Latin directed triple systems

A. Drápal[*], A. Kozlik[†]
Department of Algebra
Charles University
Sokolovská 83
186 75 Praha 8
CZECH REPUBLIC

T. S. Griggs
Department of Mathematics and Statistics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM

Abstract

It is well known that given a Steiner triple system then a quasigroup can be formed by defining an operation $\cdot$ by the identities $x \cdot x = x$ and $x \cdot y = z$ where z is the third point in the block containing the pair $\{x, y\}$. The same is true for a Mendelsohn triple system where the pair (x, y) is considered to be ordered. But it is not true in general for directed triple systems. However directed triple systems which form quasigroups under this operation do exist. We call these Latin directed triple systems and in this paper begin the study of their existence and properties.

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1 Introduction

The equivalence between Steiner triple systems, on the one hand, and Steiner quasigroups and Steiner loops, on the other hand, is well know in both the combinatorial and the algebraic communities, see for example [6, page 24] and [16, page 124]. Recall the definitions. A *Steiner triple system* of order n, STS(n), is a pair $(V, \mathcal{B})$ where V is a set of n points and $\mathcal{B}$ is a collection of triples of distinct points, also called blocks, taken from V such that every pair of distinct points from V appears in precisely one block. Such systems exist if and only if $n \equiv 1$ or $3 \pmod 6$ [11]. A *Steiner quasigroup* or *squag* is a pair $(Q, \cdot)$ where Q is a set and $\cdot$ is an operation on Q satisfying the identities

$$x \cdot x = x, \qquad y \cdot (x \cdot y) = x, \qquad x \cdot y = y \cdot x.$$

If $(V, \mathcal{B})$ is an STS(n), then a Steiner quasigroup $(Q, \cdot)$ is obtained by letting $Q = V$ and defining $x \cdot y = z$ where $\{x, y, z\} \in \mathcal{B}$. The process is reversible; if Q is a Steiner quasigroup, then a Steiner triple system is obtained by letting $V = Q$ and $\{x, y, z\} \in \mathcal{B}$ where $x \cdot y = z$ for all $x, y \in Q$, $x \neq y$. Thus there is a one-one correspondence between all Steiner triple systems and all Steiner quasigroups [16, Theorem V.1.11]. A Steiner quasigroup is also known as an *idempotent totally symmetric quasigroup* [1, Remark 2.12]. A *Steiner loop* or *sloop* is a pair $(L, \cdot)$ where

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L is a set containing an identity element, say e, and $\cdot$ is an operation on L satisfying the identities

$$e \cdot x = x, \qquad x \cdot x = e, \qquad y \cdot (x \cdot y) = x, \qquad x \cdot y = y \cdot x.$$

If $(V, \mathcal{B})$ is an STS(n), then a Steiner loop $(L, \cdot)$ is obtained by letting $L = V \cup \{e\}$ and defining $x \cdot y = z$ where $\{x, y, z\} \in \mathcal{B}$. Again the process is reversible.

Less well known is the following correspondence. A *Mendelsohn triple system* of order n, MTS(n), is a pair $(V, \mathcal{B})$ where V is a set of n points and $\mathcal{B}$ is a collection of cyclically ordered triples of distinct points taken from V such that every ordered pair of distinct points from V appears in precisely one triple. Such systems exist if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 6$ [14]. Quasigroups and loops can be obtained from Mendelsohn triple systems by precisely the same procedures as described above for Steiner triple systems. Note that the law $y \cdot (x \cdot y) = x$ is usually called semi-symmetric. So the quasigroups are known as *idempotent semisymmetric quasigroups* [1, Remark 2.12]. However the algebraic structures might also appropriately be called *Mendelsohn quasigroups* and *Mendelsohn loops*; they satisfy the same properties as their Steiner counterparts with the exception of commutativity. Similarly there is a one-one correspondence between Mendelsohn triple systems, Mendelsohn quasigroups and Mendelsohn loops.

A *directed triple system* of order n, DTS(n), is a pair $(V, \mathcal{B})$ where V is a set of n points and $\mathcal{B}$ is a collection of transitively ordered triples of distinct points taken from V such that every ordered pair of distinct points from V appears in precisely one triple. Such systems exist if and only if $n \equiv 0$ or $1 \pmod 3$ [10]. Given a DTS(n), an algebraic structure $(V, \cdot)$ can be obtained as above by defining $x \cdot x = x$ and $x \cdot y = z$ for all $x, y \in V$, $x \neq y$ where z is the third element in the transitive triple containing the ordered pair (x, y). However the structure obtained need not necessarily be a quasigroup. If $\langle u, x, y\rangle$ and $\langle y, v, x\rangle \in \mathcal{B}$ then $u \cdot x = v \cdot x = y$. But as we will see, some DTS(n)s do yield quasigroups. Such a DTS(n) will be called a *Latin directed triple system*, and denoted by LDTS(n), to reflect the fact that in this case the operation table forms a Latin square. We call the quasigroup so obtained a *DTS-quasigroup*. In an analogous way to that described above for Steiner triple systems we may also construct a loop from a LDTS(n); called a *DTS-loop*.

2 Properties

First we derive a necessary and sufficient condition for a directed triple system to be Latin.

Proposition 2.1 *Let $D = (V, \mathcal{B})$ be a DTS(n). Denote by $S_{a,b}$ the set of ordered pairs (x, y) in positions a and b respectively of the triples of $\mathcal{B}$. Then D is a LDTS(n) if and only if $S_{1,2} = S_{3,2}$, $S_{2,3} = S_{2,1}$, and $S_{1,3} = S_{3,1}$.*

Proof. Let D be a LDTS(n) and suppose that $\langle x, y, z\rangle \in \mathcal{B}$. Then $y \cdot z = x$. Now there exists w such that precisely one of $\langle y, x, w\rangle$, $\langle y, w, x\rangle$, or $\langle w, y, x\rangle \in \mathcal{B}$. In the first two cases $y \cdot w = x$ and so $w = z$ which is impossible. Therefore $\langle w, y, x\rangle \in \mathcal{B}$ and $S_{1,2} \subset S_{3,2}$. Further $x \cdot y = z$. Similarly one of $\langle w, z, y\rangle$, $\langle z, w, y\rangle$, or $\langle z, y, w\rangle \in \mathcal{B}$.

Again in the first two cases $w \cdot y = z$ and so $w = x$, which is also impossible. Therefore $\langle z, y, w\rangle \in \mathcal{B}$ and so $S_{3,2} \subset S_{1,2}$. Therefore $S_{1,2} = S_{3,2}$. It further follows that $S_{2,3} = S_{2,1}$. Finally since $S_{1,2} \cup S_{1,3} \cup S_{2,3} = S_{3,2} \cup S_{3,1} \cup S_{2,1}$ and all of the sets $S_{a,b}$ are disjoint, it follows that $S_{1,3} = S_{3,1}$.

Conversely suppose that $\langle x, y, z\rangle \in \mathcal{B}$. Then $x \cdot y = z$. For D to be a LDTS(n) we require that the equations $\alpha \cdot y = z$, $x \cdot \beta = z$, and $x \cdot y = \gamma$ have unique solutions, namely x, y, and z respectively for α, β, and γ. Clearly z is the unique solution for γ by definition. If $x \cdot \beta = z$ then precisely one of $\langle x, z, \beta\rangle$, $\langle x, \beta, z\rangle$, or $\langle z, x, \beta\rangle \in \mathcal{B}$. In the first case no such block exists, in the second case $\beta = y$, and in the third case no such block exists because $S_{1,3} = S_{3,1}$. If $\alpha \cdot y = z$ then precisely one of $\langle \alpha, y, z\rangle$, $\langle \alpha, z, y\rangle$, or $\langle z, \alpha, y\rangle \in \mathcal{B}$. In the first case $\alpha = x$ and in the other two cases no such block exists because $S_{2,3} = S_{2,1}$. Further if $\langle x, y, z\rangle \in \mathcal{B}$ then $x \cdot z = y$ and $y \cdot z = x$ and we need to show that for each equation, given any two of the parameters, the third is uniquely determined. The proof is similar to the case for the equation $x \cdot y = z$. □

The conditions for a LDTS(n) given in the above proposition can be simplified but we have chosen to present them in this form because they are reminiscent of those ($S_{1,2} = S_{2,1}$, $S_{2,3} = S_{3,2}$, and $S_{1,3} = S_{3,1}$) for another class of directed triple systems, so called *Mendelsohn directed triple systems*, the existence of which was discussed in [9]. A more succinct necessary and sufficient condition is given in the next theorem

Theorem 2.2 *Let $D = (V, \mathcal{B})$ be a DTS(n). Then D is a LDTS(n) if and only if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle w, y, x\rangle \in \mathcal{B}$ for some $w \in V$.*

Proof. In the notation of Proposition 2.1, the condition in this theorem is $S_{1,2} \subset S_{3,2}$ which is trivially implied by the conditions in the proposition. We need to show that the reverse is also true. Since the cardinalities of the sets $S_{1,2}$ and $S_{3,2}$ are equal it follows that $S_{1,2} = S_{3,2}$ which, as observed in the proof of the proposition, implies the other two conditions. □

Before discussing existence and enumeration results for DTS-quasigroups and DTS-loops, it is important to point out two fundamental differences between these and their Steiner and Mendelsohn counterparts. The first concerns flexibility. The *flexible law* states that $x \cdot (y \cdot x) = (x \cdot y) \cdot x$. As is easily verified, both Steiner quasigroups and loops and Mendelsohn quasigroups and loops all satisfy this law. But this is not the case for DTS-quasigroups and loops. Next we state and prove a necessary and sufficient condition for a DTS-quasigroup or loop to satisfy the flexible law.

Theorem 2.3 *A DTS-quasigroup or DTS-loop obtained from a LDTS(n), $D = (V, \mathcal{B})$ satisfies the flexible law if and only if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle x, z \cdot x, y \cdot x\rangle \in \mathcal{B}$.*

Proof. Suppose that $\langle x, y, z\rangle \in \mathcal{B}$. Then there exists $\alpha, \beta, \gamma \in V$ such that $\langle z, y, \alpha\rangle$, $\langle z, \beta, x\rangle$, $\langle \gamma, y, x\rangle \in \mathcal{B}$. Here we allow any of the equalities $\alpha = x$, $\beta = y$, $\gamma = z$ to be satisfied in which case all three are. Consider the six possibilities.
(a) $x \cdot (y \cdot x) = x \cdot \gamma$; $(x \cdot y) \cdot x = z \cdot x = \beta$; hence we require $x \cdot \gamma = \beta$.
(b) $y \cdot (x \cdot y) = y \cdot z = x$; $(y \cdot x) \cdot y = \gamma \cdot y = x$.

(c) $y \cdot (z \cdot y) = y \cdot \alpha = z$; $(y \cdot z) \cdot y = x \cdot y = z$.
(d) $z \cdot (y \cdot z) = z \cdot x = \beta$; $(z \cdot y) \cdot z = \alpha \cdot z$; hence we require $\alpha \cdot z = \beta$.
(e) $z \cdot (x \cdot z) = z \cdot y = \alpha$; $(z \cdot x) \cdot z = \beta \cdot z$; hence we require $\beta \cdot z = \alpha$.
(f) $x \cdot (z \cdot x) = x \cdot \beta$; $(x \cdot z) \cdot x = y \cdot x = \gamma$; hence we require $x \cdot \beta = \gamma$.
Thus the flexible law is satisfied if and only if (i) $\langle x, \beta, \gamma \rangle = \langle x, z \cdot x, y \cdot x \rangle \in \mathcal{B}$ and (ii) $\langle \alpha, \beta, z \rangle = \langle z \cdot y, z \cdot x, z \rangle \in \mathcal{B}$. To complete the proof we need to show that the second condition can be derived from the first. We have that $\langle z, y, \alpha \rangle \in \mathcal{B}$ and the first condition implies that $\langle z, \alpha \cdot z, y \cdot z \rangle = \langle z, \alpha \cdot z, x \rangle \in \mathcal{B}$ so that $\alpha \cdot z = \beta$, i.e. $\langle \alpha, \beta, z \rangle = \langle z \cdot y, z \cdot x, z \rangle \in \mathcal{B}$. $\square$

By analogy we will say that a LDTS(n) is flexible if the DTS-quasigroup and DTS-loop obtained from it satisfies the flexible law. Later, we will also use partial LDTS(n). We define these as partial DTS(n) which satisfy the conditions of Proposition 2.1 (<u>not</u> Theorem 2.2). These are not the same for partial systems; the set of directed triples $\langle x, a, y \rangle$, $\langle y, a, z \rangle$, $\langle z, a, x \rangle$ which are a partial DTS(4) satisfy the condition of Theorem 2.2 but not the conditions of Proposition 2.1 and so are <u>not</u> a partial LDTS(4). If they are augmented by directed triples $\langle y, b, x \rangle$, $\langle z, b, y \rangle$, $\langle x, b, z \rangle$ then we have a partial LDTS(5). Partial LDTS(n) will be called flexible or non-flexible depending on whether they satisfy the condition of Theorem 2.3.

The second difference between Latin directed triple systems and Steiner or Mendelsohn triple systems is that with the former there is <u>not</u> a one-one correspondence between the triple systems and the associated quasigroups or loops. Suppose that we are given the operation table of a DTS-quasigroup or DTS-loop. We wish to recover the LDTS(n), $(V, \mathcal{B})$, from which it came. Choose x, y, z, $x \neq y \neq z \neq x$ with $x \cdot y = z$. Then $\langle x, y, z \rangle$ or $\langle x, z, y \rangle$ or $\langle z, x, y \rangle \in \mathcal{B}$. In order to identify which of these three possibilities is the correct one perform a number of tests:

- if $x \cdot z \neq y$, then $\langle z, x, y \rangle \in \mathcal{B}$.
- if $z \cdot y \neq x$, then $\langle x, y, z \rangle \in \mathcal{B}$.
- if $y \cdot z \neq x$ and $z \cdot x \neq y$, then $\langle x, z, y \rangle \in \mathcal{B}$.

Otherwise, $x \cdot z = y$, $z \cdot y = x$, and either $y \cdot z = x$ or $z \cdot x = y$. The only inference that can be made is that the set $\mathcal{B}$ contains one of the six directed triples formed by ordering the three points x, y, z, together with its reverse.

In a DTS(n), $(V, \mathcal{B})$, any directed triple $\langle x, y, z \rangle \in \mathcal{B}$ for which also $\langle z, y, x \rangle \in \mathcal{B}$ will be called *bidirectional*. The set $\{x, y, z\}$ will be called a *Steiner triple*. Other directed triples will be called *unidirectional*. From the above discussion, if a LDTS(n) contains a pair of bidirectional directed triples, then these can be replaced by a different pair of bidirectional triples to form a potentially non-isomorphic LDTS(n) yet both will generate the same quasigroup and loop. This is illustrated in the following example. Here and in other places throughout the rest of this paper, where there is no danger of confusion, for simplicity we omit set brackets and commas from directed triples.

Example 2.4 Let $V = \{0, 1, 2, 3, 4, 5, 6\}$.
Define $\mathcal{B} = \{102, 201, 304, 403, 506, 605, 315, 416, 514, 613, 326, 425, 523, 624\}$,
and $\mathcal{B}' = \{012, 210, 034, 430, 056, 650, 315, 416, 514, 613, 326, 425, 523, 624\}$.
Both $(V, \mathcal{B})$ and $(V, \mathcal{B}')$ are LDTS(7)s but are clearly non-isomorphic as consideration

of the distribution of points in the middle position of the directed triples shows. However both give the same DTS-quasigroup.

	0	1	2	3	4	5	6
0	0	2	1	4	3	6	5
1	2	1	0	6	5	3	4
2	1	0	2	5	6	4	3
3	4	5	6	3	0	1	2
4	3	6	5	0	4	2	1
5	6	4	3	2	1	5	0
6	5	3	4	1	2	0	6

The automorphism group of the DTS-quasigroup is the dihedral group $\mathcal{D}_4$ of order 8 generated by the permutations (3 5 4 6) and (1 2)(5 6). Note however that this is not necessarily the automorphism group of the LDTS(7)s. The same group is the automorphism group of $(V, \mathcal{B})$ but not of $(V, \mathcal{B}')$ which has only the identity automorphism.

In view of the above, for purposes of enumeration it makes more sense to count DTS-quasigroups (or DTS-loops; these are in one-one correspondence) rather than the Latin directed triple systems from which they come. Where there are bidirectional triples, the block set $\mathcal{B}$ of a LDTS(n) will be expressed as the union of a set of Steiner triples, $\mathcal{T}$, and a set of unidirectional directed triples, $\mathcal{D}$. Denote the cardinality of $\mathcal{T}$ by t, (so that the number of bidirectional triples is $2t$), and the cardinality of $\mathcal{D}$ by d.

A directed triple system, $(V, \mathcal{B})$, is said to be *pure* if $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle z, y, x\rangle \notin \mathcal{B}$. Pure LDTS($n$) give anti-commutative DTS-quasigroups and, because there are no Steiner triples, there does exist a one-one correspondence between these. At the other extreme, commutative DTS-quasigroups correspond to the situation where every directed triple is bidirectional, i.e. where the LDTS(n) consists of the blocks of a Steiner triple system, each in some order, together with their reverse. In short, commutative DTS-quasigroups and Steiner quasigroups are the same.

In the next section we present some enumeration results for DTS-quasigroups of small order. Then in the rest of the paper we discuss existence results. A necessary condition for the existence of a LDTS(n) is $n \equiv 0, 1 \pmod 3$ and the number of directed triples is $n(n-1)/3$. For $n \equiv 1, 3 \pmod 6$, there exist Steiner quasigroups of these orders and, except for $n = 3$ or 9, by choosing a Steiner triple system containing a Pasch configuration $\{a, b, c\}, \{a, y, z\}, \{x, b, z\}, \{x, y, c\}$ and replacing these Steiner triples by directed triples $\langle a, b, c\rangle, \langle a, y, z\rangle, \langle x, b, z\rangle, \langle x, y, c\rangle, \langle z, y, x\rangle, \langle c, b, x\rangle, \langle c, y, a\rangle, \langle z, b, a\rangle$ a DTS-quasigroup which is non-commutative is obtained. Replacing a Pasch configuration by the above set of directed triples is an important technique which will be used extensively in the next two sections. Note that the set of directed triples is a partial LDTS(6) and is flexible. We will denote it by $\mathcal{P}$.

But replacing a single Pasch configuration means that most of the triples will still be bidirectional. It would be of more interest to construct pure LDTS(n) or at least ones with relatively few bidirectional triples. In Section 4 we construct flexible LDTS(n) for $n \equiv 1, 3 \pmod 6$ in which the number of unidirectional triples is asymptotic to $n^2/3$. Then in Section 5 we turn our attention to non-flexible systems and determine the complete spectrum for the existence of such LDTS(n). Again in the systems that we construct the number of unidirectional triples is asymptotic to $n^2/3$. We leave existence results for flexible LDTS(n) of even order and pure LDTS(n) to a future paper.

3 Enumeration

We present the enumeration results for DTS-quasigroups of small order in the following theorem.

Theorem 3.1 *The numbers of non-isomorphic DTS-quasigroups of order* $n = 3, 4, 6, 7, 9, 10, 12$ *are* $1, 0, 0, 2, 4, 0, 2$ *respectively.*

We consider each order in turn.
$n = 3$. Trivially the only DTS-quasigroup of order 3 is the Steiner quasigroup of this order.

$n = 4$. Let $V = \{0, 1, 2, 3\}$. Without loss of generality there exists a directed triple $\langle 0, 1, 2\rangle$. Therefore there also exists a directed triple $\langle 2, 1, 0\rangle$ or directed triples $\langle 2, 1, \cdot\rangle$, $\langle 2, \cdot, 0\rangle$, $\langle \cdot, 1, 0\rangle$ where the dots, both here and in other places later, represent yet to be assigned points. Neither of these two possibilities can be completed to form a LDTS(4).

$n = 6$. Let $V = \{0, 1, 2, 3, 4, 5\}$. There will be 10 directed triples in any LDTS(6). So without loss of generality there are directed triples $\langle 0, 1, 2\rangle$, $\langle 0, 3, 4\rangle$, $\langle \cdot, 0, 5\rangle$. But now the unassigned first element in the last block must also be 5.

$n = 7$. Let $V = \{0, 1, 2, 3, 4, 5, 6\}$. Given any directed triple system DTS(n), if the ordering of the points in the blocks is suppressed then a twofold triple system TTS(n) is obtained. There exist 4 non-isomorphic TTS(7)s which are listed in [6, page 61]. It is a straightforward exercise to take each of these in turn and try to construct LDTS(7)s by ordering the blocks. Perhaps it is appropriate to note here that there are 2368 non-isomorphic DTS(7)s, [7], but the extra constraint on Latin directed triple systems makes the exercise considerably easier. However the enumeration can be shortened as follows. In a LDTS(n), $(V, \mathcal{B})$, for $x \in V$, denote by $f(x), m(x), l(x)$, the number of occurrences of the point x in the first, middle, and last positions respectively in <u>unidirectional</u> triples of $\mathcal{B}$. Obviously $f(x) = l(x)$ for all x. Also $\Sigma_{x\in V} f(x) = \Sigma_{x\in V} m(x) = n(n-1)/3 - 2t$, where t is the number of Steiner triples.

Now consider the 4 non-isomorphic TTS(7)s from [6] in turn. It will be convenient to do so in reverse order. System #4 has $t = 0$. So for each point x, $(f(x), m(x)) = (3, 0), (2, 2), (1, 4)$ or $(0, 6)$. But neither $m(x) = 2$ nor $f(x) = 1$ as this would imply that the directed triples come from Steiner triples. So $f(x) = 3$ or 0. But the number of unidirectional triples, 14, is not divisible by 3 and so there is no LDTS(7) from this possibility.

System #3 has one Steiner triple $\{0, 1, 2\}$. So for the three points 0, 1, 2 we have $(f(x), m(x)) = (2, 0)$ or $(0, 4)$ and for the other four points $(f(x), m(x)) = (3, 0)$ or $(0, 6)$. There are two possibilities. The first is that 0, 1, 2 have $(f(x), m(x)) = (2, 0)$, 3, 4 have $(f(x), m(x)) = (3, 0)$, and 5, 6 have $(f(x), m(x)) = (0, 6)$. But then the ordered pairs $(5, 6)$ and $(6, 5)$ cannot occur. The second possibility is that 0, 1, 2 have $(f(x), m(x)) = (0, 4)$ and 3, 4, 5, 6 have $(f(x), m(x)) = (3, 0)$. But this cannot be completed without introducing further Steiner triples. (The problem is equivalent to decomposing the complete directed graph on 4 vertices into three directed 4-cycles which is not possible.)

System #2 has three Steiner triples $\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}$. The six points other than 0 have $(f(x), m(x)) = (2, 0)$ or $(0, 4)$. So there are four points of the first type and two points, say 1 and 2, of the latter type. Without loss of generality the unidirectional triples are $\langle 3, 1, 5\rangle$, $\langle 4, 1, 6\rangle$, $\langle 5, 1, 4\rangle$, $\langle 6, 1, 3\rangle$, $\langle 3, 2, 6\rangle$, $\langle 4, 2, 5\rangle$, $\langle 5, 2, 3\rangle$, $\langle 6, 2, 4\rangle$ and the DTS-quasigroup is the one given in the example in the previous section. It is flexible.

Finally system #1 has seven Steiner triples, i.e. it is two copies of identical STS(7)s and gives the Steiner quasigroup of order 7.

$n = 9$. It is possible, but extremely tedious and time-consuming, to enumerate DTS-quasigroups of order 9 by hand. Perhaps a better approach is to adopt the same technique as for order 7 and use a computer. There exist 36 non-isomorphic TTS(9)s, [15], [13]. These are listed in [6, page 63]. It is a straightforward procedure to take each of them in turn and attempt to order the blocks in order to construct a LDTS(9). We find that there are in fact four DTS-quasigroups of order 9, including the Steiner quasigroup of this order. Details of the other three are given below, referenced as examples.

Example 3.2 Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.
Define $\mathcal{T} = \{\{0, 1, 8\}, \{2, 5, 8\}, \{3, 6, 8\}, \{4, 7, 8\}, \{2, 4, 6\}, \{3, 5, 7\}\}$ and
$\mathcal{D} = \{207, 706, 605, 504, 403, 302, 213, 314, 415, 516, 617, 712\}$.
Then $(V, \mathcal{B})$ is a flexible LDTS(9) with $d = 12$ and $2t = 12$.
The automorphism group of the DTS-quasigroup is the dihedral group $\mathcal{D}_6$ of order 12 generated by the permutations (2 3 4 5 6 7) and (0 1)(2 3)(4 7)(5 6).

Example 3.3 Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.
Define $\mathcal{T} = \{\{0, 1, 8\}, \{2, 3, 4\}, \{2, 7, 8\}, \{3, 6, 8\}, \{4, 5, 8\}, \{5, 6, 7\}\}$ and
$\mathcal{D} = \{026, 035, 047, 125, 137, 146, 520, 531, 621, 640, 730, 741\}$.
Then $(V, \mathcal{B})$ is a non-flexible LDTS(9) with $d = 12$ and $2t = 12$.
For example $(0 \cdot 2) \cdot 0 = 6 \cdot 0 = 4$, whilst $0 \cdot (2 \cdot 0) = 0 \cdot 5 = 3$.
The automorphism group of the DTS-quasigroup is the dihedral group $\mathcal{D}_3$ of order 6 generated by the permutations (2 3 4)(5 7 6) and (0 1)(3 4)(5 6).

Example 3.4 Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.
Define $\mathcal{T} = \{\{0, 1, 2\}, \{3, 5, 7\}, \{4, 6, 8\}\}$ and
$\mathcal{D} = \{308, 316, 324, 403, 415, 427, 504, 518, 526, 605, 617, 623, 706, 714, 728,$
$807, 813, 825\}$.
Then $(V, \mathcal{B})$ is a non-flexible LDTS(9) with $d = 18$ and $2t = 6$.
For example $(3 \cdot 4) \cdot 3 = 2 \cdot 3 = 6$, whilst $3 \cdot (4 \cdot 3) = 3 \cdot 0 = 8$.
The automorphism group of the DTS-quasigroup is the group $\mathcal{D}_3 \times \mathcal{C}_3$ of order 18 generated by the permutations (1 2)(3 4 5 6 7 8) and (0 1 2)(3 5 7).

$n = 10$. Since n is even, $m(x)$ is odd and at least 3. The number of directed triples is 30 and so it follows that for each point x, $(f(x), m(x)) = (3, 3)$ and there are no Steiner triples. The directed triples containing each point x have the format $\langle a, x, b\rangle$, $\langle b, x, c\rangle$, $\langle c, x, a\rangle$. From these form oriented triangles (a, b, c). Collectively, these triangles have the property that they contain a directed edge (α, β) iff they also contain the directed edge (β, α). Hence they can be sewn together along common edges to form an orientable surface. It will be a surface rather than a pseudosurface because $f(x) = l(x) = 3$, i.e. each vertex has valency 3. Now the Euler characteristic, #vertices + #faces − #edges = 10 + 10 − 15 which is odd; a contradiciton. Hence there is no LDTS(10).

$n = 12$. We first present a construction of LDTS(12)s based on a tetrahedron. Let the vertex set be $\{0, 1, 2, 3\}$ and choose a consistent orientation of the faces, say (0 1 2), (0 3 1), (0 2 3), (1 3 2). Each of the four 3-cycles will be regarded as a permutation $\phi_i \in \mathcal{S}_4$, with $\phi_i(i) = i$.
For every $x \in \{0, 1, 2\}$ define sets of directed triples:

$$D_x^+ = \{\langle (x, j), (x + 1, j'), (x, \phi_{j'}(j))\rangle : j, j' \in \{0, 1, 2, 3\}, j \neq j'\}$$

$$D_x^- = \{\langle (x, j), (x + 1, j'), (x, \phi_{j'}^{-1}(j))\rangle : j, j' \in \{0, 1, 2, 3\}, j \neq j'\}$$

For every $x \in \{0, 1, 2\}$ choose $D_x \in \{D_x^+, D_x^-\}$ and regard $\mathcal{D} = D_0 \cup D_1 \cup D_2$ as a set of unidirectional triples. These triples cover every pair $((x, j), (x', j'))$ from the set $\{0, 1, 2\} \times \{0, 1, 2, 3\}$ for which $j \neq j'$. By adjoining Steiner triples $\{(0, j), (1, j), (2, j)\}$ we obtain a LDTS(12).

For each $x \in \{0, 1, 2\}$ there are two choices for D_x corresponding to the chosen orientation. However for isomorphism what is important is whether, for given x and x', these are the same or opposite. There must always be two that are the same so without loss of generality let $D_0 = D_0^+$ and $D_1 = D_1^+$. There are thus two isomorphism types depending on the choice of D_2. In the example below we explicitly list the triples of these two systems, constructed as described, where the ordered pair (x, j) is represented as the integer $4x + j$ with 10 written as T and 11 as E.

Example 3.5 Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E\}$.
Define $\mathcal{T} = \{\{0, 4, 8\}, \{1, 5, 9\}, \{2, 6, T\}, \{3, 7, E\}\}$,
$D_0^+ = \{052, 063, 071, 160, 172, 143, 270, 241, 253, 342, 350, 361\}$,
$D_1^+ = \{496, 4T7, 4E5, 5T4, 5E6, 587, 6E4, 685, 697, 786, 794, 7T5\}$,
$D_2^+ = \{81T, 82E, 839, 928, 93T, 90E, T38, T09, T1E, E0T, E18, E29\}$, and
$D_2^- = \{81E, 829, 83T, 92E, 938, 90T, T39, T0E, T18, E09, E1T, E28\}$.
Let $\mathcal{D}^+ = D_0^+ \cup D_1^+ \cup D_2^+$ and $\mathcal{D}^- = D_0^+ \cup D_1^+ \cup D_2^-$.
Then $(V, \mathcal{T} \cup \mathcal{D}^+)$ and $(V, \mathcal{T} \cup \mathcal{D}^-)$ are both non-flexible LDTS(12)s with $d = 36$ and $2t = 8$.
For example in both systems $(0 \cdot 1) \cdot 0 = 7 \cdot 0 = 2$, whilst $0 \cdot (1 \cdot 0) = 0 \cdot 6 = 3$.
The permutations $(1\ 2\ 3)(5\ 6\ 7)(9\ T\ E)$ and $(0\ 1)(2\ 3)(4\ 5)(6\ 7)(8\ 9)(T\ E)$, which together generate the alternating group $\mathcal{A}_4$ of order 12, stabilize each of the sets $\mathcal{T}, D_0^+, D_1^+, D_2^+$ and D_2^- and give the full automorphism group of the DTS-quasigroup of the LDTS(12), $(V, \mathcal{T} \cup \mathcal{D}^-)$. The other DTS-quasigroup has an additional permutation automorphism $(0\ 4\ 8)(1\ 6\ E)(2\ 7\ 9)(3\ 5\ T)$ to give the full automorphism group of order 36.

In fact the two systems are the only two DTS-quasigroups of this order. We state this formally as a proposition.

Proposition 3.6 *Every DTS-quasigroup of order 12 is isomorphic to one of the two quasigroups given in Example 3.5.*

Proof. The proof was obtained by computer with the help of the model builder **Mace4**, which is part of the package **Prover9** [12]. The procedure can easily be repeated by giving an algebraic description of DTS-quasigroups, generating all models of order 12, and using the isomorphism filter. $\square$

$n \geq 13$. At $n = 13$, the combinatorial explosion takes over. The smallest anti-commutative DTS-quasigroups are of this order. There are 8444 non-isomorphic such systems and an example is given below. However none of them are flexible.

Example 3.7 Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, W\}$.
Define $\mathcal{B} = \mathcal{D} = \{103, 142, 201, 247, 2E3, 2W5, 302, 341, 3E6, 3W7, 406, 4T5, 4E9, 4W8, 504, 518, 539, 5T7, 5E2, 5W6, 605, 619, 628, 6T4, 6E7, 6W3, 709, 715, 743, 7T6, 7E8, 7W2, 807, 816, 82T, 835, 8E4, 8W9, 908, 917, 926, 93T, 9E5, 9W4, T0W, T1E, T29, T38, E0T, E1W, W0E, W1T\}$.
Then $(V, \mathcal{B})$ is a pure non-flexible LDTS(13).
For example $(2 \cdot 3) \cdot 2 = E \cdot 2 = 5$, whilst $2 \cdot (3 \cdot 2) = 2 \cdot 0 = 1$.

In addition there are 1,197,601 non-flexible and 924 flexible (including the 2 Steiner quasigroups) DTS-quasigroups which are not anti-commutative.

It remains to identify the smallest anti-commutative, flexible DTS-quasigroups. The next order to consider is $n = 15$ but first we develop some structural theory of anti-commutative, flexible DTS-quasigroups. Let $D = (V, \mathcal{B})$ be a pure flexible LDTS(n). Suppose that $\langle x, u, y \rangle \in \mathcal{B}$. Then there exists z, v such that $\langle y, u, z \rangle$, $\langle y, v, x \rangle \in \mathcal{B}$ where $z \neq x$, $v \neq u$. So $(y \cdot x) \cdot y = v \cdot y$ and $y \cdot (x \cdot y) = y \cdot u = z$. Therefore $v \cdot y = z$, i.e. $\langle z, v, y \rangle \in \mathcal{B}$. It follows that $\mathcal{B}$ partitions into subsets $\{\langle x_1, u, x_2 \rangle, \langle x_2, u, x_3 \rangle, \ldots, \langle x_{n-1}, u, x_n \rangle, \langle x_n, u, x_1 \rangle, \langle x_2, v, x_1 \rangle, \langle x_3, v, x_2 \rangle, \ldots, \langle x_n, v, x_{n-1} \rangle, \langle x_1, v, x_n \rangle\}, n \geq 3$, which we will call *components*, with each point $u, v, x_1, x_2, \ldots, x_n$ distinct. These components can be thought of as spheres with u and v at the poles, both joined to $x_1, x_2, \ldots, x_n$ around the equator. In the notation used above for the case $n = 7$, for each point x of a LDTS(n), $m(x) \neq 1$, and further, if it is pure $m(x) \neq 2$. Also $n - 1 - m(x)$ is divisible by 2 and the above argument shows that if it is also pure and flexible $n - 1 - m(x)$ is divisible by 4. We now have the following result.

Proposition 3.8 *There is no anti-commutative, flexible DTS-quasigroup of order 15.*

Proof. The constraints that $14 - m(x)$ is divisible by 4 and $m(x) \neq 2$ implies that $m(x) = 14, 10$ or 6. Suppose that there are λ, μ and ν points with each of these three counts, respectively. Then

$$14\lambda + 10\mu + 6\nu = 70 \quad \text{and} \quad \lambda + \mu + \nu = 15.$$

Hence $8\lambda + 4\mu = -20$ which is a contradiction because the coefficients cannot be negative. $\square$

However for $n = 16$, there does exist an anti-commutative flexible DTS-quasigroup. It was found by computer using the package **Paradox** [4].

Example 3.9 Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$.
Define $\mathcal{B} = \mathcal{D} = \{801, 107, 70E, E05, 50F, F0B, B08, 198, 791, E97, 59E, F95, B9F, 89B, B36, 638, 83D, D37, 73F, F3E, E3B, 64B, 846, D48, 74D, F47, E4F, B4E, 1E6, 6EA, AE8, 8E2, 2ED, DEC, CE1, 6F1, AF6, 8FA, 2F8, DF2, CFD, 1FC, 03C, C39, 93A, A30, C40, 94C, A49, 04A, 312, 214, 413, 253, 452, 354, 026, 629, 920, 6D0, 9D6, 0D9, 56C, C67, 765, C85, 78C, 587, 27A, A7B, B72, AC2, BCA, 2CB, 1AF, FAD, DA1, FB1, DBF, 1BD\}$.
Then $(V, \mathcal{B})$ is a pure, flexible LDTS(16). It has only the identity automorphism.

The next order to consider is $n = 18$ and again we can use the theory developed above to prove that there is no pure, flexible LDTS(n) of this order.

Proposition 3.10 *There is no anti-commutative, flexible DTS-quasigroup of order 18.*

Proof. Since 4 divides $17 - m(x)$ and $m(x) \neq 1$ then $m(x) = 17, 13, 9$ or 5. Suppose that there are λ, μ, ν and ρ points with each of these four counts, respectively. Then

$$17\lambda + 13\mu + 9\nu + 5\rho = 102 \text{ and } \lambda + \mu + \nu + \rho = 18.$$

Further $\lambda = 0$ or 1.
If $\lambda = 1$ then

$$13\mu + 9\nu + 5\rho = 85 \text{ and } \mu + \nu + \rho = 17.$$

Hence $8\mu + 4\nu = 0$ and the only solution is $(\lambda, \mu, \nu, \rho) = (1, 0, 0, 17)$. With this distribution, it is not possible to construct a pure, flexible LDTS(18) composed of components as required.
If $\lambda = 0$ then

$$13\mu + 9\nu + 5\rho = 102 \text{ and } \mu + \nu + \rho = 18.$$

Hence $8\mu + 4\nu = 12$ so $(\lambda, \mu, \nu, \rho) = (0, 0, 3, 15)$ or $(0, 1, 1, 16)$. Again it is not possible to construct a pure, flexible LDTS(18) composed of components. $\square$

For $n = 19$, the equations lead to a unique distribution. We have that 4 divides $18 - m(x)$ and since $m(x) \neq 2$ it follows that $m(x) = 18, 14, 10$ or 6. Proceeding as before let there be λ, μ, ν and ρ points with each of these four counts, respectively. Then

$$18\lambda + 14\mu + 10\nu + 6\rho = 114 \text{ and } \lambda + \mu + \nu + \rho = 19$$

with again $\lambda = 0$ or 1.
If $\lambda = 1$ then

$$14\mu + 10\nu + 6\rho = 96 \text{ and } \mu + \nu + \rho = 18.$$

Hence $8\mu + 4\nu = -12$ and there is no solution.
If $\lambda = 0$ then

$$14\mu + 10\nu + 6\rho = 114 \text{ and } \mu + \nu + \rho = 19.$$

Hence $8\mu + 4\nu = 0$ and the only solution is $(\lambda, \mu, \nu, \rho) = (0, 0, 0, 19)$. This leaves open the possibility of an anti-commutative, flexible DTS-quasigroup with a cyclic automorphism and indeed such a system does exist.

Example 3.11 Let $V = \mathcal{Z}_{19}$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i \mapsto i + 1$.
The starter blocks for $\mathcal{B} = \mathcal{D}$ are $\langle 0, 1, 6\rangle$, $\langle 6, 1, 9\rangle$, $\langle 9, 1, 0\rangle$, $\langle 6, 2, 0\rangle$, $\langle 0, 2, 9\rangle$, $\langle 9, 2, 6\rangle$.
Then $(V, \mathcal{B})$ is a pure, flexible LDTS(19).

4 Flexible LDTS

Our constructions of flexible LDTS(n) are of two types. The first of these uses the well-known so-called "doubling" construction for Steiner triple systems and is particularly simple. It deals with the residue classes 3, 7 (mod 12). The details are given in the proof of the following proposition.

Proposition 4.1 *There exists a flexible LDTS(n) for all $n \equiv$ 3, 7 (mod 12).*

Proof. Put $m = (n - 1)/2$ and choose an STS(m), $(V, \mathcal{B})$. Let $V' = \{x' : x \in V\}$ and $W = V \cup V' \cup \{\infty\}$. Construct a collection of triples $\mathcal{B}'$ as follows. For all $\{x, y, z\} \in \mathcal{B}$, assign $\{x, y, z\}, \{x, y', z'\}, \{x', y, z'\}, \{x', y', z\} \in \mathcal{B}'$. Further let $\{x, x', \infty\} \in \mathcal{B}'$ for all $x \in V$. Then $(W, \mathcal{B}')$ is an STS(n). In order to obtain a LDTS(n) replace each Pasch configuration as above by the set $\mathcal{P}$ of directed triples and retain the sets containing the point ∞ as Steiner triples. Because the LDTS(n) is constructed of flexible components, i.e. just the flexible partial LDTS(6), $\mathcal{P}$, and the trivial squag on 3 points, it is also flexible. The number of unidirectional triples, $d = (n - 1)(n - 3)/3$ and the number of bidirectional triples, $2t = n - 1$. $\square$

The second construction of LDTS(n) uses a standard technique (Wilson's fundamental construction). For this we need the concept of a *group divisible design* (GDD). Recall that a 3-GDD of type g^u is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where V is a base set of cardinality $v = gu$, $\mathcal{G}$ is a partition of V into u subsets of cardinality g called *groups* and $\mathcal{B}$ is a family of triples called *blocks* which collectively have the property that every pair of elements from different groups occur in precisely one block but no pair of elements from the same group occur at all. We will also need 3-GDDs of type $g^u m^1$. These are defined analogously, with the base set V being of cardinality $v = gu + m$ and the partition G being into u subsets of cardinality g and one set of cardinality m. Necessary and sufficient conditions for 3-GDDs of type g^u were determined in [3] and for 3-GDDs of type $g^u m^1$ in [5]; a convenient reference is [8] where the existence of all the GDDs that are used can be verified.

We will also need the following system.

Example 4.2 Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, W\}$.
Define $\mathcal{T}$=$\{\{0, 4, 5\}, \{1, 7, 9\}, \{1, T, W\}, \{3, 5, 8\}, \{3, 7, W\}, \{5, 9, T\}\}$ and
$\mathcal{D} = \{103, 142, 156, 18E, 201, 243, 257, 28W, 302, 341, 60E, 629, 63T, 647, 65W, 681, 706, 74T, 75E, 782, 80T, 849, 908, 92E, 936, 94W, T07, T26, T3E, T48, E0W, E2T, E39, E46, E51, E87, W09, W4E, W52, W86\}$.
Then $(V, \mathcal{B})$ is a flexible LDTS(13) with $d = 40$ and $2t = 12$.

We can now prove the following proposition.

Proposition 4.3 *There exists a flexible LDTS(n) for all $n \equiv$ 1, 9 (mod 12).*

Proof. The proof is divided into different residue classes.
(a) $n \equiv 1 \pmod{12}$. Take a 3-GDD of type $6^s, s \geq 3$. Inflate each point by a factor 2 and adjoin an extra point ∞. On each inflated group, together with the point ∞, place a flexible LDTS(13) given in Example 4.2. On each inflated block place the set $\mathcal{P}$ of directed triples $\langle a,b,c\rangle$, $\langle a,y,z\rangle$, $\langle x,b,z\rangle$, $\langle x,y,c\rangle$, $\langle z,y,x\rangle$, $\langle c,b,x\rangle$, $\langle c,y,a\rangle$, $\langle z,b,a\rangle$, with the three sets of points $\{a,x\}$, $\{b,y\}$, $\{c,z\}$ as the inflated points in the three groups. We will use $\mathcal{P}$ in this manner throughout. This simple construction gives a flexible LDTS($12s+1$), $s \geq 3$. A count shows that $d = (n-1)(n-3)/3$ and $2t = n-1$.

(b) $n \equiv 9 \pmod{24}$. Take a 3-GDD of type $4^{3s+1}, s \geq 1$. Inflate each point by a factor 2 and adjoin an extra point ∞. On each inflated group, together with the point ∞, place a flexible LDTS(9) given in Example 3.2. On each inflated block place the set of directed triples $\mathcal{P}$. This gives a flexible LDTS($24s+9$), $s \geq 1$ with $d = (n-1)(2n-9)/6$ and $2t = 3(n-1)/2$.

(c) $n \equiv 21 \pmod{24}$. Take a 3-GDD of type $4^{3s+1}6^1, s \geq 1$. Inflate each point by a factor 2 and adjoin an extra point ∞. On each inflated group of cardinality 8, together with the point ∞, place a flexible LDTS(9) given in Example 3.2 and on the inflated group of cardinality 12, together with the point ∞, place a flexible LDTS(13) given in Example 4.2. On each inflated block place the set of directed triples $\mathcal{P}$. This gives a flexible LDTS($24s+21$), $s \geq 1$ with $d = (2n^2 - 11n + 45)/6$ and $2t = 3(n-5)/2$.

(d) The above constructions complete the proof of the proposition except for the two values $n = 21$ in (c) and $n = 25$ in (a). These too can be constructed by GDD techniques. For $n = 21$ take a 3-GDD of type 3^3. Inflate each point by a factor 2 and adjoin three extra points $\infty_1, \infty_2, \infty_3$. On each inflated group, together with the three extra points, place a flexible LDTS(9) given in Example 3.2 in such a way that the triple $\{\infty_1, \infty_2, \infty_3\}$ is identified with the same Steiner triple in each LDTS(9). On each inflated block place the set of directed triples $\mathcal{P}$. This gives a flexible LDTS(21) with $d = 108$ and $2t = 32$. For $n = 25$ take a 3-GDD of type 4^3. Inflate each point by a factor 2 and adjoin an extra point ∞. On each inflated group, together with the point ∞, place a flexible LDTS(9) given in Example 3.2. On each inflated block place the set of directed triples $\mathcal{P}$. This gives a flexible LDTS(25) with $d = 164$ and $2t = 36$. $\square$

Combining the results of the above two propositions we have proved the following result.

Theorem 4.4 *There exists a flexible LDTS(n) for all $n \equiv$ 1, 3 (mod 6).*

5 Non-flexible LDTS

Our constructions of non-flexible LDTS(n) use a variety of techniques and divide into different residue classes. The first proposition deals with the case where n is divisible by 3 and is a modification of the well-known Bose construction. First we recall some basic definitions.

Two Latin squares L and M are said to be mutually orthogonal if $L(x,y) =$

$L(x', y')$ and $M(x, y) = M(x', y')$ implies that $x = x'$ and $y = y'$. A Latin square L is said to be self-orthogonal if it is mutually orthogonal to its transpose L'. The diagonal of a self-orthogonal Latin square is a *transversal*, i.e. it contains every element precisely once; thus by relabelling the elements, a self-orthogonal Latin square can be made *idempotent*, i.e. $L(i, i) = i$.

Proposition 5.1 *There exists a non-flexible LDTS(n) for all $n \equiv 0 \pmod 3$, except $n = 3, 6$.*

Proof. Let $m = n/3$ and L be a self-orthogonal Latin square of side m, with the rows, columns, and entries in $\mathbb{Z}_m$ and labelled in such a way as to be idempotent. Such a square exists for all $m \neq 2, 3, 6$, [2]. Denote the entry in row x, column y by $x \star y$.
Let $V = \mathbb{Z}_m \times \mathbb{Z}_3$. Let $\mathcal{D}$, the set of unidirectional triples, be

$$\langle (x, i), (x \star y, i + 1), (y, i) \rangle, \; x, y \in \mathcal{Z}_m, \; x \neq y, \; i \in \mathcal{Z}_3$$

and $\mathcal{T}$, the set of Steiner triples, be

$$\{(x, 0), (x, 1), (x, 2)\}, \; x \in \mathcal{Z}_m.$$

Then $(V, \mathcal{B}) = (V, \mathcal{D} \cup \mathcal{T})$ is a LDTS(n). For $m = 1$ it produces the squag of order 3. We show that for $m \neq 1$ it is not flexible. Choose any $x, y \in \mathcal{Z}_m$, $x \neq y$.
Now $[(x, i) \cdot (y, i)] \cdot (x, i) = (x \star y, i + 1) \cdot (x, i) = (z, i)$ where $z \star x = x \star y$.
Also $(x, i) \cdot [(y, i) \cdot (x, i)] = (x, i) \cdot (y \star x, i + 1) = (w, i)$ where $x \star w = y \star x$.
If $w = z$ then $(x \star y, y \star x) = (z \star x, x \star z)$ which violates L being self-orthogonal. Hence $w \neq z$ and the LDTS(n) is non-flexible. The number of unidirectional triples, $d = 3m(m-1) = n(n-3)/3$ and the number of bidirectional triples, $2t = 2m = 2n/3$.

It remains to consider the three values of m for which there does not exist a self-orthogonal Latin square. By Theorem 3.1, for $m = 2$, there is no LDTS(6). For $m = 3$, non-flexible LDTS(9)s are given in Examples 3.3 and 3.4. For $m = 6$, we remark that the full force of self-orthogonality is not required in the above construction. Using the idempotent anti-symmetric Latin square below will produce a LDTS(18) which is non-flexible.

	0	1	2	3	4	5
0	0	5	4	1	3	2
1	4	1	5	0	2	3
2	3	0	2	5	1	4
3	2	4	1	3	5	0
4	5	3	0	2	4	1
5	1	2	3	4	0	5

$[(0, 0) \cdot (1, 0)] \cdot (0, 0) = (5, 1) \cdot (0, 0) = (4, 0)$, whilst $(0, 0) \cdot [(1, 0) \cdot (0, 0)] = (0, 0) \cdot (4, 1) = (2, 0)$. □

Next we deal with the case where $n \equiv 1 \pmod 6$. The following example is a non-flexible LDTS(13).

Example 5.2 Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, W\}$.
Define $\mathcal{T}$=$\{\{0, 7, 8\}, \{1, 8, T\}, \{3, 8, 9\}, \{6, 8, W\}, \{4, 7, W\}, \{4, 9, T\}\}$ and

$\mathcal{D} = \{012, 046, 053, 0E9, 0WT, 145, 1WE, 213, 240, 256, 2T7, 2E8, 2W9, 310, 34E, 357, 3T6, 3W2, 548, 5ET, 5W1, 619, 643, 650, 6T2, 6E7, 716, 759, 7T3, 7E2, 842, 8E5, 917, 952, 9E6, 9W0, TE0, TW5, E41, EW3\}$.
Then $(V, \mathcal{B})$ is a non-flexible LDTS(13) with $d = 40$ and $2t = 12$.
For example $(0 \cdot 1) \cdot 0 = 2 \cdot 0 = 4$, whilst $0 \cdot (1 \cdot 0) = 0 \cdot 3 = 5$.

Proposition 5.3 *There exists a non-flexible LDTS(n) for all $n \equiv 1 \pmod 6$, except $n = 7$.*

Proof. We have already noted that there is no non-flexible LDTS(7) and a non-flexible LDTS(13) is given in the above example. Let $m \geq 3$ and put $n = 6m+1$. Let $(V, \mathcal{B}) = (V, \mathcal{D} \cup \mathcal{T})$ be a non-flexible LDTS($3m$), constructed as in the proof of the previous proposition. We form a LDTS($6m + 1$) as follows. Let $V' = \{x' : x \in V\}$ and $W = V \cup V' \cup \{\infty\}$. Construct a collection of triples $\mathcal{B}'$ as follows. For all $\langle x, y, z\rangle \in \mathcal{D}$, assign $\langle x, y, z\rangle, \langle x, y', z'\rangle, \langle x', y, z'\rangle, \langle x', y', z\rangle \in \mathcal{D}'$. In addition for all $\{x, y, z\} \in \mathcal{T}$ assign $\langle x, y, z\rangle, \langle x, y', z'\rangle, \langle x', y, z'\rangle, \langle x', y', z\rangle, \langle z', y', x'\rangle, \langle z, y, x'\rangle, \langle z, y', x\rangle, \langle z', y, x\rangle \in \mathcal{D}'$. Further let $\{x, x', \infty\} \in \mathcal{T}'$, the set of Steiner triples in the LDTS($6m + 1$), for all $x \in V$. Let $\mathcal{B}' = \mathcal{D}' \cup \mathcal{T}'$. Then $(W, \mathcal{B}')$ is a non-flexible LDTS(n) with $d = (n - 1)(n - 3)/3$ and $2t = n - 1$. □

Next we deal with the case where $n \equiv 4 \pmod{12}$. First we give three examples for the cases $n = 16, 28, 40$. The first of these is used in the proposition below, the proof of which again uses GDD techniques. The other two examples give the values which the method misses.

Example 5.4 Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$.
Define $\mathcal{T}$=$\{\{0, 1, 2\}, \{0, 3, 4\}\}$ and
$\mathcal{D} = \{135, 14A, 1C6, 236, 25C, 297, 2D8, 2F4, 468, 47B, 4D2, 4FC, 506, 537, 541, 5AF, 5B9, 5ED, 60F, 631, 692, 6A7, 6CD, 6E5, 705, 71F, 732, 78E, 796, 7AD, 80C, 819, 83A, 852, 86B, 8D4, 90D, 91E, 938, 945, 9BF, 9CA, A08, A2B, A39, A4E, AC1, B0A, B18, B2E, B3D, B64, B7C, C0E, C3B, C58, C74, CF2, D07, D1B, D3F, DA5, DC9, DE6, E0B, E17, E2A, E3C, E49, E8F, F09, F1D, F3E, F87, FA6, FB5\}$.
Then $(V, \mathcal{B})$ is a non-flexible LDTS(16) with $d = 76$ and $2t = 4$.
For example $(1 \cdot 3) \cdot 1 = 5 \cdot 1 = 4$, whilst $1 \cdot (3 \cdot 1) = 1 \cdot 6 = C$.

Example 5.5 Let $V = \mathcal{Z}_{14} \times \mathcal{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $(i, j) \mapsto (i + 1, j)$.
The starter blocks for $\mathcal{T}$ are $\{(0, 0), (1, 0), (3, 0)\}$ and $\{(0, 0), (4, 0), (0, 1)\}$ and for $\mathcal{D}$ are $\langle(0, 0), (9, 0), (12, 1)\rangle$, $\langle(0, 0), (1, 1), (7, 0)\rangle$, $\langle(0, 0), (6, 1), (11, 1)\rangle$, $\langle(0, 0), (7, 1), (5, 1)\rangle$, $\langle(0, 0), (8, 1), (4, 1)\rangle$, $\langle(0, 0), (9, 1), (8, 0)\rangle$, $\langle(0, 0), (13, 1), (6, 0)\rangle$, $\langle(0, 1), (11, 0), (13, 1)\rangle$, $\langle(0, 1), (12, 0), (3, 0)\rangle$, $\langle(0, 1), (2, 1), (10, 0)\rangle$, $\langle(0, 1), (4, 1), (7, 1)\rangle$, $\langle(0, 1), (8, 1), (9, 0)\rangle$, $\langle(0, 1), (9, 1), (1, 1)\rangle$, $\langle(0, 1), (11, 1), (2, 0)\rangle$.
Then $(V, \mathcal{B})$ is a non-flexible LDTS(28) with $d = 196$ and $2t = 56$.
For example $[(0, 0) \cdot (1, 1)] \cdot (0, 0) = (7, 0) \cdot (0, 0) = (8, 1)$, whilst $(0, 0) \cdot [(1, 1) \cdot (0, 0)] = (0, 0) \cdot (6, 0) = (13, 1)$.

Example 5.6 Let $V = \mathcal{Z}_{20} \times \mathcal{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $(i,j) \mapsto (i+1,j)$.
The starter blocks for $\mathcal{T}$ are $\{(0,0),(1,0),(3,0)\}$, $\{(0,0),(4,0),(9,0)\}$, $\{(0,0),(8,0),(0,1)\}$ and for $\mathcal{D}$ are $\langle(0,0),(1,1),(10,0)\rangle$, $\langle(0,0),(2,1),(7,0)\rangle$, $\langle(0,0),(3,1),(6,0)\rangle$, $\langle(0,0),(4,1),(19,1)\rangle$, $\langle(0,0),(9,1),(10,1)\rangle$, $\langle(0,0),(11,1),(7,1)\rangle$, $\langle(0,0),(14,1),(13,0)\rangle$, $\langle(0,0),(15,1),(13,1)\rangle$, $\langle(0,0),(16,1),(14,0)\rangle$, $\langle(0,0),(17,1),(6,1)\rangle$, $\langle(0,1),(2,0),(7,1)\rangle$, $\langle(0,1),(12,0),(10,1)\rangle$, $\langle(0,1),(15,0),(3,1)\rangle$, $\langle(0,1),(2,1),(13,0)\rangle$, $\langle(0,1),(4,1),(1,0)\rangle$, $\langle(0,1),(5,1),(17,1)\rangle$, $\langle(0,1),(6,1),(10,0)\rangle$, $\langle(0,1),(8,1),(14,0)\rangle$, $\langle(0,1),(11,1),(7,0)\rangle$, $\langle(0,1),(19,1),(13,1)\rangle$.
Then $(V,\mathcal{B})$ is a non-flexible LDTS(40) with $d = 400$ and $2t = 120$.
For example $[(0,0)\cdot(1,1)]\cdot(0,0) = (10,0)\cdot(0,0) = (11,1)$, whilst $(0,0)\cdot[(1,1)\cdot(0,0)] = (0,0)\cdot(7,0) = (2,1)$.

Proposition 5.7 *There exists a non-flexible LDTS(n) for all $n \equiv 4 \pmod{12}$, except $n = 4$.*

Proof. We have already noted that there is no LDTS(4) and non-flexible LDTS(n) for $n = 16, 28, 40$ are given above. Take a 3-GDD of type $6^s 8^1$, $s \geq 3$. Inflate each point by a factor 2. On each inflated group of cardinality 12 place a non-flexible LDTS(12) constructed as in the proof of Proposition 5.1 and on the inflated group of cardinality 16 place a non-flexible LDTS(16) given in Example 5.4. On each inflated block place the set of directed triples $\mathcal{P}$. This gives a non-flexible LDTS($12s+16$), $s \geq 3$ with $d = (n^2 - 3n + 20)/3$ and $2t = 2(n-10)/3$. □

Now we come to the final case where $n \equiv 10 \pmod{12}$. This in turn divides into three different residue classes, for one of which we will need the following example of a non-flexible LDTS(22).

Example 5.8 Let $V = \mathcal{Z}_{11} \times \mathcal{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $(i,j) \mapsto (i+1,j)$.
The starter blocks for $\mathcal{T}$ are $\{(0,0),(1,0),(3,0)\}$ and $\{(0,0),(4,0),(0,1)\}$ and for $\mathcal{D}$ are $\langle(0,0),(5,1),(8,1)\rangle$, $\langle(5,0),(0,0),(3,1)\rangle$, $\langle(1,1),(0,0),(10,1)\rangle$, $\langle(2,1),(0,0),(5,0)\rangle$, $\langle(3,1),(0,0),(2,1)\rangle$, $\langle(3,1),(0,1),(4,1)\rangle$, $\langle(4,1),(0,0),(6,1)\rangle$, $\langle(6,1),(0,0),(1,1)\rangle$, $\langle(9,1),(5,1),(0,0)\rangle$, $\langle(10,1),(0,0),(4,1)\rangle$.
Then $(V,\mathcal{B})$ is a non-flexible LDTS(22) with $d = 110$ and $2t = 44$.
For example $[(1,1)\cdot(0,0)]\cdot(1,1) = (10,1)\cdot(1,1) = (6,0)$, whilst $(1,1)\cdot[(0,0)\cdot(1,1)] = (1,1)\cdot(6,1) = (2,0)$.

Proposition 5.9 *There exists a non-flexible LDTS(n) for all $n \equiv 10 \pmod{12}$ except $n = 10$ and possibly except $n = 58$.*

Proof. We deal with the different residue classes in turn.
(a) $n \equiv 34 \pmod{36}$. Take three copies of a non-flexible LDTS($12s+12$), $s \geq 0$, constructed as in the proof of Proposition 5.1 on point sets $\{\infty, (i,0) : 0 \leq i \leq 12s+10\}$, $\{\infty, (i,1) : 0 \leq i \leq 12s+10\}$, $\{\infty, (i,2) : 0 \leq i \leq 12s+10\}$ respectively. Now take an idempotent, antisymmetric Latin square of side $12s+11$, for example a self-orthogonal Latin square. Adjoin the Steiner triples $\{(x,0),(x,1),(x,2)\}$, $x \in$

$\mathbb{Z}_{12s+11}$ and unidirectional triples $\langle (x,0),(y,1),(x\star y,2)\rangle$ and $\langle (y\star x,2),(y,1),(x,0)\rangle$, $x,y\in\mathbb{Z}_{12s+11}$, $x\neq y$. This gives a non-flexible LDTS$(36s+34)$, $s\geq 0$, with $d=(n^2-5n-2)/3$ and $2t=2(2n+1)/3$.

(b) $n\equiv 10 \pmod{36}$. This case is similar to the previous one but starting with three copies of a non-flexible LDTS$(12s+4)$, $s\geq 1$, constructed as in the proof of Proposition 5.7. This gives a non-flexible LDTS$(36s+10)$, $s\geq 1$, with $d=(n^2-5n+58)/3$ and $2t=2(2n-29)/3$, $n\geq 154$.

(c) $n\equiv 22 \pmod{36}$. The method used in the previous two cases is inapplicable here because of the non-existence of a LDTS$(12s+8)$. We revert to a GDD technique. Take a 3-GDD of type $9^{2s}11^1, s\geq 2$. Inflate each point by a factor 2. On each inflated group of cardinality 18 place a non-flexible LDTS(18) constructed as in the proof of Proposition 5.1 and on the inflated group of cardinality 22 place a non-flexible LDTS(22) given in Example 5.8. On each inflated block place the set of directed triples $\mathcal{P}$. This gives a non-flexible LDTS$(36s+22)$, $s\geq 2$, with $d=(n+8)(n-11)/3$ and $2t=2(n+44)/3$ and just leaves the value $n=58$ undecided. □

It remains only to consider $n=58$. We first need the following example which is of a non-flexible LDTS(24) which contains a LDTS(7) as a subsystem. In fact the LDTS(24) contains three disjoint LDTS(7)s but we will not need this additional property.

Example 5.10 Let $V=\{\mathcal{Z}_7\times\mathcal{Z}_3\}\cup\{\infty_1,\infty_2,\infty_3\}$.
The three disjoint LDTS(7)s are defined by the triples obtained from the following starter blocks under the action of the mapping $(i,j)\mapsto(i,j+1)$ with $\infty_1,\infty_2,\infty_3$ as fixed points.
The starter blocks for the Steiner triples $\mathcal{T}_1$ are $\{(0,0),(4,0),(6,0)\}$, $\{(1,0),(5,0),(6,0)\}$, $\{(2,0),(3,0),(6,0)\}$ and for the unidirectional triples $\mathcal{D}_1$ are $\langle(1,0),(0,0),(3,0)\rangle$, $\langle(1,0),(4,0),(2,0)\rangle$, $\langle(2,0),(0,0),(1,0)\rangle$, $\langle(2,0),(4,0),(5,0)\rangle$, $\langle(3,0),(0,0),(5,0)\rangle$, $\langle(3,0),(4,0),(1,0)\rangle$, $\langle(5,0),(0,0),(2,0)\rangle$, $\langle(5,0),(4,0),(3,0)\rangle$.
The starter blocks for the remaining Steiner triples $\mathcal{T}_2$ are $\{(0,0),(3,1),(3,2)\}$, $\{(3,0),(4,1),(6,2)\}$, $\{(2,0),(6,1),(4,2)\}$, $\{(0,0),(0,1),(0,2)\}$, $\{\infty_1,\infty_2,\infty_3\}$ and for the unidirectional triples $\mathcal{D}_2$ are $\langle(1,0),(0,1),(1,2)\rangle$, $\langle(1,0),(2,1),(5,2)\rangle$, $\langle(1,0),(0,2),(6,1)\rangle$, $\langle(1,0),(2,2),(1,1)\rangle$, $\langle(1,0),(4,2),(5,1)\rangle$, $\langle(2,0),(0,1),(2,2)\rangle$, $\langle(3,0),(1,1),(4,2)\rangle$, $\langle(3,0),(2,2),(6,1)\rangle$, $\langle(4,0),(0,1),(4,2)\rangle$, $\langle(4,0),(0,2),(5,1)\rangle$, $\langle(4,0),(2,2),(3,1)\rangle$, $\langle(5,0),(0,1),(5,2)\rangle$, $\langle(5,0),(2,1),(4,2)\rangle$, $\langle(5,0),(6,1),(1,2)\rangle$, $\langle(5,0),(2,2),(5,1)\rangle$, $\langle(5,0),(3,2),(1,1)\rangle$, $\langle(6,0),(0,1),(6,2)\rangle$, $\langle(6,0),(2,1),(1,2)\rangle$, $\langle(6,0),(5,1),(3,2)\rangle$, $\langle(1,0),(6,2),\infty_1\rangle$, $\langle(3,0),(2,1),\infty_1\rangle$, $\langle(5,0),(0,2),\infty_1\rangle$, $\langle(1,0),(3,1),\infty_2\rangle$, $\langle(2,0),(0,2),\infty_2\rangle$, $\langle(5,0),(4,1),\infty_2\rangle$, $\langle(3,0),(5,2),\infty_3\rangle$, $\langle(4,0),(1,2),\infty_3\rangle$, $\langle(6,0),(0,2),\infty_3\rangle$, $\langle\infty_1,(0,0),(1,2)\rangle$, $\langle\infty_1,(2,0),(3,1)\rangle$, $\langle\infty_1,(6,0),(5,2)\rangle$, $\langle\infty_2,(0,0),(2,2)\rangle$, $\langle\infty_2,(3,0),(5,1)\rangle$, $\langle\infty_2,(4,0),(1,1)\rangle$, $\langle\infty_3,(0,0),(4,2)\rangle$, $\langle\infty_3,(1,1),(3,2)\rangle$, $\langle\infty_3,(5,0),(6,2)\rangle$, $\langle(4,0),\infty_1,(4,1)\rangle$, $\langle(6,0),\infty_2,(6,1)\rangle$, $\langle(2,0),\infty_3,(2,1)\rangle$.
Putting $\mathcal{T}=\mathcal{T}_1\cup\mathcal{T}_2$ and $\mathcal{D}=\mathcal{D}_1\cup\mathcal{D}_2$, then $(V,\mathcal{B})=(V,\mathcal{D}\cup\mathcal{T})$ is a non-flexible LDTS(24) containg three disjoint LDTS(7)s subsystems and with $d=144$ and $2t=40$. For example $[(1,0)\cdot(0,1)]\cdot(1,0)=(1,2)\cdot(1,0)=(2,1)$, whilst $(1,0)\cdot[(0,1)\cdot(1,0)]=(1,0)\cdot\infty_1=(6,2)$.

Proposition 5.11 *There exists a non-flexible LDTS(58).*

Proof. Define sets $\mathcal{N} = \{(\infty, j) : 0 \leq j \leq 6\}$, $\mathcal{M}_k = \{(i,k) : 0 \leq i \leq 16\}$, $k = 0,1,2$. Take three copies of a non-flexible LDTS(24) containing a LDTS(7) as a subsystem, constructed as in Example 5.10 on point sets $\mathcal{N} \cup \mathcal{M}_0$, $\mathcal{N} \cup \mathcal{M}_1$, $\mathcal{N} \cup \mathcal{M}_2$ respectively with in each case the LDTS(7) on the set $\mathcal{N}$. Now take an idempotent, antisymmetric Latin square of side 17, for example a self-orthogonal Latin square. Adjoin the Steiner triples $\{(x,0),(x,1),(x,2)\}, x \in \mathbb{Z}_{17}$ and unidirectional triples $\langle (x,0),(y,1),(x \star y,2)\rangle$ and $\langle (y \star x,2),(y,1),(x,0)\rangle$, $x,y \in \mathbb{Z}_{17}$, $x \neq y$. This gives a non-flexible LDTS(58), with $d = 960$ and $2t = 142$. □

Collecting together all the results in this section gives the following theorem.

Theorem 5.12 *The existence spectrum of non-flexible LDTS(n) is* $n \equiv 0, 1 \pmod 3$, $n \neq 3,4,6,7,10$.

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