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On P_4-transversals of Chordal Graphs

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Abstract

A P_4-transversal of a graph G is a set of vertices T which meets every P_4 of G. A P_4-transversal T is called stable if there are no edges in the subgraph of G induced by T. It has been previously shown by Hoàng and Le that it is NP-complete to decide whether a comparability (and hence perfect) graph G has a stable P_4-transversal. In the following we show that the problem is NP-complete for chordal graphs. We apply this result to show that two related problems of deciding whether a chordal graph has a P_3-free P_4-transversal, and deciding whether a chordal graph has a P_4-free P_4-transversal (also known as a *two-sided* P_4-transversal) are both NP-complete. Additionally, we strengthen the main results to strongly chordal graphs.

Key words: chordal graphs, P_4-transversal

1 Introduction and results

A graph is *perfect* if for every induced subgraph H of G, the chromatic number $\chi(H)$ is equal to the clique number $\omega(H)$. The Strong Perfect Graph Theorem states that a graph is perfect if and only if it has no induced odd cycle or its complement [2]. This result had been conjectured by Berge [1]. In the long history of this conjecture, the study of the structure of P_4's in a graph has been found to play an important role. In [8], the authors define the notion of a *P_4-transversal* to be a subset of vertices of a graph meeting every P_4. They show that if a graph has a P_4-transversal with certain properties it is guaranteed to be perfect. They also investigate the complexity of finding a P_4-transversal with various properties. In particular they investigate stable P_4-transversals, i.e., P_4-transversals which form a *stable set* – a set of vertices inducing a subgraph with no edges. They show that for comparability graphs

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(and therefore also for perfect graphs) it is NP-complete to decide whether a graph has a stable P_4-transversal. In [9], the authors consider a related problem of P_4-colourings. They show that finding a P_4-free P_4-transversal (a "P_4-free 2-colouring" in their terminology) is NP-complete for comparability graphs, P_5-free graphs and (C_4, C_5)-free graphs.

Here we first show that the problem of finding a stable P_4-transversal remains NP-complete when restricted to chordal graphs:

Theorem 1.1 *It is NP-complete to decide whether a given chordal graph has a stable P_4-transversal.*

We apply this result to derive the following consequences:

Theorem 1.2 *It is NP-complete to decide whether a given chordal graph has a P_3-free P_4-transversal.*

Theorem 1.3 *It is NP-complete to decide whether a given chordal graph has a P_4-free P_4-transversal.*

Note that Theorem 1.3 also improves on the results from [9] mentioned above. We contrast Theorem 1.1 with the result of [4], which can be reformulated as follows:

Theorem 1.4 [4] *For chordal graphs, a stable P_3-transversal can be found in polynomial time.*

We note that the NP-completeness of these kinds of partition problems for general graphs has been proved in [6].

In the last section of the paper we discuss some extensions of these results.

2 Preliminaries

A graph G is called *H-free* if G contains no induced subgraph isomorphic to a graph H. In particular, a P_4-free graph is called a *cograph.* (Recall that P_4 is the path with four vertices and three edges.) It has been shown [3] that any P_4-free graph can be constructed from a single vertex using the operations of disjoint union and join. (The *join* of two graphs is constructed by taking their disjoint union and adding all possible edges between the two graphs.) The construction of a cograph G can be therefore represented as a rooted tree T in which the leaves are the vertices of the graph G, and the internal nodes are labeled either 0 or 1, denoting the operations of disjoint union and join respectively. T shall be referred to as a *tree representation* of G. It could be easily seen that two vertices of G are adjacent if and only if their least common ancestor in T is labeled 1. Note that T is not necessarily unique. We call T a *cotree* if the labels of the internal nodes of T strictly alternate on any path in T. It is known[3] that every cograph has a unique cotree (up to isomorphism). If a tree representation T of a cograph G is not a cotree, one can easily transform it into an equivalent cotree by identifying consecutive vertices of T having the same label. Hence for simplicity, we shall refer to any tree representation of a cograph as a cotree.

A graph is *chordal* if it does not contain an induced cycle of length 4 or more. It is known [7] that a graph is *chordal* if and only if there exists a linear ordering $\prec$ of its vertices such that if v, w are two neighbours of u with $u \prec v$, $u \prec w$, then v and w are adjacent. Such an ordering is called a *perfect elimination ordering.*

A *literal* is a variable v_i or its negation $\neg v_i$ (often written as $\bar{v}_i$). A *clause* is a disjunction of literals. A propositional formula is in *conjunctive normal form* if it is written as a conjunction of clauses. The set of all variables of the formula φ is denoted by $var(\varphi)$. The truth assignment τ for the set of variables $var(\varphi)$ is a mapping $\tau : var(\varphi) \to \{true, false\}$. The *3-satisfiability* problem $3SAT$ is the problem of finding a satisfying truth assignment for all variables of a given formula in conjunctive normal form in which every clause has exactly 3 literals. It is known to be NP-complete.

It should be noted that all the problems mentioned in section 1 are clearly in NP; this follows from the fact that testing whether a graph contains a P_4, a P_3, or is a stable set, can be done in polynomial time.

3 Stable P_4-transversals

To prove Theorem 1.1, we describe a polynomial time reduction from the problem $3SAT$. Let φ be a formula in conjunctive normal form with exactly three literals in any clause, i.e. $\varphi = \bigwedge_{j=1}^{m} C_j$ where $C_j = l_1^j \vee l_2^j \vee l_3^j$ where l_1^j, l_2^j, l_3^j are literals. Let $var(\varphi) = \{v_1, v_2, \ldots, v_n\}$ be all variables appearing in φ. Let $J_i^{(+)}$ be the indices of clauses which contain the literal v_i and $J_i^{(-)}$ be the indices of clauses which contain the literal $\neg v_i$.

Let $C(C_j)$ be the graph shown in Figure 1, and let $G(\varphi)$ for the formula φ be the graph G_n inductively defined as follows:

(1) Let G_0 be the disjoint union of the graphs $\{C(C_j)\}_{j=1}^{m}$ (see Figure 1).

(2) Let G_i (see Figure 1) be the graph created from G_{i-1} as follows. Add two adjacent vertices v_i and $\overline{v}_i$ and make them completely adjacent to all

vertices of G_{i-1}. For every $j \in J_i^{(+)}$ add a vertex v_i^j adjacent to v_i and adjacent to the vertex l_k^j of $C(C_j)$ if v_i is the k-th literal of the clause C_j. Similarly, for every $j \in J_i^{(-)}$ add a vertex $\overline{v}_i^j$ adjacent to $\overline{v}_i$ and adjacent to the vertex l_k^j in $C(C_j)$ if $\neg v_i$ is the k-th literal of the clause C_j. (Note that we can assume that a literal occurs in a clause only once.)

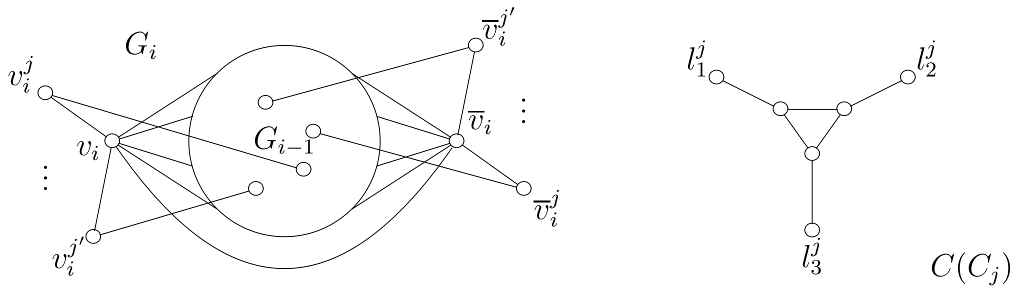

Fig. 1. The graphs G_i and $C(C_j)$. Note that the circle in the center is the graph G_{i-1} completely adjacent to v_i and $\overline{v}_i$; and each of the vertices v_i^j and $\overline{v}_i^j$ is adjacent to a single vertex in G_{i-1}. The graph $C(C_j)$ corresponds to the clause $C_j = l_1^j \vee l_2^j \vee l_3^j$.

We can also describe the graph $G(\varphi)$ in the following (non-inductive) way:

(1) $G(\varphi)$ contains the vertices $v_i, \overline{v}_i$ for every i, the vertices v_i^j (resp. $\overline{v}_i^j$) for every occurrence of the literal v_i (resp. $\neg v_i$) in the clause C_j, and the vertices of $C(C_j)$ for every clause C_j containing among others the vertices l_1^j, l_2^j and l_3^j.
(2) The vertex v_i (resp. $\overline{v}_i$) is adjacent to all vertices of $C(C_j)$ for every j, to the vertices $v_{i'}, \overline{v}_{i'}$ for all i', to all vertices v_i^j (resp. $\overline{v}_i^j$) that may exist, and to all vertices $v_{i'}^j, \overline{v}_{i'}^j$ that may exist, for each i' with $i' < i$.
(3) The vertex v_i^j (resp. $\overline{v}_i^j$) is adjacent to the vertex v_i (resp. $\overline{v}_i$), to the vertex l_k^j such that v_i (resp. $\neg v_i$) is the k-th literal of the clause C_j, and to all vertices $v_{i'}, \overline{v}_{i'}$ for all i' with $i' > i$.
(4) The vertex l_k^j is adjacent to its only neighbour in $C(C_j)$, to the vertex v_i^j (resp. $\overline{v}_i^j$) such that v_i (resp. $\neg v_i$) is the k-th literal of the clause C_j, and to the vertices $v_{i'}, \overline{v}_{i'}$ for all i'.
(5) The remaining vertices of $C(C_j)$ are only adjacent to their respective neighbours in $C(C_j)$ and to the vertices $v_i, \overline{v}_i$ for all i.

First we need the following proposition and its corollary:

Proposition 3.1 *For all i, the graph G_i is chordal.*

Proof. We prove this by induction. For $i = 0$, observe that the graph $C(C_j)$ is chordal for every j, hence G_0 is chordal. For $i > 0$, suppose that G_{i-1} is chordal; let π be a perfect elimination ordering of its vertices. Now it is not difficult to see that $v_i^1, v_i^2, \ldots, \overline{v}_i^1, \overline{v}_i^2, \ldots, \pi, v_i, \overline{v}_i$ is a perfect elimination ordering of G_i. □

Corollary 3.2 *The graph $G(\varphi)$ is chordal.*

We make the following observations about the graph $G(\varphi)$ and its subgraphs:

Observation 3.3 *Every stable P_4-transversal of the graph $C(C_j)$ contains at least one of the vertices l_1^j,l_2^j or l_3^j. Every maximal stable set of $C(C_j)$ is a P_4-transversal.*

Proof. The proof is by inspection. □

Proposition 3.4 *Let S be a stable P_4-transversal of $G(\varphi)$. Then the following holds:*

(1) For all i, $v_i \notin S$ and $\overline{v}_i \notin S$.
(2) For any j, $v_i^j \notin S$ implies for all j', $\overline{v}_i^{j'} \in S$

Proof.

(1) The vertex v_i is adjacent to all vertices of $C(C_j)$, and so if it belongs to the stable set S, then no vertex of $C(C_j)$ can be in S. Therefore $G(\varphi) \setminus S$ contains all vertices of $C(C_j)$ and hence contains a P_4, contrary to S being a P_4-transversal. The same holds for $\overline{v}_i$.
(2) Suppose that $v_i^j \notin S$ and also $\overline{v}_i^{j'} \notin S$ for some j, j'. Then by the previous argument also $v_i \notin S$ and $\overline{v}_i \notin S$, and hence S cannot be a P_4-transversal since the vertices $v_i^j, v_i, \overline{v}_i, \overline{v}_i^{j'}$ form a P_4 in $G(\varphi) \setminus S$.

□

Lemma 3.5 *The formula φ is satisfiable if and only if the graph $G(\varphi)$ has a stable P_4-transversal.*

Proof. First suppose that τ is a satisfying truth assignment of φ. We use τ to construct a stable P_4-transversal of $G(\varphi) = G_n$.

Let S_0^j be any maximal stable set in $C(C_j)$ with the following property. For all k, the vertex $l_k^j \in S_0^j$ if and only if v_i is the k-th literal of the clause C_j and $\tau(v_i) = true$, or $\neg v_i$ is the k-th literal of the clause C_j and $\tau(v_i) = false$. Clearly since τ satisfies φ and therefore also satisfies the clause C_j, at least one of the vertices l_1^j, l_2^j, l_3^j is in S_0^j. By Observation 3.3, S_0^j is a P_4-transversal in $C(C_j)$. Now let $S_0 = \bigcup_{j=1}^m S_0^j$. Since the graphs $C(C_j)$ in G_0 are vertex disjoint, it follows that S_0 is a stable P_4-transversal of G_0.

Now let $S = S_0 \cup \bigcup_{i=1}^n S_i^+$ where

$$S_i^+ = \begin{cases} \{\overline{v}_i^j\}_{j \in J_i^{(-)}} & \text{if } \tau(v_i) = true \\ \{v_i^j\}_{j \in J_i^{(+)}} & \text{if } \tau(v_i) = false \end{cases}$$

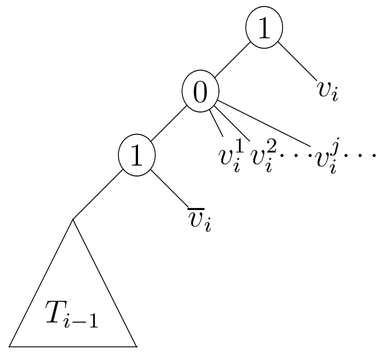

Fig. 2. The cotree for the graph $G_i \setminus S_i$

We show that S is a stable P_4-transversal of $G(\varphi)$. First let $S_i = S_0 \cup \bigcup_{j=1}^{i} S_j^+$. Clearly $S_i = S_{i-1} \cup S_i^+$ and $S_{i-1} \subseteq S_i$. We show by induction that S_i is a stable P_4-transversal of G_i.

For $i = 0$ the claim follows from the above. Therefore suppose that $i > 0$ and S_{i-1} is a stable P_4-transversal of the graph G_{i-1}. Without loss of generality, we may assume that $\tau(v_i) = true$. Then $S_i = S_{i-1} \cup \{\overline{v}_i^j\}_{j \in J_i^{(-)}}$, where S_{i-1} is a stable set. Using the fact that $\tau(v_i) = true$ and the definition of S_0^j, we have that $l_k^j \notin S_0^j$ whenever $\neg v_i$ is the k-th literal of the clause C_j. Since in that case l_k^j and $\overline{v}_i$ are the only neighbours of the vertex $\overline{v}_i^j$ in G_i, it easily follows that S_i is a stable set. Now we only need to show that S_i is a P_4-transversal of G_i, that is that $G_i \setminus S_i$ is a P_4-free graph. From the induction hypothesis $G_{i-1} \setminus S_{i-1}$ is already a P_4-free graph. Therefore there exists a cotree T_{i-1} for this graph. To show the claim we construct a cotree for $G_i \setminus S_i$. As in the previous argument, it follows that $l_k^j \in S_0^j$, whenever v_i is the k-th literal of the clause C_j. Since in that case the vertex v_i^j is only adjacent to the vertex v_i in $G_i \setminus S_i$, we obtain the cotree for $G_i \setminus S_i$ as shown in Figure 2.

It follows that $S = S_n$ is a stable P_4-transversal of the graph $G(\varphi) = G_n$.

Now suppose that $G(\varphi)$ has a stable P_4-transversal, say S. We construct the truth assignment τ for the formula φ in the following way: for every variable v_i we set $\tau(v_i) = true$ just if for some j the vertex $v_i^j \notin S$. We show that τ satisfies φ.

Consider the clause C_j of φ. Since S is a stable P_4-transversal of $G(\varphi)$, the set $S \cap C(C_j)$ is a stable P_4-transversal of $C(C_j)$. It follows from Observation 3.3 that the vertex $l_k^j \in S \cap C(C_j)$ for some k, and hence $l_k^j \in S$. If v_i is the k-th literal of the clause C_j, it follows that v_i^j is not in the stable set S. (Recall that the vertex v_i^j is a neighbour of l_k^j.) Therefore by the definition of τ, we have that $\tau(v_i) = true$, and therefore τ satisfies C_j. If $\neg v_i$ is the k-th literal of the clause C_j, we deduce that $\overline{v}_i^j \notin S$. By Proposition 3.4, we must have for all j', $v_i^{j'} \in S$. Therefore it follows from the definition of τ that $\tau(v_i) = false$, and we again conclude that τ satisfies C_j.

Clearly, since τ satisfies all clauses C_j, it satisfies the formula φ; this concludes the proof. □

Proof. [Theorem 1.1] One can easily see that the graph $G(\varphi)$ can be constructed in polynomial time. Hence the claim follows from Lemma 3.5. □

4 P_3-free and P_4-free P_4-transversals

We now proceed to prove Theorem 1.2 and 1.3.

Proposition 4.1 *Let Y be the graph shown in Figure 3.*

(1) Every P_3-free P_4-transversal S of the graph Y has the property that $u \in S$ and $v \notin S$ (see Figure 3a).
(2) Every P_4-free P_4-transversal S of the graph Y has the property that either $u \in S$ and $v \notin S$, or $u \notin S$ and $v \in S$ (see Figure 3a,3b).

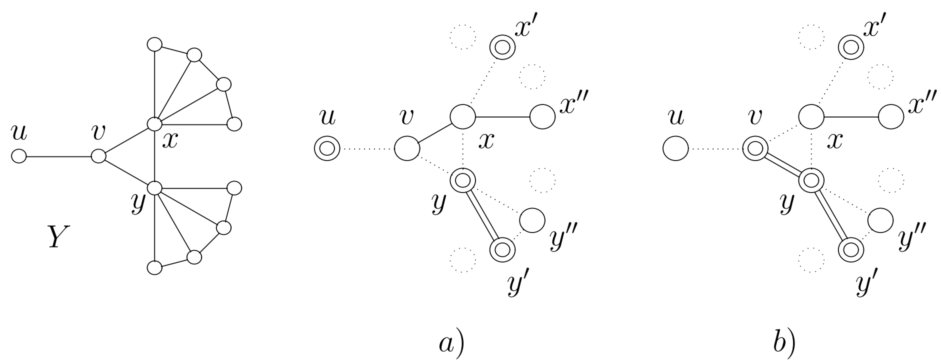

Fig. 3. The graph Y and a sketch of some possible P_4-transversals (the doubly circled vertices)

Proof. Observe that among the neighbours of x and the neighbours of y there must always be a vertex $x' \in S$ and a vertex $x'' \notin S$, and similarly a vertex $y' \in S$ and a vertex $y'' \notin S$. (Clearly both neighbourhoods contain a P_4, so they cannot be entirely in S, nor entirely not in S.) It follows that x and y cannot both be in S, and cannot both be not in S. (In the former case the vertices y', y, x, x' form a P_4 in S, and in the latter case, the vertices y'', y, x, x'' form a P_4 in $Y \setminus S$.) Without loss of generality we may assume that $y \in S$ and $x \notin S$.

First suppose that S is P_3-free. Then we have $v \notin S$, since otherwise the vertices v, y, y' form a P_3 in S. Moreover u must be in S, since otherwise the vertices u, v, x, x'' form a P_4 in $Y \setminus S$. This proves the first part of the claim.

Now suppose that S is P_4-free. Then either $v \notin S$ and we similarly find that $u \in S$, or $v \in S$ and then we have that $u \notin S$, since otherwise the vertices u, v, y, y' form a P_4 in S. This concludes the proof. □

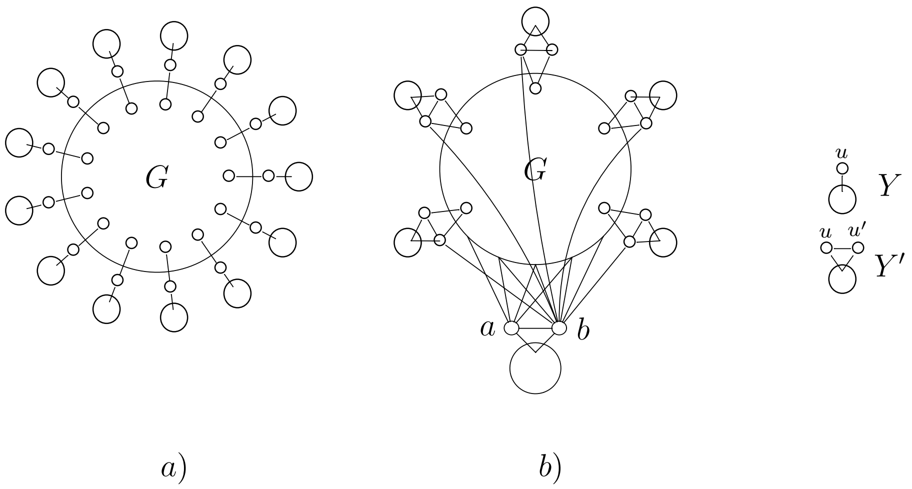

Fig. 4. The construction of the graph G' used in the proof of Theorem 1.2 (a) and Theorem 1.3 (b)

Proof. [Theorem 1.2] In order to prove the NP-completeness of the problem of recognizing the existence of a P_3-free P_4-transversal, we construct a polynomial time reduction from the stable P_4-transversal problem for chordal graphs.

Let G be a chordal graph. Let G' be the graph constructed from G in the following way (see Figure 4a). For every vertex $w \in V(G)$ we add a copy Y_w of the graph Y (see Figure 3) in which we change the labels of the vertices u and v to u_w and v_w respectively, and we make the vertices w and u_w adjacent. Observe that G' is chordal, since both G and Y are chordal. We now prove that G has a stable P_4-transversal if and only if G' has a P_3-free P_4-transversal.

Suppose that G has a stable P_4-transversal S. Let S_w be any P_3-free transversal of Y_w. Let $S' = S \cup \bigcup_{w \in V(G)} S_w$. By Proposition 4.1, we have $u_w \in S'$ for all $w \in V(G)$. Since for every w the vertex u_w is a cut-vertex in G', it easily follows that S' is a P_4-transversal of G'. Moreover, since S is stable, and for every w the vertex $v_w \notin S'$, it follows that S' is P_3-free.

Now suppose that G' has a P_3-free P_4-transversal S'. Let $S = S' \cap V(G)$. Clearly S is a P_4-transversal of G. We show that S is also stable, thus proving the claim. Suppose that there are two adjacent vertices $w, w' \in S$. By Proposition 4.1, we have $u_w \in S'$ and $u_{w'} \in S'$. Therefore the vertices $u_w, w, w', u_{w'}$ form a P_4 in S' contrary to S' being P_3-free. □

Proof. [Theorem 1.3] As in the previous proof, we construct a polynomial time reduction from the stable P_4-transversal problem for chordal graphs. Let G be a chordal graph. Let Y' be the graph obtained from Y by adding an additional vertex u' adjacent to both u and v. Let G' be the graph obtained from G as follows (see Figure 4b). For every vertex $w \in V(G)$ we add a copy Y'_w of the graph Y' in which we change the labels of the vertices u, u' and v to u_w, u'_w and v_w respectively, and we make w adjacent to u_w and u'_w. Moreover, we add a copy Y'' of the graph Y', in which we change the labels of the vertices u, u' and v to a, b and f respectively. We make a and b adjacent to all vertices of G, and make b adjacent to u_w for every $w \in V(G)$.

Observe that G' is chordal. Indeed, since G is chordal, it has a perfect elimination ordering π. Similarly, since Y'_w is chordal, it also has a perfect elimination ordering π_w. It is easy to see, by inspection, that we may choose π_w to end with the vertices u'_w and u_w, in that order. Lastly, since Y'' is chordal, let π'' be any perfect elimination ordering of Y''. Now one can easily verify that $\pi_{w_1}, \pi_{w_2}, \ldots, \pi, \pi''$ is a perfect elimination ordering of G', where $w_1, w_2, \ldots$ is an enumeration of the vertices of G.

We now prove that G has a stable P_4-transversal if and only if G' has a P_4-free P_4-transversal. Suppose that G has a stable P_4-transversal S. Let S_w be a P_4-free P_4-transversal of Y'_w satisfying $u_w, u'_w \in S_w$. Let S'' be a P_4-free P_4-transversal of Y'' satisfying $a, b \notin S''$; it also follows that $f \in S''$. Now let $S' = S \cup S'' \cup \bigcup_{w \in V(G)} S_w$. We show that S' is a P_4-free P_4-transversal of the graph G'.

Since S_w is P_4-free, let T_w be the cotree representing $S_w \setminus \{u_w, u'_w\}$. Similarly, let T'' be the cotree for S'', $\overline{T}$ the cotree for $G \setminus S$, $\overline{T}_w$ the cotree for $Y_w \setminus S_w$, and $\overline{T}''$ the cotree for $Y'' \setminus (S'' \cup \{a, b\})$. Let T' and $\overline{T}'$ be the cotrees depicted on Figure 5.

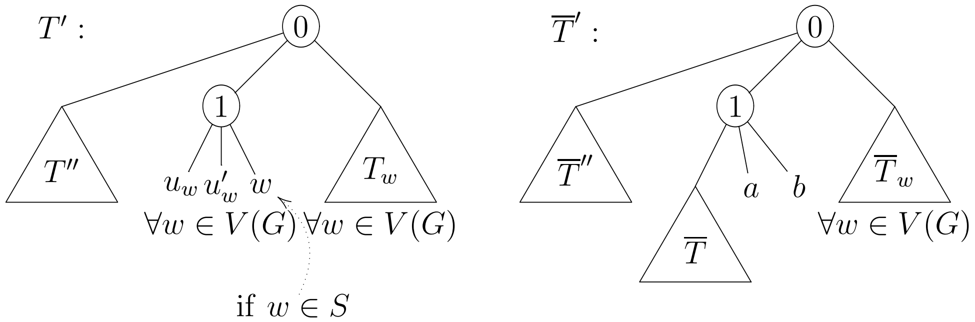

Fig. 5. The cotrees corresponding to the subgraphs of G' induced on S' and $G' \setminus S'$. (The subtrees marked with $\forall w \in V(G)$ indicate that for every vertex in G, such subtree is added.)

One can easily verify that T' and $\overline{T}'$ are exactly the cotrees of the subgraphs of G' induced on S' and $G' \setminus S'$ respectively. That shows that S' and $G' \setminus S'$ are both P_4-free, and hence S' is a P_4-free P_4-transversal of the graph G'.

Now suppose that S' is a P_4-free P_4-transversal of G'. We may assume that the vertex f of Y'' is in S'. (Otherwise we consider $G' \setminus S'$ in place of S' since both S' and $G' \setminus S'$ are P_4-free.) Now it follows from Proposition 4.1 that $a, b \notin S'$. Similarly, it follows that for every w either $u_w, u'_w \in S'$ and $v_w \notin S'$, or $u_w, u'_w \notin S'$ and $v_w \in S'$. Since the latter would create a P_4 in $G' \setminus S'$ (i.e., the vertices a, b, u_w, u'_w form a P_4 in G'), it follows that $u_w, u'_w \in S'$ and $v_w \notin S'$ for every $w \in V(G)$.

Now let $S = S' \cap V(G)$. We show that S is a stable P_4-transversal of the graph G. Clearly S is a P_4-transversal of G. We only need to show that S is also stable. Suppose otherwise, i.e., let $w, w' \in S$ be adjacent. Then the vertices $u_w, w, w', u_{w'}$ clearly form a P_4 in S' (recall that $u_w \in S'$ for all $w \in V(G)$), which leads to a contradiction since S' is P_4-free. Hence S must be stable. □

5 Further Results

A graph G is *strongly chordal* if it is chordal and there exists a perfect elimination ordering $\prec$ of the vertices of G such that if $u \prec v \prec w \prec z$ and (u, z), (u, w) and (v, w) are edges of G then also (v, z) is an edge (such ordering is called *strong elimination ordering*). Strongly chordal graphs form an interesting subclass of chordal graph as there are several difficult combinatorial graph problems that are polynomially solvable in strongly chordal graphs, but are NP-complete for chordal graphs [5].

In the previous sections we proved that it is NP-complete to decide whether a chordal graph has a stable P_4-transversal, a P_3-free P_4-transversal or a P_4-free transversal. It is easy to check that the perfect elimination orderings of $G(\varphi)$ and G' used in the proofs of these results are in fact strong elimination orderings (provided G is strongly chordal in the case of G'). Note that it suffices to show this for $G(\varphi)$ and G' from Theorem 1.3 since G' from Theorem 1.2 is an induced subgraph of G' from Theorem 1.3. Hence we obtain the following stronger result.

Theorem 5.1 *(1) It is NP-complete to decide whether a strongly chordal graph has a stable P_4-transversal.*

(2) It is NP-complete to decide whether a strongly chordal graph has a P_3-free P_4-transversal.

(3) It is NP-complete to decide whether a strongly chordal graph has a P_4-free P_4-transversal.

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