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Research Article

$\mathscr{PT}$-Symmetric Laplace Operators on Star Graphs: Real Spectrum and Self-Adjointness

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How ideas of $\mathscr{PT}$-symmetric quantum mechanics can be applied to quantum graphs is analyzed, in particular to the star graph. The class of rotationally symmetric vertex conditions is analyzed. It is shown that all such conditions can effectively be described by circulant matrices: real in the case of odd number of edges and complex having particular block structure in the even case. Spectral properties of the corresponding operators are discussed.

1. Introduction

Writing this paper we got inspiration from two rapidly developing areas of modern mathematical physics: *$\mathscr{PT}$-symmetric quantum mechanics* and *quantum graphs*. Both areas attract interest of both mathematicians and physicists for the last two decades with numerous conferences organized and articles published. The first area grew up from the simple observation that a quantum mechanical Hamiltonian "often" has real spectrum even if it possesses combined parity and time-reversal symmetry instead of self-adjointness [1–6]. Considering such operators increases the set of physical phenomena that could be modeled and raises up new interesting mathematical questions [7–10]. It appears that spectral theory of such $\mathscr{PT}$-symmetric operators (see precise definition below) can be well-understood using framework of self-adjoint operators in Krein spaces [11].

The theory of quantum graphs—differential operators on metric graphs—can be used to model quantum or acoustic systems where motion is confined to a neighborhood of a set of (one-dimensional) intervals [12–14]. Until now quantum graphs were mostly studied in the context of self-adjoint or dissipative operators. Our key idea is to look at differential operators on metric graphs under more general symmetry assumptions reminding those in $\mathscr{PT}$-symmetric theory. Surprisingly spectral properties of operators on graphs with symmetries have not been paid much attention. We mention here just two papers [15, 16], where symmetries of graphs were used to construct counterexamples showing that inverse problems are not necessarily uniquely solvable.

In quantum graphs motion along the edges is described by ordinary differential equations, which are coupled together by certain vertex conditions connecting together values of the functions at the end points of the intervals building the underlined metric graph. The role of vertex conditions is twofold: to describe how the waves are penetrating through the vertices and to make the differential operator self-adjoint. If the requirement of self-adjointness is waved then such conditions should instead ensure that the resolvent set is not empty; that is, the resolvent for the corresponding differential operator exists for some λ. The later condition is not very precise and one of the goals of the current paper is to understand it in the case of the simplest merit graph with symmetries—the star graph. It can be considered as a building block to define differential operators on arbitrary metric graphs. To avoid discussing properties of the differential operator we limit our studies to the Laplace operator. Moreover the graph formed by semi-infinite edges is considered in order to avoid influence from the peripheral vertices. We just mention here that star graphs formed by finite edges but with standard conditions (see (2)) at the central vertex were considered recently in the framework of

$\mathscr{PT}$-symmetry [17–19]. Our focus will be precisely on the vertex conditions at the central vertex. We are not interested in generalizing formalism of quantum graphs for the sake of generalization, but we look for new spectral phenomena that can be observed.

Current paper grew up from the Master Thesis of Maria Astudillo written in 2008 [20] and already attracted attention of mathematical physics community (see references in [21]). The current paper is just the first step to understand spectral structure of differential operators on graphs with nonstandard symmetries.

The paper is organized as follows: in the first two sections we introduce basic notations and discuss how to generalize the notion of $\mathscr{PT}$-symmetry for the case of star graph. The main difficulty is that the notion of $\mathscr{PT}$-symmetry may be generalized in two different ways and we decided to follow the definition giving new spectral structure (Section 3). We call corresponding operators $\mathscr{RT}$-symmetric. All possible Robin conditions leading to $\mathscr{RT}$-symmetric Laplacians are described in Section 4. It appears that the structure of matrices A in the Robin condition depends on whether the number of edges is odd or even. In the first case the matrices are real circulant, while in the second case they may be complex but are block-circulant. Possibility to obtain non-self-adjoint operators with N real eigenvalues is studied in the last section.

2. Notations and Elementary Properties

Our goal is to generalize ideas that originated from $\mathscr{PT}$-symmetry for the case of the star graph. More precisely we will confine our studies to the case of the Laplace operator with the domain given by generalized Robin conditions at the central vertex. Consider the star graph Γ_N formed by N semi-infinite edges $E_n = [0,\infty)$ joined together at one central vertex. The corresponding Hilbert space $L_2(\Gamma_N)$ can be identified with the space of vector valued functions $u \equiv \vec{u}(x)$ on $x \in [0,\infty)$ with the values in $\mathbb{C}^N$: $L_2(\Gamma_N) = L_2([0,\infty),\mathbb{C}^N)$.

Definition 1. The operator $L_A = -d^2/dx^2$ is defined on the set of functions from the Sobolev space $W_2^2([0,\infty);\mathbb{C}^N)$ satisfying generalized Robin conditions:

$$\vec{u}'(0) = A\vec{u}(0), \tag{1}$$

where A is a certain $N \times N$ matrix.

The operator L_A can be seen as a certain point perturbation of the self-adjoint standard Laplace operator defined on the domain of functions from the same Sobolev space satisfying standard vertex conditions:

$$\begin{gathered} u_1(0) = u_2(0) = \cdots = u_N(0), \\ \sum_{j=1}^{N} u_j'(0) = 0. \end{gathered} \tag{2}$$

In fact all such point perturbations are defined by vertex conditions of a more general form (8), but Theorem 2 implies that only conditions of form (1) are important for our goal.

The operator adjoint to L_A is again the Laplace operator but is defined by vertex conditions (1) with the matrix A substituted by the matrix A^* that is the operator L_{A^*}:

$$(L_A)^* = L_{A^*}. \tag{3}$$

This can be proven by integration by parts for $\vec{u} \in \mathrm{Dom}(L_A)$, $\vec{v} \in \mathrm{Dom}(L_A^*)$:

$$\begin{aligned} \langle L_A\vec{u},\vec{v}\rangle_{L_2(\Gamma_N)} &= \int_0^\infty \left\langle -\vec{u}''(x), \vec{v}(x)\right\rangle_{\mathbb{C}^N} dx \\ &= \left\langle \vec{u}'(0), \vec{v}(0)\right\rangle_{\mathbb{C}^N} - \left\langle \vec{u}(0), \vec{v}'(0)\right\rangle_{\mathbb{C}^N} \\ &\quad + \int_0^\infty \left\langle \vec{u}(x), -\vec{v}''(x)\right\rangle_{\mathbb{C}^N} dx \\ &= \left\langle \vec{u}(0), A^*\vec{v}(0) - \vec{v}'(0)\right\rangle_{\mathbb{C}^N} \\ &\quad + \int_0^\infty \left\langle \vec{u}(x), -\vec{v}''(x)\right\rangle_{\mathbb{C}^N} dx, \end{aligned} \tag{4}$$

where we used the fact that $\vec{u}$ satisfies (1). This formula defines a bounded functional with respect to $\vec{u}$ if and only if $A^*\vec{v}(0) - \vec{v}'(0) = 0$ and $v \in W_2^2([0,\infty);\mathbb{C}^N)$.

It follows that the operator L_A is self-adjoint if and only if A is a Hermitian matrix $A^* = A$. In this paper we are not interested in the case where L_A is self-adjoint.

The spectrum of the operator L_A may contain up to N isolated eigenvalues. The corresponding eigenfunction is a solution to the differential equation

$$-u''(x) = \lambda u(x) \tag{5}$$

satisfying Robin conditions (1). Any square integrable solution to the differential equation is given by

$$\vec{u}(x) = \vec{a}\exp(ikx), \quad k^2 = \lambda, \; \vec{a} \in \mathbb{C}^N \tag{6}$$

with $\mathrm{Im}\, k > 0$. This function satisfies Robin condition if and only if

$$\det(A - ik) = 0. \tag{7}$$

The last equation has at most N distinct solutions (in the correct half-plane). Observe that not all solutions lead to eigenfunctions, since one needs to meet the condition $\mathrm{Im}\, k > 0$.

As the theory of $\mathscr{PT}$-symmetric operators indicates the most interesting case is when the spectrum of the operator is pure real, the operator itself is not self-adjoint. We are going to look closer at such operators. If the operator L_A has N real eigenvalues, then the matrix A has N negative eigenvalues, but it does not imply that it is Hermitian.

Note that possible vertex conditions are not limited to those described by (1). More generally one may consider the Laplace operator $L_{B,C} = -d^2/dx^2$ defined on the domain of functions from $W_2^2([0,\infty);\mathbb{C}^N)$ satisfying the following vertex conditions:

$$B\vec{u}'(0) + C\vec{u}(0) = 0, \tag{8}$$

where B, C are certain $N\times N$ matrices, such that $\operatorname{rank}(B,C) = N$. Here we follow ideas from [22]. The following theorem shows that a Laplace operator described by the vertex conditions of the form (8) has N real eigenvalues if either the vertex conditions are of the form (1) or the spectrum covers the whole complex plane.

Theorem 2. *Consider the operator $L_{B,C}$ defined by vertex condition (8). If $L_{B,C}$ has N real eigenvalues (counting multiplicities), then either B is invertible or the spectrum of $L_{B,C}$ is the whole complex plane.*

Proof. A function u is an eigenfunction of $L_{B,C}$ if and only if it satisfies the differential equation (5), which has solution (6). Substituting in (8), we get that for a certain $\vec{a} \neq \vec{0}$

$$(C + ikB)\vec{a} = 0. \tag{9}$$

Conversely, if this holds for some k with $\operatorname{Im} k > 0$ then k^2 is an eigenvalue of $L_{B,C}$.

The last equation has nontrivial solution if and only if

$$\det(C + ikB) = 0. \tag{10}$$

Let us now consider two invertible matrices S and T. If (10) is satisfied then also

$$\det(SCT + ikSBT) = 0. \tag{11}$$

Let us take S and T such that

$$(SBT)_{i,k} = \begin{cases} 1, & i = k \leq \operatorname{rank} B, \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

Let us introduce the following function: $q(k) =: \det(SCT + ikSBT)$. Because of (12), q is a polynomial in k of degree at most $\operatorname{rank} B$. If $q \equiv 0$, then any k with $\operatorname{Im} k > 0$ gives an eigenvalue k^2 and as a result the spectrum is the whole complex plane. Otherwise q has at most $\operatorname{rank} B$ zeros. Suppose now that each eigenvalue has multiplicity 1, then if $\operatorname{rank} B \neq N$, the operator will have less than N eigenvalues. Therefore we must have the fact that B is invertible. To complete the proof, the case of multiple eigenvalues has to be considered. Let us assume that k_0^2 is an eigenvalue of multiplicity m. To see that B must be invertible if $L_{B,C}$ has N eigenvalues, it is enough to prove that $q(k)$ has a zero of order at least m at k_0. To prove this, we consider the eigenfunctions $\vec{u}^1 = \vec{a}^1 e^{ik_0x}, \dots, \vec{u}^m = \vec{a}^m e^{ik_0x}$. Here, all constant vectors $\vec{a}^j$, $j = 1, \dots, m \leq N$ are linearly independent. Let us then choose vectors $\vec{a}^{m+1}, \dots, \vec{a}^N$ such that $D = [\vec{a}^1, \vec{a}^2, \dots, \vec{a}^N]$ has determinant equal to 1. It then follows that

$$\begin{aligned}\det(C + ikB) &= \det((C + ikB)D) = \det\Big(\big[(C + ikB) \\ &\quad \cdot \vec{a}^1, (C + ikB)\vec{a}^2, \dots, (C + ikB)\vec{a}_N\big]\Big)\end{aligned} \tag{13}$$

has a zero of order at least m at $k = k_0$. □

Our method to obtain nontrivial operators with nonstandard symmetries is a certain generalization of the method of point interactions originally developed in the framework of self-adjoint Hamiltonians [23, 24]. The phenomenon described in Theorem 2 in connection with $\mathcal{PT}$-symmetric point interaction was first observed in [25], following [26].

This theorem implies that the class of operators L_A defined by Robin vertex conditions (1) is rather wide; therefore in what follows we focus our attention on this class only.

3. Pseudo-Hermitian and Pseudoreal Operators

Our studies are inspired by recent papers devoted to investigation of the so-called $\mathcal{PT}$-symmetric operators in one dimension. An operator L is called $\mathcal{PT}$-symmetric if it satisfies the following relation:

$$\mathcal{PT}L = L\mathcal{PT}, \tag{14}$$

where $\mathcal{P}$ is the spacial symmetry operator (parity symmetry),

$$(\mathcal{P}u)(x) = u(-x), \tag{15}$$

and $\mathcal{T}$ is the antilinear operator of complex conjugation (time-reversal symmetry),

$$(\mathcal{T}u)(x) = \overline{u(x)}. \tag{16}$$

$\mathcal{PT}$-symmetry (like usual operator symmetry) is not enough to guarantee that the corresponding operator is physically relevant: it might happen that its spectrum is empty (take, e.g., the second derivative operator on the interval $[-a, a]$ with *both* Dirichlet and Neumann conditions imposed on both end points of the interval). In conventional quantum theory the notion of a self-adjoint operator substitutes simple symmetry property (of course any self-adjoint operator is symmetric, but not the other way around). Therefore it appears natural to substitute relation (14) with the following one:

$$L^* = \mathcal{P}L\mathcal{P}^{-1} = \mathcal{P}L\mathcal{P}^*, \tag{17}$$

where we use the fact that $\mathcal{P}$ is unitary $\mathcal{P}^{-1} = \mathcal{P}^*$. In the case of conventional $\mathcal{PT}$-symmetric theory the operator $\mathcal{P}$ is not only unitary, but also self-adjoint $\mathcal{P}^* = \mathcal{P}$. It follows that the operator satisfying (17) is pseudo-self-adjoint; that is, it is self-adjoint not in the original Hilbert space but in the Krein space with the sesquilinear form $[\cdot,\cdot]$ defined by $\mathcal{P}$ as Gram operator:

$$[f, g] = \langle \mathcal{P}f, g \rangle, \tag{18}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in L_2.

The goal of the current paper is to generalize $\mathcal{PT}$-symmetry for the case of operators on metric graphs, more precisely for the star graph Γ_N. Operator on such a star graph can be considered as a building block to define operators on arbitrary graphs. We are going to substitute the operator of spacial symmetry $\mathscr{P}$ with the rotation operator $\mathscr{R}$ defined as follows:

$$\mathscr{R}\begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \\ \vdots \\ u_{N-1}(x) \\ u_N(x) \end{pmatrix} = \begin{pmatrix} u_N(x) \\ u_1(x) \\ u_2(x) \\ \vdots \\ u_{N-2}(x) \\ u_{N-1}(x) \end{pmatrix}. \tag{19}$$

We are not going to distinguish the rotation operators in $\mathbb{C}^N$ and in the space of vector-valued functions hoping that this will not lead to any misunderstanding. Observe that the operator $\mathscr{R}$ is unitary but not self-adjoint (of course provided $N \neq 2$).

We would like to understand whether ideas from $\mathscr{P}\mathscr{T}$-symmetric theory may lead to an interesting new class of operators L_A, which are not self-adjoint and not $\mathscr{P}\mathscr{T}$-symmetric with a suitably defined self-adjoint spacial symmetry operator $\mathscr{P}$. For example, if N is even then such a spacial symmetry operator can be defined as

$$\mathscr{P} = \mathscr{R}^{N/2}. \tag{20}$$

Using $\mathscr{P}$ we may introduce $\mathscr{P}\mathscr{T}$-symmetric operators on Γ_N, but this would be just a vector version of conventional one-dimensional theory (see Lemma 4 below).

Formulas (14) and (17) suggest that we look closer at the following two possible generalizations of the notion of $\mathscr{P}\mathscr{T}$-symmetry:

(i) *pseudoreal* operators

$$\mathscr{R}\mathscr{T}L = L\mathscr{R}\mathscr{T}, \tag{21}$$

and

(ii) *pseudo-Hermitian* operators

$$L^* = \mathscr{R}L\mathscr{R}^{-1} = \mathscr{R}L\mathscr{R}^*. \tag{22}$$

We are going to reserve the term pseudo-self-adjoint for operators which are pseudo-Hermitian with respect to a self-adjoint operator $\mathscr{P}$ as in (17).

Surprisingly pseudo-Hermitian operators do not define any new interesting class as follows from the two lemmas below.

Lemma 3. *Let N be an odd number; then the operator L_A is $\mathscr{R}$-pseudo-Hermitian only if it is self-adjoint; that is, $A = A^*$.*

Proof. Iterating formula (22) one gets the following set of equations:

$$\begin{aligned} L_A^* &= \mathscr{R}^m L_A \mathscr{R}^{-m}, \quad m \text{ is odd}; \\ L_A &= \mathscr{R}^m L_A \mathscr{R}^{-m}, \quad m \text{ is even}. \end{aligned} \tag{23}$$

Taking into account that $\mathscr{R}^N$ is the identity operator we arrive at the following relation for any odd N:

$$L_A^* = L_A \tag{24}$$

proving our statement. □

Lemma 4. *Let N be an even number.*

(i) *If in addition $N/2$ is an odd number, then the operator L_A is $\mathscr{R}$-pseudo-Hermitian only if it is pseudo-self-adjoint with respect to $\mathscr{P} = \mathscr{R}^{N/2}$.*

(ii) *If in addition $N/2$ is an even number, then the operator L_A is $\mathscr{R}$-pseudo-Hermitian only if it commutes with the self-adjoint rotation $\mathscr{P} = \mathscr{R}^{N/2}$ and therefore is unitarily equivalent to an orthogonal sum of Laplace operators on $\Gamma_{N/2}$ with Robin conditions at the central vertices.*

Proof. We just apply formula (23) to get

$$\begin{aligned} L_A^* &= \mathscr{R}^{N/2} L_A \mathscr{R}^{-N/2}, \quad \frac{N}{2} \text{ is odd}; \\ L_A &= \mathscr{R}^{N/2} L_A \mathscr{R}^{-N/2}, \quad \frac{N}{2} \text{ is even}. \end{aligned} \tag{25}$$

In the first case the operator L_A is pseudo-self-adjoint with respect to $\mathscr{P} = \mathscr{R}^{N/2}$.

In the second case ($N = 4n$, $n \in \mathbb{N}$) the operator L_A commutes with the self-adjoint operator $\mathscr{P} = \mathscr{R}^{N/2}$. Since $\mathscr{P}^2 = \mathscr{R}^N = \mathscr{I}$ is the identity operator, its spectrum is ± 1. The corresponding subspaces coincide with the sets of functions satisfying $u_{j+N/2}(x) = \pm u_j(x)$. Each of the eigensubspaces can be identified with the Hilbert space $L_2([0,\infty);\mathbb{C}^{N/2})$. Since L_A commutes with $\mathscr{P}$ it can be written as an orthogonal sum of two Laplace operators $L_\pm = -d^2/dx^2$, each acting in $L_2([0,\infty);\mathbb{C}^{N/2})$ simply because $L_A\mathscr{P} = \mathscr{P}L_A$ implies that $\vec{u} \in \mathrm{Dom}(L_A) \Leftrightarrow \mathscr{P}\vec{u} \in \mathrm{Dom}(L_A)$. The operators $L_\pm$ are then defined by Robin conditions of form $\vec{u}'_\pm(0) = A_\pm \vec{u}_\pm(0)$, where $\vec{u}_\pm \in W_2^2([0,\infty);\mathbb{C}^{N/2})$. Finally, the operators $L_\pm$ can be seen as Laplace operators on start graphs $\Gamma_{N/2}$ with $N/2$ edges with Robin conditions at the central vertices. □

The operators $L_\pm$ appearing in the previous Lemma satisfy symmetry properties similar to (22), but with the "rotation" operators $\mathscr{R}_\pm$ of lower size ($N/2$ instead of N). If $\mathscr{R}_+$ is the standard rotation operator in the space $\mathbb{C}^{N/2}$, the operator $\mathscr{R}_-$ is a certain modified rotation operator:

$$\mathscr{R}_- = \mathrm{diag}\,(1,1,\ldots,1,-1)\,\mathscr{R}_+. \tag{26}$$

It might be interesting to understand the symmetry of L_- in more detail, but we may conclude already now that in most cases the operator L_A is pseudo-Hermitian only if it is also pseudo-self-adjoint (with respect to another symmetry operator). Therefore in what follows we focus on the studies of pseudo-real realisations of the Laplace operator on the star graph Γ_N. Therefore we are going to use the following definition.

Definition 5. An operator L in $L_2(\Gamma_N)$ is called $\mathcal{RT}$-*symmetric* if and only if it satisfies the following relation:

$$\mathcal{RT}L = L\mathcal{RT}, \tag{27}$$

where $\mathcal{T}$ is the antilinear operator of complex conjugation.

This definition guarantees that the spectrum of the operator is symmetric with respect to the real axis. Really if $\psi(x)$ is an eigenfunction corresponding to the eigenvalue λ, then $\varphi(x) = \mathcal{RT}\psi$ is also an eigenfunction but corresponding to the eigenvalue $\overline{\lambda}$, provided (27) holds:

$$L\varphi = L\mathcal{RT}\psi = \mathcal{RT}L\psi = \mathcal{RT}\lambda\psi = \overline{\lambda}\mathcal{RT}\psi = \overline{\lambda}\varphi. \tag{28}$$

Hence with Definition 5 we always have spectrum which is symmetric with respect to the real axis as in the classical $\mathcal{PT}$-symmetric theory.

4. $\mathcal{RT}$-Symmetry of Point Interactions

In the current section we are going to describe the structure of $\mathcal{RT}$-symmetric operators. It appears that the corresponding matrices A belong to the class of circulant matrices which we describe now. A *circulant matrix* is a special case of a Toeplitz matrix.

Definition 6. An $N \times N$ matrix $A = \{a_{ik}\}$ is called *circulant* if the value of the entry a_{ik} depends only on the difference $(k - i) \bmod N$; that is,

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \cdots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix} =: \operatorname{circ}\left(a_0, a_1, \ldots, a_{N-1}\right). \tag{29}$$

Definition 7. An $N \times N$ matrix A is called $k \times k$ *block circulant* if $A = \operatorname{circ}(B_0, B_1, \ldots, B_{k-1})$, where B_i, $i = 0, \ldots, k-1$, are block matrices of the same size $n \times n$, $N = kn$.

The following theorem describes matrices A leading to $\mathcal{RT}$-symmetric operators on Γ_N.

Theorem 8. *Consider the operator L_A determined by Definition 1.*

If N is odd, then the operator L_A is $\mathcal{RT}$-symmetric if and only if A is a real circulant matrix; that is,

$$A = \operatorname{circ}\left(a_0, a_1, \ldots, a_{N-1}\right), \quad a_j \in \mathbb{R}, \; j = 0, \ldots, N-1. \tag{30}$$

If N is even, then the operator L_A is $\mathcal{RT}$-symmetric if and only if A is a complex $N/2 \times N/2$ block circulant matrix formed by the following 2×2 blocks:

$$A = \left(\begin{array}{cc|cc|c|cc} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-2} & a_{N-1} \\ \overline{a}_{N-1} & \overline{a}_0 & \overline{a}_1 & \overline{a}_2 & \cdots & \overline{a}_{N-3} & \overline{a}_{N-2} \\ \hline a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-4} & a_{N-3} \\ \overline{a}_{N-3} & \overline{a}_{N-2} & \overline{a}_{N-1} & \overline{a}_0 & \cdots & \overline{a}_{N-5} & \overline{a}_{N-4} \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline a_2 & a_3 & a_4 & a_5 & \cdots & a_0 & a_1 \\ \overline{a}_1 & \overline{a}_2 & \overline{a}_3 & \overline{a}_4 & \cdots & \overline{a}_{N-1} & \overline{a}_0 \end{array}\right), \quad a_j \in \mathbb{C}, \; j = 0, \ldots, N-1. \tag{31}$$

Proof. Let us consider the operators L_A as defined in Definition 1. Suppose that the function $u \in \operatorname{Dom}(L_A)$ and therefore satisfies the boundary condition (1). The boundary condition for the function $\mathcal{RT}u$ is given by

$$\begin{aligned} \mathcal{RT}\vec{u}'(0) &= A\mathcal{RT}\vec{u}(0) \Longrightarrow \\ \vec{u}'(0) &= \mathcal{TR}A\mathcal{RT}\vec{u}(0) \Longrightarrow \\ \vec{u}'(0) &= \mathcal{R}^{-1}\overline{A}\mathcal{R}\vec{u}(0). \end{aligned} \tag{32}$$

This condition should be identical with (1) leading to

$$\begin{aligned} A &= \mathcal{R}^{-1}\overline{A}\mathcal{R} \Longleftrightarrow \\ A &= \mathcal{R}\overline{A}\mathcal{R}^{-1}, \end{aligned} \tag{33}$$

where we used the fact that the rotation matrix $\mathcal{R}$ has real entries.

Let us denote the entries of the matrix A by $a_{i,k}$; then the last equality implies

$$a_{i,k} = \overline{a}_{i-1 \bmod N, k-1 \bmod N}. \tag{34}$$

The structure of the matrix A is as follows: every next row in the matrix is equal to the previous one shifted to the right one step and conjugated. It is clear then that A is determined by N complex numbers, for example, those building the first row. If there would be no complex conjugation or the entries would be real, then A would be circulant.

Now, we consider the cases when N is odd and even separately.

N Is Odd. We get the following chain of equalities:

$$\begin{aligned} a_{i \bmod N, k \bmod N} &= \overline{a}_{i+1 \bmod N, k+1 \bmod N} \\ &= a_{i+2 \bmod N, k+2 \bmod N} = \cdots \\ &= \overline{a}_{i+N \bmod N, k+N \bmod N} \\ &= \overline{a}_{i \bmod N, k \bmod N}, \end{aligned} \tag{35}$$

implying that

$$a_{i,k} = \overline{a}_{i,k} \Longrightarrow a_{i,k} \in \mathbb{R}, \quad i,k = 0, \ldots, N-1. \tag{36}$$

Therefore

$$a_{i,k} = a_{i-1 \bmod N, k-1 \bmod N} \tag{37}$$

and we see that the matrix A is a real circulant matrix. It is determined by N real parameters:

$$\begin{aligned} a_j &:= a_{1,j+1}, \\ A &= \operatorname{circ}\left(a_0, a_1, a_2, \ldots, a_{N-1}\right). \end{aligned} \tag{38}$$

N Is Even. We again consider (34) to obtain the following:

$$\begin{aligned} a_{i \bmod N, k \bmod N} &= \overline{a}_{i+1 \bmod N, k+1 \bmod N} \\ &= a_{i+2 \bmod N, k+2 \bmod N} = \cdots \\ &= a_{i+N \bmod N, k+N \bmod N}. \end{aligned} \tag{39}$$

We do not obtain any restriction on $a_{i,k}$ and hence the entries of the first row can be chosen arbitrarily among complex numbers. But we still have the following property, which reminds us of circulant matrices:

$$a_{i,k} = a_{i',k'}, \tag{40}$$

provided $i' = (i + 2s) \bmod N$ and $k' = (k + 2s) \bmod N$, $s = 1, \ldots, N/2$.

Let us denote the first row in A by a_j, $j = 0, \ldots, N-1$; then each row of the matrix A is the conjugation of the previous row shifted to the right; that is,

$$A = \left(\begin{array}{cc|cc|c|cc} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-2} & a_{N-1} \\ \overline{a}_{N-1} & \overline{a}_0 & \overline{a}_1 & \overline{a}_2 & \cdots & \overline{a}_{N-3} & \overline{a}_{N-2} \\ \hline a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-4} & a_{N-3} \\ \overline{a}_{N-3} & \overline{a}_{N-2} & \overline{a}_{N-1} & \overline{a}_0 & \cdots & \overline{a}_{N-5} & \overline{a}_{N-4} \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline a_2 & a_3 & a_4 & a_5 & \cdots & a_0 & a_1 \\ \overline{a}_1 & \overline{a}_2 & \overline{a}_3 & \overline{a}_4 & \cdots & \overline{a}_{N-1} & \overline{a}_0 \end{array}\right), \tag{41}$$

$$a_j \in \mathbb{C}, \quad j = 0, \ldots, N-1,$$

as claimed. We see that A is a 2×2 block circulant matrix. The blocks forming A are not chosen arbitrarily but depend on N complex parameters. □

5. On the Spectrum of the $\mathcal{RT}$-Symmetric Operators

We can now look at the discrete spectra of the constructed $\mathcal{RT}$-symmetric operators on the star graph Γ_N. We have already observed that, as in the case of all $\mathcal{PT}$-symmetric operators, the nonreal eigenvalues of a $\mathcal{RT}$-symmetric operator always appear in conjugate pairs (28). That is, if λ is an eigenvalue of a $\mathcal{RT}$-symmetric operator $\mathcal{A}$, then $\overline{\lambda}$ is also an eigenvalue of the operator $\mathcal{A}$. We noted that the operator L_A may have at most N distinct eigenvalues (counting multiplicities). The most interesting case is when the spectrum of the operator is real; also the operator itself is not self-adjoint.

The eigenvalues of L_A are given by solutions of (7) with $\operatorname{Im} k > 0$. Negative eigenvalues correspond to k on the upper part of the imaginary axis. In what follows we study the case where the operator has precisely N (negative) real eigenvalues. Since the structures of the matrices are different in the cases when the number of edges N is even or odd, these cases will be studied separately.

5.1. An Odd Number of Edges. Before we study the discrete spectrum of the operator L_A, we recall some known results about the eigenvalues of a circulant matrix which can be nicely calculated as follows [27].

Proposition 9. *Let $A = \operatorname{circ}(a_0, a_1, \ldots, a_{N-1})$ be a circulant matrix; then its eigenvalues are given by*

$$\mu_j = \sum_{j=0}^{N-1} a_j e^{2\pi i/N}, \quad j = 0, \ldots, N-1. \tag{42}$$

Proof. The key idea is to write arbitrary circulant matrix A as a sum of powers of the rotation matrix $\mathcal{R}$. The rotation matrix $\mathcal{R}$ can be seen as an elementary circulant matrix:

$$\mathcal{R} = \operatorname{circ}(0, 1, 0, \ldots, 0). \tag{43}$$

We have, similarly,

$$\mathcal{R}^j = \operatorname{circ}\Big(0, 0, \ldots, \underbrace{1}_{j+1}, \ldots, 0\Big), \quad \mathcal{R}^N = \mathcal{I}. \tag{44}$$

Hence any circulant A possesses the representation

$$A = a_0 \mathcal{I} + \sum_{j=1}^{N-1} a_j \mathcal{R}^j. \tag{45}$$

The eigenvalues of the rotation matrix $\mathcal{R}$ are $e^{j(2\pi i/N)}$, $j = 0, 1, \ldots, N-1$, with the eigenvectors

$$\left(1, e^{j(2\pi i/N)}, e^{2j(2\pi i/N)}, \ldots, e^{(N-1)j(2\pi i/N)}\right). \tag{46}$$

It follows that (42) holds. □

Theorem 10. *Assume that N is odd; then any $\mathcal{RT}$-symmetric operator L_A on the star-graph Γ_N has N real eigenvalues only if it is self-adjoint.*

Proof. First, we note that $\lambda < 0$ is an eigenvalue of L_A if and only if $\mu = -\sqrt{-\lambda}$ is an eigenvalue of the matrix A. If N is odd, then in accordance with Theorem 8 A is a circulant matrix with real entries. Then its eigenvalues are nothing else other than a discrete Fourier transform of $\{a_n\}_{n=0}^{N-1}$:

$$\mu_j = \sum_{n=0}^{N-1} e^{nj(2\pi i/N)} a_n, \quad j = 0, 1, \ldots, N-1,$$
$$a_m = \frac{1}{N}\sum_{j=0}^{N-1} e^{-mj(2\pi i/N)} \mu_j, \quad m = 0, 1, \ldots, N-1. \tag{47}$$

It follows that $\overline{a_n} = a_{N-n}$, indeed

$$\overline{a_n} = \frac{1}{N}\sum_{j=0}^{N-1} \overline{\mu_j} e^{nj(2\pi i/N)} = \frac{1}{N}\sum_{j=0}^{N-1} \mu_j e^{-(N-n)j(2\pi i/N)} = a_{N-n}, \tag{48}$$

where we used that μ_j are all real. Taking into account that A is circulant we conclude that it is Hermitian; hence the operator L_A is self-adjoint. □

Of course, if the number of real eigenvalues is less than N the operator L_A does not need to be self-adjoint. The proof of the later theorem may give an impression that whether the size of A is even or odd does not play any essential role. The next subsection shows that the difference is tremendous.

5.2. An Even Number of Edges. We recall from Theorem 8 that in case the star graph Γ_N has even number of edges the operator L_A is $\mathscr{RT}$-symmetric if and only if $A = \text{circ}(B_0, B_1, \ldots, B_K)$, where $K = N/2$, and

$$B_i = \begin{pmatrix} a_{2i} & a_{(2i+1) \bmod N} \\ \overline{a}_{(2i-1) \bmod N} & \overline{a}_{2i} \end{pmatrix}. \tag{49}$$

A vector $\vec{d} \neq 0$ is an eigenvector of A with eigenvalue μ if and only if

$$A\vec{d} = \mu\vec{d}. \tag{50}$$

Following the ideas in [28], we will look for eigenvectors of A of the following form (this is a natural suggestion given the structure of the eigenvectors for the case when N is odd):

$$\vec{d} = \vec{d}(\omega, \nu) = \left(\nu, \omega\nu, \omega^2\nu, \ldots, \omega^{K-1}\nu\right), \tag{51}$$

where ν is a nonzero two-dimensional vector and ω is a fixed Kth root of the unity; that is,

$$\omega \in \left\{e^{(2\pi i/K)j}\right\}, \quad j = 0, \ldots, K-1. \tag{52}$$

One may prove that all eigenvectors of A are of this form, since A is commuting with $\mathscr{R}^2$.

Extending (50) with $A = \text{circ}(B_0, B_1, \ldots, B_K)$, as explained above, the following set of K equations is obtained:

$$\begin{gathered}
\left(B_0 + B_1\omega + B_2\omega^2 + B_3\omega^3 + \cdots + B_{K-1}\omega^{K-1}\right)\nu = \mu\nu, \\
\left(B_{K-1} + B_0\omega + B_1\omega^2 + B_2\omega^3 + \cdots + B_{K-2}\omega^{K-1}\right)\nu = \omega\mu\nu, \\
\vdots \\
\left(B_1 + B_2\omega + B_3\omega^2 + B_4\omega^3 + \cdots + B_0\omega^{K-1}\right)\nu = \omega^{K-1}\mu\nu.
\end{gathered} \tag{53}$$

Dividing the jth equation by ω^j, $j = 1, \ldots, K-1$, it reduces to the first one. Hence we have just one equation. Let us now rewrite it as an eigenvector equation:

$$H\nu = \mu\nu, \tag{54}$$

where the square matrix $H = H(w)$ is

$$\begin{aligned}
H &= B_0 + B_1\omega + B_2\omega^2 + B_3\omega^3 + \cdots + B_{K-1}\omega^{K-1} \\
&= \begin{pmatrix} a_0 & a_1 \\ \overline{a}_{N-1} & \overline{a}_0 \end{pmatrix} + \begin{pmatrix} a_2 & a_3 \\ \overline{a}_1 & \overline{a}_2 \end{pmatrix}\omega + \cdots + \begin{pmatrix} a_{N-2} & a_{N-1} \\ \overline{a}_{N-3} & \overline{a}_{N-2} \end{pmatrix}\omega^{K-1} \\
&= \begin{pmatrix} a_0 + a_2\omega + \cdots + a_{N-2}\omega^{K-1} & a_1 + a_3\omega + \cdots + a_{N-1}\omega^{K-1} \\ \overline{a}_{N-1} + \overline{a}_1\omega + \cdots + \overline{a}_{N-3}\omega^{K-1} & \overline{a}_0 + \overline{a}_2\omega + \cdots + \overline{a}_{N-2}\omega^{K-1} \end{pmatrix}.
\end{aligned} \tag{55}$$

Each eigenvector $\nu = \nu(\omega)$ of $H(\omega)$ will generate an eigenvector $\vec{d}$ of A with the same eigenvalue. If for every ω the 2×2 matrix $H(\omega)$ has two negative eigenvalues, then the matrix A has precisely N negative eigenvalues and so the operator L_A.

The following theorem is a counterpart of Theorem 10 for the case when N is even.

Theorem 11. *Let the number N of edges of the star-graph Γ_N be even. Then among $\mathscr{RT}$-symmetric operators L_A (given by Definition 1) there are some with N real eigenvalues that are not self-adjoint.*

Proof. To prove the theorem it is enough to present an example of such a matrix A for arbitrary even N. It is enough to find A such that all $H(\omega)$ have two negative eigenvalues. As by Theorem 8 if N is even then L_A is $\mathscr{RT}$-symmetric if and only if A is block circulant. Consider such block circulant matrix A given by

$$\begin{gathered}
a_0 = -8K + 2 + i, \\
a_1 = a_3 = \cdots = a_{N-1} = 1 + i, \\
a_2 = a_4 = \cdots = a_{N-2} = 2 + i.
\end{gathered} \tag{56}$$

Equivalently the matrix A can be presented as

$$A = -4N + \operatorname{circ}\left(\begin{pmatrix} 2+i & 1+i \\ 1-i & 2-i \end{pmatrix}, \begin{pmatrix} 2+i & 1+i \\ 1-i & 2-i \end{pmatrix}, \dots\right)$$
$$= \begin{pmatrix} -4N+2+i & 1+i & 2+i & \cdots & 2+i & 1+i \\ 1-i & -4N+2-i & 1-i & \cdots & 1-i & 2-i \\ 2+i & 1+i & -4N+2+i & \cdots & 2+i & 1+i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2+i & 1+i & 2+i & \cdots & -4N+2+i & 1+i \\ 1-i & 2-i & 1-i & \cdots & 1-i & -4N+2-i \end{pmatrix}. \quad (57)$$

Using entries in (56) the corresponding matrices H, which appeared first in (55), are given by

$$H(\omega) = \begin{cases} N\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}, & \omega \neq 1, \\ \dfrac{N}{2}\begin{pmatrix} -6+i & 1+i \\ 1-i & -6-i \end{pmatrix}, & \omega = 1. \end{cases} \quad (58)$$

Here we just used the fact that

$$1 + \omega + \omega^2 + \cdots + \omega^{N/2-1} = \begin{cases} 0, & \omega \neq 1, \\ \dfrac{N}{2}, & \omega = 1. \end{cases} \quad (59)$$

Each of the matrices $H(\omega)$ given by (58) has two negative eigenvalues implying that the corresponding A as well as L_A has N negative eigenvalues. □

The theorem implies that the class of $\mathscr{RT}$-symmetric operators is much richer in the case of even N.

6. Conclusions

The main result of this paper is the description of all $\mathscr{RT}$-symmetric Laplace operators on a star graph with the most general coupling condition at the central vertex. Essentially the same result holds if the edges forming the graph are compact and Dirichlet, Neumann, or any other identical Hermitian conditions are introduced at the remote vertices. It might be interesting to extend our studies assuming different (not necessarily Hermitian) conditions at the remote vertices.

Our results on the discrete spectrum can be extended in two ways:

(i) describing in the case N is even the whole family of $\mathscr{RT}$-symmetric operators leading to N real eigenvalues (not only providing a counterexample as is done here);

(ii) studying the case of compact star graphs (defined as above).

The current paper opens a new direction in the studies of quantum graphs, namely, investigation of differential operators on metric graphs possessing generalised symmetries, not necessarily self-adjoint ones.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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