** u and a **regular value** y. The $\deg(u,\Omega,y)$ is independent of y provided y stays in a connected component of $\mathbb{R}^n\backslash u(\partial\Omega)$. This allows to define $\deg(u,\Omega,y)$ for any y satisfying (1.7). Next one defines $\deg(u,\Omega,y)$ for any continuous map u by a variant of Lemma 1. The two notions above are closely connected; for example if $\Omega = B^n$ the unit ball of $\mathbb{R}^n$ and $y \notin u(\partial\Omega) = u(S^{n-1})$, then

$$\deg(u,\Omega,y) = \deg\left(\frac{u-y}{|u-y|},\ S^{n-1}\right) \tag{1.8}$$

where the degree on the righthand side of (1.8) refers to the degree of a map from S^{n-1} to S^{n-1}; see e.g. H. Brezis and L. Nirenberg [3].

A very important extension of degree theory to **infinite dimensional spaces** was discovered by J. Leray and J. Schauder [1] in the thirties. It requires continuity and some kind of compactness. It has many applications, in particular, in the study of nonlinear partial differential equations (see e.g. H. Brezis and L. Nirenberg [3]).

In what follows I propose to describe some recent extensions of degree theory to a class of maps in finite dimensional spaces, which are possibly **discontinuous**. This was first done for Sobolev maps, in some limiting cases of the Sobolev imbedding.

2. Degree theory for maps in the Sobolev class $H^1(S^2,S^2)$

In 1982 I was working with J. M. Coron on a problem raised by M. Giaquinta and S. Hildebrandt [1] concerning harmonic maps. Let Ω be unit disc in $\mathbb{R}^2$ and consider maps $u:\Omega\to\mathbb{R}^3$ satisfying the system

$$\begin{cases} -\Delta u = u|\nabla u|^2 & \text{in } \Omega \\ |u| = 1 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Solutions of (2.1) correspond to critical points of the functional $\int_\Omega |\nabla u|^2$ subject to the constraint

$$u \in H^1_g(\Omega,S^2) = \{u\in H^1(\Omega,\mathbb{R}^3), |u|=1 \quad \text{a.e. in } \Omega \text{ and } u=g \text{ on } \partial\Omega\}$$

where H^1 refers to the usual Sobolev space, $g:\partial\Omega\to S^2$ is a given (smooth) map. It is easy to see that (2.1) has at least one solution, namely by considering

$$\operatorname*{Min}_{u\in H^1_g(\Omega,S^2)} \int |\nabla u|^2.$$

Call such a minimizer u_0. If $g=C$ is a constant then $u_0=C$ is the only solution of (2.1). The question of M. Giaquinta and S. Hildebrandt was whether (2.1) has at least 2 solutions whenever $g \not\equiv$ Const. In support of their conjecture they considered the special case

$$g(x,y) = (Rx, Ry, (1-R^2)^{1/2}) \quad \text{with } 0 < R < 1. \tag{2.2}$$

In this case one may write down explicitly two solutions of (2.1), namely

$$\underline{u}(x,y) = \frac{2\lambda}{\lambda^2 + r^2}(x,y,\lambda) + (0,0,-1)$$

and

$$\overline{u}(x,y) = \frac{2\mu}{\mu^2 + r^2}(x,y,-\mu) + (0,0,1)$$

where $r^2 = x^2 + y^2$, $\lambda = \dfrac{1}{R} + (\dfrac{1}{R^2} - 1)^{1/2}$ and $\mu = \dfrac{1}{R} - (\dfrac{1}{R^2} - 1)^{1/2}$.

(Note that $\underline{u}$ and $\overline{u}$ are simply rescaled stereographic projections from the north and south pole respectively.)

We did answer positively the question of M. Giaquinta and S. Hildebrandt:

Theorem (see H. Brezis and J. M. Coron [1] and also J. Jost [1]). *If $g \not\equiv$ Const, then the system (2.1) has at least two solutions.*

The starting point in our proof is the observation that the space $H^1_g(\Omega, S^2)$ is **not** connected. In fact, it has infinitely many connected components and they are classified using degree theory. Unfortunately, we **cannot** use the classical degree theory because maps in H^1 need **not** be continuous in 2 dimensions.

We were first led with J. M. Coron to investigate the class of maps $\varphi \in H^1(S^2, S^2)$ and try to define their degree. The natural strategy is to consider the integral in (1.4) namely

$$\frac{1}{4\pi}\int_{S^2} \det J_\varphi. \tag{2.3}$$

Note that the integral is well defined because $\varphi \in H^1$ implies $\det J_\varphi \in L^1$ (recall that det refers to the determinant of a 2×2 matrix). To have an interesting degree we would like to know that the quantity in (2.3) is an integer (in $\mathbb{Z}$). This is a direct consequence of the following:

Lemma 2 (R. Schoen and K. Uhlenbeck [1]). *Given any $\varphi \in H^1(S^2, S^2)$ there is a sequence (φ_j) in $C^1(S^2, S^2)$ such that $\varphi_j \to \varphi$ in H^1.*

I would like to sketch the proof, because it is quite interesting and plays an important role in Section 4. Consider a smoothing process, by convolution or just by averaging for simplicity, say

$$\overline{\varphi}_\varepsilon(x) = \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} \varphi(y)dy$$

where $B_\varepsilon(x)$ is a geodesic disc on S^2 of radius ε centered at x. Clearly $|\overline{\varphi}_\varepsilon| \le 1$, but $\overline{\varphi}_\varepsilon$, in general, does not take its values in S^2. **If** we happen to know that φ is **also** continuous then $\overline{\varphi}_\varepsilon \to \varphi$ uniformly, as $\varepsilon \to 0$, and then we may consider

$$\varphi_\varepsilon = \overline{\varphi}_\varepsilon / |\overline{\varphi}_\varepsilon| \tag{2.4}$$

which has all the required properties. Unfortunately, if $\varphi \in H^1$ only, then φ need **not** be continuous, and (2.4) does not even make sense because (in principle) $\overline{\varphi}_\varepsilon$ could vanish. The key observation is that $\overline{\varphi}_\varepsilon$ does **not** vanish (for ε small) and in fact we have

$$|\overline{\varphi}_\varepsilon(x)| \longrightarrow 1 \qquad \textbf{uniformly on } S^2. \tag{2.5}$$

The proof of (2.5) relies on Poincaré's inequality

$$\int_{B_\varepsilon(x)} |\varphi(y) - \overline{\varphi}_\varepsilon(x)| dy \le C |B_\varepsilon(x)|^{1/2} \int_{B_\varepsilon(x)} |\nabla \varphi| \tag{2.6}$$

and thus

$$\frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |\varphi(y) - \overline{\varphi}_\varepsilon(x)| dy \le C \left[\int_{B_\varepsilon(x)} |\nabla \varphi|^2 \right]^{1/2}. \tag{2.7}$$

Note that, for every y,

$$|\varphi(y) - \overline{\varphi}_\varepsilon(x)| \ge \operatorname{dist}(\overline{\varphi}_\varepsilon(x), S^2) = 1 - |\overline{\varphi}_\varepsilon(x)|$$

and thus

$$\operatorname{dist}(\overline{\varphi}_\varepsilon(x), S^2) \le C \left[\int_{B_\varepsilon(x)} |\nabla \varphi|^2 \right]^{1/2} \to 0 \quad \text{uniformly as } \varepsilon \to 0.$$

In particular, $|\overline{\varphi}_\varepsilon(x)| \to 1$ **uniformly** and then it is not difficult to prove that $\varphi_\varepsilon = \overline{\varphi}_\varepsilon / |\overline{\varphi}_\varepsilon|$ converges to φ in H^1.

Remark 1. One may ask a more general question: Is $C^1(M, N)$ dense in the Sobolev space $W^{1,p}(M, N)$ where M and N are compact manifolds (N has no boundary but M may have a boundary). If $p \ge \dim M$, the answer is positive (for every N) and the proof is the same as above. If $p < \dim M$, a deep result of F. Bethuel [1] asserts that there is density if and only if $\Pi_{[p]}(N) = 0$.

Let me now explain briefly how the H^1 degree is used to decompose $H^1_g(\Omega, S^2)$ into its components. Fix a "reference" map, for example u_0 (the absolute minimizer). Given any $u \in H^1_g(\Omega, S^2)$, we "glue" the two maps u and u_0 together and define a map $\varphi \in H^1(S^2, S^2)$ by $\varphi = (u, u_0)$ (one copy of Ω is identified with S^2_+, the upper half-hemisphere, and the other copy of Ω is identified with S^2_-). One may then write

$$H^1_g(\Omega, S^2) = \bigcup_{k=-\infty}^{+\infty} \mathcal{E}_k$$

where $\mathcal{E}_k = \{\varphi = (u, u_0);\ \deg \varphi = k\}$. These are the connected components of $H^1_g(\Omega, S^2)$. It is then natural to try to minimize the energy Ω on each $\mathcal{E}_k$:

$$\operatorname*{Inf}_{u \in \mathcal{E}_k} \int_\Omega |\nabla u|^2. \tag{2.8}$$

It is not clear at all that this infimum is achieved (because a minimizing sequence converges weakly in H^1; the classes $\mathcal{E}_k$ are closed for the strong H^1 topology, but **not** for the weak H^1 topology). Using a delicate analysis we were able to prove that the infimum in (2.8) is achieved at least when $k = +1$ or $k = -1$. For more details see H. Brezis and J. M. Coron [1], H. Brezis [1],[2].

Remark 2. In the case of the **special** boundary condition g given by (2.2) one can prove that the infimum in (2.8) is achieved only when $k = 0$ and $k = -1$ (and the corresponding minimizers are given by $\underline{u}$ and $\overline{u}$). It is a beautiful **open problem** to determine whether, for this g, $\underline{u}$ and $\overline{u}$ are the **only** solutions of (2.1).

The method described above for $H^1(S^2, S^2)$ easily extends to $W^{1,n}(S^n, S^n)$ and allows to define the degree of any map $\varphi \in W^{1,n}(S^n, S^n)$. It is given by the formula

$$\deg \varphi = \frac{1}{|S^n|} \int_{S^n} \det J_\varphi.$$

The fact that $\deg \varphi \in \mathbb{Z}$ is proved as above using the property that $C^1(S^n, S^n)$ is dense in $W^{1,n}(S^n, S^n)$. Recall that $W^{1,n}(S^n, S^n)$ is **not** contained in $C^0(S^n, S^n)$. This is again a limiting case for the Sobolev imbedding.

3. Degree theory for maps in the Sobolev class $H^{1/2}(S^1, S^1)$

In 1985 L. Boutet de Monvel and O. Gabber observed that maps in the Sobolev class $H^{1/2}(S^1, S^1)$ have a well-defined degree. (Note that the space $H^{1/2}$ in one dimension is **not** contained in C^0. Once more this is a limiting case for the Sobolev imbedding!) Their motivation came from the Ginzburg-Landau model and their argument is presented as an Appendix in A. Boutet de Monvel-Berthier, V. Georgescu and R. Purice [1] (another application of the $H^{1/2}$ degree, also connected to the Ginzburg-Landau theory is presented in F. Bethuel, H. Brezis and F. Hélein [1]).

The original argument of L. Boutet de Monvel and O. Gabber was the following. If $\varphi \in C^1(S^1, S^1)$, then

$$\deg \varphi = \frac{1}{2\pi i} \int_{S^1} \frac{\dot{\varphi}}{\varphi} = \frac{1}{2\pi i} \int_{S^1} \overline{\varphi} \dot{\varphi}. \tag{3.1}$$

However, this last integral makes sense if one merely assumes that $\varphi \in H^{1/2}$ because $\dot{\varphi} \in H^{-1/2}$ and the integral viewed as a scalar product in the duality between $H^{-1/2}$ and $H^{1/2}$ has a meaning. One may wonder whether the resulting number is an integer.

The answer is again positive because $C^1(S^1, S^1)$ is dense in $H^{1/2}(S^1, S^1)$; the argument is essentially the same as in the proof of Lemma 2, except that here one uses the fact that

$$\int_{S^1}\int_{S^1} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^2} dxdy < \infty \tag{3.2}$$

to deduce that $|\overline{\varphi}_\varepsilon(x)| \to 1$ uniformly (see Section 4). Recall that (3.2) is one of the definitions of the space $H^{1/2}$ (see e.g. R. A. Adams [1]).

There are two other approaches which lead to the fact that maps in $H^{1/2}(S^1, S^1)$ have a degree—each one with a different flavor.

First, via the **Fourier series**. This grew out of a question of I. M. Gelfand. Given a complex-valued function φ on S^1 let (a_j) denote its Fourier coefficients. Then $\varphi \in H^{1/2}(S^1, \mathbb{C})$ if and only if

$$\|\varphi\|^2_{H^{1/2}} = \sum_{j=-\infty}^{+\infty} |j|\,|a_j|^2 < \infty. \tag{3.3}$$

On the other hand, if $\varphi \in C^1(S^1, S^1)$, its degree, given by (3.1) takes the form

$$\deg \varphi = \sum_{j=-\infty}^{+\infty} j|a_j|^2. \tag{3.4}$$

The fact that the sum of the series in (3.4) belongs to $\mathbb{Z}$ for **any** $\varphi \in H^{1/2}(S^1, S^1)$ is established, as above, via a density argument. It would be good to have a direct and simple proof of that property. More precisely, if (a_j) is a sequence of complex numbers satisfying (3.3),

$$\sum_{j=-\infty}^{+\infty} |a_j|^2 = 1 \tag{3.5}$$

and

$$\sum_{j=-\infty}^{+\infty} a_j \overline{a}_{j+k} = 0 \quad \forall k \neq 0, \tag{3.6}$$

then

$$\sum_{j=-\infty}^{+\infty} j|a_j|^2 \in \mathbb{Z}.$$

Note that (3.5) and (3.6) correspond to the fact that the map

$$\varphi(\theta) = \sum_{j=-\infty}^{+\infty} a_j e^{ij\theta}$$

takes its values in S^1. (Indeed $|\varphi| \equiv 1$ is equivalent to the property that

$$\int_{S^1} |\varphi|^2 e^{ik\theta} = 0 \quad \forall k \neq 0 \text{ and } \int_{S^1} |\varphi|^2 = 2\pi,$$

which may be written as (3.5) and (3.6).)

In view of the formula (3.4) it is now quite natural that maps in $H^{1/2}$ have a degree. But it is far from obvious that maps in C^0 have a degree! A general map φ in $C^0(S^1, S^1)$ need not belong to $H^{1/2}(S^1, S^1)$ and thus the series

$$\sum_{j=-\infty}^{+\infty} |j| \, |a_j|^2$$

may be divergent. If this happens

$$\deg \varphi = \sum_{j=+1}^{+\infty} j|a_j|^2 + \sum_{j=-\infty}^{-1} j|a_j|^2 = +\infty - \infty$$

has no clear meaning. Since $\deg \varphi$ makes sense for $\varphi \in C^0(S^1, S^1)$, there must be some kind of cancellation of the two infinite quantities, leaving us with a "principal value". In fact it would be very interesting to understand what summation process (if any) may be used to compute

$$\sum_{j=-\infty}^{+\infty} j|a_j|^2$$

for a general $\varphi \in C^0(S^1, S^1)$. In particular can one use any of the standard methods, for example,

$$\lim_{n\to\infty} \sum_{j=-n}^{+n} j|a_j|^2$$

or

$$\lim_{r\uparrow 1} \sum_{j=-\infty}^{+\infty} j|a_j|^2 r^{|j|} \quad ?$$

Here is still another approach which shows that maps in $H^{1/2}$ have a degree. It relies on the characterization of $H^{1/2}$ as **trace** space for H^1. More precisely, given some $\varphi \in H^{1/2}(S^1, S^1)$ we consider it as a map in $H^{1/2}(S^1, \mathbb{C})$ and then we may extend it to the unit disc B^2 in $\mathbb{R}^2$ by a map $u \in H^1(B^2, \mathbb{C})$. We then recall the formula (1.6) which here takes the form

$$\deg \varphi = \frac{1}{\pi} \int_{B^2} \det J_u. \tag{3.7}$$

Formula (3.7) holds for smooth maps. But, again, we observe that the right hand side in (3.7) makes sense provided $u \in H^1$ and to have such u it suffices to assume that $\varphi \in H^{1/2}$.

Finally, a density argument, as above, shows that the integral on the right hand side of (3.7) belongs to $\mathbb{Z}$ for any $\varphi \in H^{1/2}(S^1, S^1)$.

This last approach also works in higher dimensions. Suppose $\varphi \in W^{s,p}(S^n, S^n)$ for some fractional Sobolev space. We may extend φ inside the unit ball B^{n+1} by some $u \in W^{s+1/p,p}(B^{n+1}, \mathbb{R}^{n+1})$ and then use the formula (1.6)

$$\deg \varphi = \frac{1}{|B^{n+1}|} \int_{B^{n+1}} \det J_u. \tag{3.8}$$

The integral on the right hand side of (3.8) makes sense when $u \in W^{1,n+1}$ and so we may take $p = n+1 \quad s = 1 - \dfrac{1}{p} = \dfrac{n}{n+1}$. We now reach the conclusion that maps $\varphi \in W^{\frac{n}{n+1},n+1}(S^n, S^n)$ have a degree (in $\mathbb{Z}$). This class falls again in the category of the limiting Sobolev exponent and it is slightly larger than $W^{1,n}(S^n, S^n)$ (via the fractional Sobolev imbedding). For example when $n = 2$, there is a well-defined degree for maps $\varphi \in W^{\frac{2}{3},3}(S^2, S^2)$; this class is a little bigger than the class $H^1(S^2, S^2)$ considered in Section 2.

At this stage the situation was becoming rather confusing and we decided, with Louis Nirenberg, to investigate a suggestion of L. Boutet de Monvel and O. Gabber, namely, to define a degree for VMO maps. Such a degree is **not defined via an integral formula but rather via approximation** (in the same manner as one extends degree theory from C^1 to C^0). As we shall see in the next Section such a class includes C^0 maps as well as **all Sobolev maps** in the limiting case of the Sobolev exponent.

4. Degree theory for maps in $\mathrm{VMO}(S^n, S^n)$

Here, and throughout the rest of this paper we present our recent work with Louis Nirenberg; see H. Brezis and L. Nirenberg [1],[2].

Let us first recall the definition of BMO; this is a celebrated space introduced by F. John and L. Nirenberg [1].

An integrable function $f : S^n \to \mathbb{R}$ belongs to BMO if

$$\|f\|_{\mathrm{BMO}} = \operatorname*{Sup}_{B \subset S^n} ⨍_B |f - ⨍_B f| < \infty \tag{4.1}$$

where the Sup is taken over all geodesic balls on B on S^n, with radius $r \le 1$. Formula (4.1) defines a semi-norm or a norm on BMO modulo constants. BMO is complete under this norm. A **very useful** equivalent norm is given by

$$\|f\|_* = \operatorname*{Sup}_{B \subset S^n} ⨍_B ⨍_B |f(y) - f(z)| dy dz. \tag{4.2}$$

In fact its easy to check that

$$\|f\|_{\mathrm{BMO}} \le \|f\|_* \le 2\|f\|_{\mathrm{BMO}}. \tag{4.3}$$

Sometimes it is convenient to take the Sup in (4.1) (or (4.2)) over all balls with radius $r \leq r_0$; this yields a norm which is equivalent to the original BMO norm.

Clearly,

$$L^\infty \subset \mathrm{BMO}$$

and

$$\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{L^\infty}.$$

But the converse is not true: the well-known example is the log function. More precisely, fix a point x_0 on S^n then

$$f(x) = \zeta(x) \log|x - x_0| \in \mathrm{BMO} \tag{4.4}$$

where ζ is a smooth cut-off function supported near x_0. An important inequality, due to F. John and L. Nirenberg [1] asserts that

$$\mathrm{BMO} \subset L^p \quad \forall 1 \leq p < \infty.$$

More precisely,

$$\|f - \fint_{S^n} f\|_{L^p} \leq C\|f\|_{\mathrm{BMO}} \tag{4.5}$$

where C depends only on p and n. In fact, a sharper form asserts that $e^{c|f|} \in L^1$ whenever $\|f\|_{\mathrm{BMO}} \leq 1$, where c depends only on n.

It turns out that $C^0(S^n)$ is **not** dense in $\mathrm{BMO}(S^n)$. Since we plan to define the degree via approximation by smooth maps, it is essential to deal with maps which can be regularized. Hence we will work with

$$\mathrm{VMO}(S^n) = \text{the closure of } C^0(S^n) \text{ in } \mathrm{BMO}(S^n)$$

i.e., a function $f \in \mathrm{BMO}(S^n)$ belongs to $\mathrm{VMO}(S^n)$ if there is a sequence (f_j) in $C^0(S^n)$ such that $\|f_j - f\|_{\mathrm{BMO}} \to 0$; without loss of generality, using (4.5), we may also assume that $f_j \to f$ in $L^p \quad \forall p < \infty$ and $f_j \to f$ a.e.

The space VMO (for functions on $\mathbb{R}^n$) has been introduced by D. Sarason [1] who also established a useful characterization.

Lemma 3. *A function of $f \in BMO(S^n)$ belongs to $VMO(S^n)$ if and only if*

$$\lim_{|B| \to 0} \fint_B |f - \fint_B f| = 0 \tag{4.6}$$

or equivalently

$$\lim_{|B| \to 0} \fint_B \fint_B |f(y) - f(z)| dy dz = 0. \tag{4.7}$$

The fact that $f \in \mathrm{VMO}(S^n) \Rightarrow$ (4.6) is easy. Indeed we may write, for any function g,

$$\fint_B |f - \fint_B f| \le \|f-g\|_{\mathrm{BMO}} + \fint_B |g - \fint_B g|.$$

Given ε we may choose a continuous function g such that $\|f-g\|_{\mathrm{BMO}} < \varepsilon$. Then we may find a $\delta > 0$ such that

$$\fint_B |g - \fint_B g| < \varepsilon \quad \forall B \text{ with } |B| < \delta.$$

The converse (i.e., (4.6) $\Rightarrow f \in \mathrm{VMO}$) is more delicate (see D. Sarason [1] or H. Brezis and L. Nirenberg [1]).

Examples:

1) The function f in (4.4) does **not** belong to $\mathrm{VMO}(S^n)$. However $|f|^\alpha \in \mathrm{VMO}(S^n)$ for any $0 < \alpha < 1$. Hence, there are unbounded functions in VMO. Similarly, the function $\zeta(x)\log\big|\log|x-x_0|\big|$ belongs to $\mathrm{VMO}(S^n)$.

2) The Sobolev space $W^{1,n}(S^n) \subset \mathrm{VMO}(S^n)$. This follows easily from the Poincaré inequality

$$\int_B |f - \fint_B f| \le C|B|^{1/n}\int_B |\nabla f| \tag{4.8}$$

where C depends only on n. From (4.8) we deduce that

$$\fint_B |f - \fint_B f| \le C\left(\int_B |\nabla f|^n\right)^{1/n} \tag{4.9}$$

and we may then apply Lemma 3 to infer that $f \in \mathrm{VMO}$.

3) More generally, functions in the fractional Sobolev space $W^{s,p}(S^n)$ with $0 < s < n$ and $sp = n$ belong to $\mathrm{VMO}(S^n)$ (note that the condition $sp = n$ is limiting for the Sobolev imbedding). To prove this, it suffices to consider the case $0 < s < 1$ (when $s \ge 1$, $W^{s,p} \subset W^{1,n}$ and we are reduced to the previous example).

Recall (see e.g. R. Adams [1]) that a function f belongs to $W^{s,p}$ provided

$$\|f\|^p_{W^{s,p}} = \int_{S^n}\int_{S^n} \frac{|f(y)-f(z)|^p}{|y-z|^{sp+n}}\,dydz < \infty.$$

We have, by Hölder,

$$\begin{aligned}
\fint_B\fint_B |f(y)-f(z)| &\le \frac{1}{|B|^2}\left(\int_B\int_B |f(y)-f(z)|^p\right)^{1/p} |B|^{2/p'} \\
&\le \frac{1}{|B|^{2/p}}\left(\int_B\int_B \frac{|f(y)-f(z)|^p}{|y-z|^{2n}}\right)^{1/p} (2r)^{2n/p} \\
&\le C\left(\int_B\int_B \frac{|f(y)-f(z)|^p}{|y-z|^{sp+n}}\right)^{1/p},
\end{aligned}$$

(since $|y - z| \le 2r$, for $y, z \in B =$ a ball of radius r). Applying Lemma 3 once more we conclude that $f \in \mathrm{VMO}(S^n)$.

Finally, we say that a vector-valued function $u : S^n \to \mathbb{R}^k$ belongs to $\mathrm{VMO}(S^n, \mathbb{R}^k)$ if all its component are in VMO. If Σ is a closed subset of $\mathbb{R}^k$ we say that $u \in \mathrm{VMO}(S^n, \Sigma)$ provided $u \in \mathrm{VMO}(S^n, \mathbb{R}^k)$ and $u(x) \in \Sigma$ a.e. on S^n.

Our main result is the following

Theorem 1. *Any map $u \in VMO(S^n, S^n)$ has a well-defined degree in $\mathbb{Z}$. If $u \in C^0(S^n, S^n)$ or if u belongs to one of the Sobolev classes described in Sections 2 and 3, the new degree coincides with the degree previously defined.*

The properties of this new degree are very similar to the properties of the standard degree:

Property 1. The degree is stable under small BMO perturbation, i.e., if $u \in \mathrm{VMO}(S^n, S^n)$ and (u_j) is a sequence in $\mathrm{VMO}(S^n, S^n)$ such that

$$\|u_j - u\|_{\mathrm{BMO}} \to 0 \tag{4.10}$$

then

$$\deg u_j = \deg u \quad \text{for } j \text{ sufficiently large.}$$

As a consequence, the degree is constant under homotopy within VMO, i.e., if $H(x,t) \in C([0,1],\, \mathrm{VMO}(S^n, S^n))$ then

$$\deg(H(\cdot,\, 0)) = \deg(H(\cdot,\, 1)).$$

Property 2. If $u \in \mathrm{VMO}(S^n, S^n)$ is such that

$$\deg u \ne 0$$

then

$$\mathrm{ess} R(u) = S^n. \tag{4.11}$$

Note that since u is only defined a.e. it does not make sense to talk about the range of u. Instead one considers the essential range which is the smallest closed set $\Sigma \subset S^n$ such that $u(x) \in \Sigma$ a.e. on S^n. Property (4.11) says that there cannot be a "hole" in the range of u, i.e., there is no open ball B in S^n such that $u(x) \in S^n \backslash B$ a.e.

Property 3 (Borsuk). If $u \in \mathrm{VMO}(S^n, S^n)$ is odd, then

$$\deg u \qquad \text{is odd}$$

Property 4 (Hopf). If $u, v \in \mathrm{VMO}(S^n, S^n)$ are such that

$$\deg u = \deg v$$

then there is a homotopy H within VMO (as in Property 1) connecting u and v.

Our definition of degree for VMO maps is extremely simple. Given u in VMO(S^n, S^n) set

$$\overline{u}_\varepsilon(x) = \fint_{B_\varepsilon(x)} u, \quad 0 < \varepsilon < 1,$$

where $B_\varepsilon(x)$ is the geodesic ball on S^n, with center x and radius ε. Note that $\overline{u}_\varepsilon \in C^0(S^n, B^{n+1})$ and also that

$$1 - |\overline{u}_\varepsilon(x)| = \operatorname{dist}(\overline{u}_\varepsilon(x), S^n) \leq \fint_{B_\varepsilon(x)} |u(y) - \overline{u}_\varepsilon(x)| dy. \tag{4.12}$$

Since $u \in$ VMO the right hand side in (4.12) tends to 0 **uniformly** in x as $\varepsilon \to 0$. Therefore, as $\varepsilon \to 0$,

$$|\overline{u}_\varepsilon(x)| \to 1 \quad \text{uniformly in } x$$

and thus we may introduce, for $\varepsilon \leq \varepsilon_0$, the map

$$u_\varepsilon(x) = \frac{\overline{u}_\varepsilon(x)}{|\overline{u}_\varepsilon(x)|}.$$

Since $u_\varepsilon \in C^0(S^n, S^n)$ we may consider

$$\deg u_\varepsilon.$$

This number is independent of ε for $\varepsilon \leq \varepsilon_0$ since we may use ε as a homotopy parameter to connect u_ε and $u_{\varepsilon'}$. By **definition** we let

$$\deg u = \deg u_\varepsilon \qquad \text{for } \varepsilon \leq \varepsilon_0.$$

For the proofs of all the above results we refer to H. Brezis and L. Nirenberg [1]. They are not very difficult, but, still, the VMO degree theory is more subtle than the usual C^0 theory. Here are some delicate points:

Remark 3. In the C^0 case, Property 1 asserts that if $u, v \in C^0(S^n, S^n)$ and

$$\|u - v\|_{L^\infty} < 1$$

then $\deg u = \deg v$. In the VMO case such a statement does not hold. More precisely, there exists **no uniform** $\delta > 0$ such that if $u, v \in \text{VMO}(S^n, S^n)$ and

$$\|u - v\|_{\text{BMO}} < \delta$$

then $\deg u = \deg v$. In fact we can construct (see Lemma 6 in H. Brezis and L. Nirenberg [1]) sequences (u_j) and (v_j) in $C^1(S^1, S^1)$ such that

$$\|u_j - v_j\|_{\text{BMO}} \to 0 \quad \text{as } j \to \infty$$

(even $\|u_j - v_j\|_{H^{1/2}} \to 0$ as $j \to \infty$) and

$$|\deg u_j - \deg v_j| \geq 1 \quad \forall j.$$

The exact formulation of Property 1 in the VMO case is:

Given any $u \in \mathrm{VMO}(S^n, S^n)$ there is some $\delta > 0$ **depending on** u such that if $v \in \mathrm{VMO}(S^n, S^n)$ and $\|v - u\|_{\mathrm{BMO}} < \delta$, then $\deg v = \deg u$.

Remark 4. As we have already pointed out the degree counts "how many times" φ covers its range (including algebraic multiplicity). If $\varphi \in C^0(S^1, S^1)$ then φ covers globally S^1 at most a **finite** number of times; this follows from the uniform continuity: there is a $\delta > 0$ such that $|x - y| < \delta \Rightarrow |\varphi(x) - \varphi(y)| < 1$ and so the number of times that φ may cover S^1 is at most of the order of $1/\delta$. When $\varphi \in \mathrm{VMO}(S^1, S^1)$ it may cover S^1 **infinitely** many times. Here is such an example. Consider a real-valued function $f(\theta)$ on $[0, 2\pi]$ which is smooth on $[0, 2\pi]$ except at $\theta = \pi$, with $\lim\limits_{\theta \to \pi} f(\theta) = +\infty$ and $f(0) = 0$, $f(2\pi) = 2\pi$. We may construct such an f which belongs to VMO, choosing, for example,

$$f(\theta) = \theta + \zeta(\theta)\big|\log|\theta - \pi|\big|$$

where ζ is a smooth cut-off function supported near $\theta = \pi$. Set

$$\varphi(\theta) = e^{if(\theta)}.$$

Then $\varphi \in \mathrm{VMO}(S^1, S^1)$ (it is easy to see, using (4.7) that the composition $L \circ f$ of a Lipschitz map L with a VMO map f lies in VMO). As $\theta \to \pi$, $\theta < \pi$, $\varphi(\theta)$ turns around S^1 **infinitely** many times in the **positive** direction. As soon as θ crosses π, $\varphi(\theta)$ turns around S^1 **infinitely** many times in the **negative** direction! Again, the degree seems to be a difference of two infinite quantities and we encounter the same kind of **cancellation phenomenon** as in Section 3 with Fourier series. Here also the degree seems to be some sort of "principal value". It would be very interesting to clarify this point.

5. Further properties of VMO maps in connection with Topology

We present here various remarks and additional results.

A. Homotopy classes and VMO

At this moment it is not clear whether VMO is the "largest" natural class on which a degree can be defined. What is certain is that bigger classes such as $L^p(S^n, S^n)$, $1 \leq p \leq \infty$, or $\mathrm{BMO}(S^n, S^n)$ do **not** have a degree. The reason is that the spaces $L^p(S^n, S^n)$, $1 \leq p \leq \infty$, and $\mathrm{BMO}(S^n, S^n)$ are **arcwise connected**; this is true even if S^n is replaced in the domain space by a manifold X and in target space by a manifold Y; see Section I.5 in H. Brezis and L. Nirenberg [1].

When dealing with continuous maps, topologists consider the **homotopy classes** say of $C^0(S^n, S^k)$. These are the connected components $\mathcal{C}_i$ of $C^0(S^n, S^k)$. One may ask what are the connected components of $\mathrm{VMO}(S^n, S^k)$? It turns out that they are of the same type as in the C^0 case. More precisely, they are the closures in BMO of the above $\mathcal{C}_i$; see Section I.5 and Lemmas A.18 - A.24 in H. Brezis and L. Nirenberg [1].

B. Lifting and VMO

Another topic of interest in Topology concerns **lifting**. For the sake of simplicity let us consider maps from S^1 into S^1. The question is whether a map $\varphi : S^1 \to S^1$ can be written as

$$\varphi = e^{if}$$

for some $f : S^1 \to \mathbb{R}$. When $\varphi \in C^0(S^1, S^1)$, a classical result asserts that there is such $f \in C^0(S^1, \mathbb{R})$ if and only if $\deg \varphi = 0$. Here is an extension to VMO:

Theorem 2. *Assume $\varphi \in VMO(S^1, S^1)$, then φ may be written as*

$$\varphi = e^{if} \quad \text{for some } f \in \mathrm{VMO}(S^1, \mathbb{R})$$

if and only if $\deg \varphi = 0$.

The proof, which is much more elaborate than in the continuous case, is presented in Section I.6 of H. Brezis and L. Nirenberg [1]. It is related to earlier work of R. Coifman and Y. Meyer [1]. We have more general results in the framework of 3 spaces X, Y, Z and F is a continuous covering map of Z to Y. Under the natural topological assumptions we prove that a map $\varphi \in \mathrm{VMO}(X, Y)$ can be lifted to Z, i.e.,

$$\varphi = F \circ f$$

for some $f \in \mathrm{VMO}(X, Z)$.

Remark 5. The question of lifting for **Sobolev maps** is more delicate than it seems and has been settled only recently. Here is the problem in a simple situation. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let $u \in W^{1,p}(\Omega, S^1)$ with $1 \le p < \infty$. Can one write

$$u = e^{if} \quad \text{for some } f \in W^{1,p}(\Omega, \mathbb{R})\ ?$$

The answer is **positive** if $p \ge 2$ and **negative** if $1 \le p < 2$. This is a result of F. Bethuel and X. Zheng [1]; a simpler proof, due to P. Mironescu is given in H. Brezis [3].

C. Toeplitz operators and VMO

Let me recall briefly the notion of Toeplitz operators. Consider the Hilbert space $L^2(S^1, \mathbb{C})$ and the closed subspace

$$\mathcal{H}^2 = \{f \in L^2(S^1, \mathbb{C}); \int_{S^1} f(\theta) e^{in\theta} d\theta = 0 \quad \forall n = 1, 2, \dots\}.$$

Let P be the orthogonal projection from L^2 onto $\mathcal{H}^2$. Given a function $\varphi \in L^\infty(S^1, \mathbb{C})$ consider the multiplication operator, defined on L^2 by

$$M_\varphi f = \varphi f.$$

By definition the Toeplitz operator T_φ, with symbol φ, is

$$T = PM_\varphi$$

considered as a bounded operator from $\mathcal{H}^2$ into itself.

A very classical result in the theory of Toeplitz operators (see e.g. R. Douglas [1]) asserts that if $\varphi \in C^0(S^1, \mathbb{C})$ and $\varphi \neq 0$ on S^1, then T_φ is Fredholm and

$$\operatorname{index}(T_\varphi) = -\deg\left(\frac{\varphi}{|\varphi|}\right).$$

Here is an extension to VMO.

Theorem 3. *Assume $\varphi \in VMO(S^1, \mathbb{C}) \cap L^\infty(S^1, \mathbb{C})$ satisfies*

$$|\varphi| \geq \alpha > 0 \quad \textit{a.e. on } S^1.$$

Then T_φ is Fredholm and

$$\operatorname{index}(T_\varphi) = -\deg\left(\frac{\varphi}{|\varphi|}\right).$$

Of course, the degree is to be understood in the sense of degree for VMO maps. The proof of Theorem 3, which is joint with P. Mironescu, is presented in Appendix 2 of H. Brezis and L. Nirenberg [2]. It uses a deep result: the Fefferman-Stein duality of $\mathcal{H}^1$ and BMO (see C. Fefferman and E. Stein [1]).

6. Degree theory for VMO maps on domains

As we have mentioned in the Introduction there is a classical notion of degree for continuous maps on domains of $\mathbb{R}^n$. Such a concept can be extended to VMO maps. We have first to define precisely what is meant by BMO and VMO on domains.

Let $\Omega \subset \mathbb{R}^n$ be an open (connected) bounded set. A function $f \in L^1_{\text{loc}}(\Omega)$ belongs to BMO(Ω) provided

$$\|f\|_{\text{BMO}} = \operatorname*{Sup}_{\overline{B} \subset \Omega} \fint_B \left| f - \fint_B f \right| < \infty, \tag{6.1}$$

where the Sup in (6.1) is taken over all balls B whose closure is contained in Ω. This notion depends (in principle) on the choice of norm in $\mathbb{R}^n$—a different norm gives rise to a different geometry of balls. A deep result of P. Jones asserts that two different norms on $\mathbb{R}^n$ yield two **equivalent** BMO norms. This is proved in H. Brezis and L. Nirenberg [2], using the methods of P. Jones [1]. The main idea is to show that if in (6.1) we consider balls B “well-inside” (i.e., $B = B_r(x)$ with $r \leq \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$) we obtain a smaller norm, which is equivalent to the BMO norm.

Now VMO(Ω) is the closure of $C^0(\overline{\Omega})$ for the BMO norm. The analogue of Lemma 3 holds provided $\overline{B} \subset \Omega$. As above, the Sobolev space $W^{s,p}(\Omega) \subset \mathrm{VMO}(\Omega)$ when $0 < s < n$ and $sp = n$.

We wish to define

$$\deg(u, \Omega, y)$$

for a map $u \in \mathrm{VMO}(\Omega, \mathbb{R}^{n+1})$ such that $y \notin u(\partial\Omega)$. This last condition does not make sense since VMO maps do **not**, in general, have a trace on the boundary. We make instead the following assumption:

$$\text{(6.2)} \qquad \begin{cases} \text{there exist constants } \delta > 0 \text{ and } r_0 > 0 \text{ such that} \\ \displaystyle{-\!\!\!\!\!\!\int_{B_r(x)}} |u(z) - y| \geq \delta \quad \forall x \in \Omega \text{ with } r = \dfrac{1}{2}\mathrm{dist}(x, \partial\Omega) \leq r_0. \end{cases}$$

Note that, if $u \in C^0(\overline{\Omega}, \mathbb{R}^{n+1})$, assumption (6.2) is equivalent to the condition that $y \notin u(\partial\Omega)$. Of course, we could also have made the stronger assumption that

$$\text{(6.3)} \qquad |u - y| \geq \delta \text{ a.e. on some neighborhood of } \partial\Omega.$$

However, such a condition would be too restrictive in our framework. For example, if $u \in W^{1,n}$, let $\varphi = u_{|\partial\Omega}$ and assume that

$$|\varphi - y| \geq \gamma > 0 \qquad \text{a.e. on } \partial\Omega,$$

then (6.2) holds, but (6.3) does not hold.

Our main result is

Theorem 4. *Assume $u \in VMO(\Omega, \mathbb{R}^{n+1})$ satisfies (6.2) then*

$$\deg(u, \Omega, y) \quad \textit{is well-defined (in } \mathbb{Z}).$$

This new degree has all the properties that one expects for a degree. Here are some:

Property 1. Assume $u \in \mathrm{VMO}(\Omega, \mathbb{R}^{n+1})$ and $(u_j) \subset \mathrm{VMO}(\Omega, \mathbb{R}^{n+1})$ are such that $u_j \to u$ in BMO and in L^1_{loc}, and (6.2) holds for u and for u_j uniformly in j (i.e., with the same δ and r_0). Then

$$\deg(u_j, \Omega, y) = \deg(u, \Omega, y) \quad \text{for } j \text{ large}.$$

Property 2. Suppose $u \in \mathrm{VMO}(\Omega, \mathbb{R}^{n+1})$ satisfies (6.2) and $\deg(u, \Omega, y) \neq 0$. Then $y \in \mathrm{ess}R(u)$ and more precisely

$$B_\delta(y) \subset \mathrm{ess}R(u).$$

In the classical theory, with $u \in C^0(\overline{\Omega}, \mathbb{R}^{n+1})$, assuming also Ω is smooth, we have

$$\text{(6.4)} \qquad \deg(u, \Omega, y) = \deg\left(\frac{u - y}{|u - y|},\ \partial\Omega,\ S^{n+1}\right).$$

Formula (6.4) does not make sense for maps u in VMO—again because they do not have a trace on $\partial\Omega$.

To get around this difficulty we were led to introduce a new class of functions f in VMO(Ω) which **does** have a trace φ on the boundary, with φ in VMO($\partial\Omega$). Our definition is the following: Let φ be a function in VMO($\partial\Omega$). Let $\widetilde{\varphi}$ be the extension of φ in a neighborhood of $\partial\Omega$, constant on normals.

Definition. A function $f \in \mathrm{VMO}(\Omega)$ belongs to $\mathrm{VMO}_\varphi(\Omega)$ if

$$\lim_{\substack{r\to 0\\ r=\frac{1}{2}\operatorname{dist}(x,\partial\Omega)}} \fint_{B_r(x)} |f-\widetilde{\varphi}| = 0.$$

There are **many natural examples** of functions in VMO_φ. Any function f in $W^{1,n}$ belongs to VMO_φ where φ = trace of f (in the sense of Sobolev spaces). The harmonic extension in Ω of a function $\varphi \in \mathrm{VMO}(\partial\Omega)$ belongs to $\mathrm{VMO}_\varphi(\Omega)$. Etc. ... . For such maps we have

Theorem 5. *Assume $u \in VMO_\varphi(\Omega, \mathbb{R}^{n+1})$ and*

$$|\varphi - y| \geq \delta > 0 \quad \textit{i.e., on } \partial\Omega.$$

Then (6.2) holds and

$$\deg(u,\Omega,y) = \det\left(\frac{\varphi-y}{|\varphi-y|}, \partial\Omega, S^{n-1}\right).$$

The proof is given in Section II.4 of H. Brezis and L. Nirenberg [2].

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