K-Theory **28:** 329–351, 2003.

A Question of Nori: Projective Generation of Ideals

S. M. BHATWADEKAR and MANOJ KUMAR KESHARI
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai-400005, India. e-mail: {smb, manoj}@math.tifr.res.in

(Received: June 2002)

Abstract. Let A be a smooth affine domain of dimension d over an infinite perfect field k and let n be an integer such that $2n \geqslant d + 3$. Let $I \subset A[T]$ be an ideal of height n. Assume that $I = (f_1, \dots, f_n) + (I^2T)$. Under these assumptions, it is proved in this paper that $I = (g_1, \dots, g_n)$ with $f_i - g_i \in (I^2T)$, thus settling a question of Nori affirmatively.

Mathematics Subject Classifications (2000): Primary 13C10, secondary 13B25.

Key words: projective modules, affine domain, unimodular elements.

1. Introduction

Let A be a commutative Noetherian ring and let I be an ideal in $A[T]$ such that I/I^2 is generated by n elements. Assume that $n \geqslant \dim(A[T]/I) + 2$. If I contains a monic polynomial, then a result of Mohan Kumar (a proof of which is implicit in the proof of [15], Theorem 5) says that I is a surjective image of a projective $A[T]$-module of rank n with trivial determinant. Subsequently, Mandal improved this result by showing that I is generated by n elements ([12], Theorem 1.2). Now suppose that A is the coordinate ring of the real three sphere and $\mathfrak{m}$ is a real maximal ideal. Let $I = \mathfrak{m}A[T]$. Then, it is easy to see that $\mu(I/I^2) = 3 = \dim(A[T]/I) + 2$. Since $\mathfrak{m}$ is not generated by three elements (see [8]), I cannot be generated by three elements. Such examples show that the above result of Mandal is not valid for an ideal I not containing a monic polynomial without further assumptions. Obviously, one such natural assumption would be that $I(0)$ is generated by n elements, where $I(0)$ denotes the ideal $\{f(0)\colon f(T) \in I\}$ of A. Even then, as shown in ([4], Example 5.2) I may not be generated by n elements. Therefore, it is natural to ask: *what further conditions are needed to conclude that I is generated by n elements*? Towards this goal, motivated by a result from topology (see Appendix by Nori in [13]), Nori posed the following question:

QUESTION. Let A be a smooth affine domain of dimension d over an infinite perfect field k and let n be an integer such that $2n \geqslant d + 3$. Let I be a prime ideal of $A[T]$ of height n such that $A[T]/I$ and $A/I(0)$ are smooth k-algebras. Let P

be a projective A-module of rank n and let ϕ: $P[T] \twoheadrightarrow I/(I^2T)$ be a surjection. Then, can we lift ϕ to a surjection from $P[T]$ to I?

In this paper, we give an affirmative answer (Theorem 4.13) to this question. More precisely, we prove the following theorem:

THEOREM. *Let k be an infinite perfect field and let A be a regular domain of dimension d which is essentially of finite type over k. Let n be an integer such that $2n \geqslant d + 3$. Let $I \subset A[T]$ be an ideal of height n and let P be a projective A-module of rank n. Assume that we are given a surjection ϕ: $P[T] \twoheadrightarrow I/(I^2T)$. Then there exists a surjection Φ: $P[T] \twoheadrightarrow I$ such that Φ is a lift of ϕ.*

Prior to our theorem, the following results were obtained: Mandal ([13], Theorem 2.1) answered the question in affirmative in the case I contains a monic polynomial even without any smoothness condition. An example is given in the case $d = n = 3$ (see [4], Example 6.4) which shows that the question does not have an affirmative answer if we do not assume that I contains a monic polynomial and drop the assumption that A is smooth.

Mandal and Varma ([14], Theorem 4) settled the question, where A is a regular k-spot (i.e. a local ring of a regular affine k-algebra). Subsequently, Bhatwadekar and Raja Sridharan ([4], Theorem 3.8) answered the question in the case $\dim A[T]/I = 1$.

A few words about the method of the proof. The essential ideas are contained in the case where $P = A^n$ is free. To simplify the notation, we denote the ring $A[T]$ by R.

Following an idea of Quillen (see [17]), we show that the collection of elements $s \in A$ such that the surjection ϕ_s can be lifted to a surjection Ψ: ${R_s}^n \twoheadrightarrow I_s$ is an ideal of A. This ideal, in view of the result of Mandal–Varma (the local case), is not contained in any maximal ideal of A and, hence, contains 1. Therefore, we are through.

Denote this collection by $\mathcal{S}$. It is obvious that $\mathcal{S}$ is an ideal if we show that for $s, t \in \mathcal{S}$, $s+t \in \mathcal{S}$. As in [17], we assume that $s + t = 1$. Since A is regular, if some power of s is in I, then, by using Quillen's splitting lemma for an automorphism of ${R_{st}}^n$ which is isotopic to identity, one can easily show that $1 = s + t \in \mathcal{S}$ (for example see [4], Lemma 3.5). The crux of the proof is to reduce the problem to this case. We indicate in brief how this reduction is achieved. First we digress a bit.

The surjection ϕ: $R^n \twoheadrightarrow I/(I^2T)$ can be lifted to Φ': $R^n \twoheadrightarrow I \cap I'$, where I' is an ideal of R of height n comaximal with I (we say I' is residual to I with respect to ϕ). A 'Subtraction principle' (see Theorem 3.7 and Corollary 4.11) says that if the surjection (induced by Φ') ϕ_1: $R^n \twoheadrightarrow I'/(I'^2T)$ has a surjective lift from R^n to I', then ϕ can be lifted to a surjection Φ: $R^n \twoheadrightarrow I$.

Now, using the fact that $t = 1 - s \in \mathcal{S}$, we first show the existence of an ideal I_1 which is residual to I with respect to ϕ and satisfying the additional property that I_1 is comaximal with Rs. Then, using the fact that $s \in \mathcal{S}$, we show that there exists an ideal I_2 which contains a power of s and is residual to I_1. Thus, the desired reduction is achieved.

Since the problem is solved for I_2, a repeated application of a 'Subtraction principle' completes the proof.

The explicit completion of the unimodular vector (a^2, b, c), given by Krusemeyer, also plays a crucial role in the above arguments.

2. Preliminaries

In this section we define some of the terms used in the paper and state some results for later use.

All rings considered in this paper are commutative and Noetherian. All modules considered are assumed to be finitely generated. For a ring A, the Jacobson radical of A is denoted by $\mathcal{J}(A)$.

Let A be a ring and let $A[T]$ be the polynomial algebra in one variable T. Then $A(T)$ denotes the ring obtained from $A[T]$ by inverting all monic polynomials. For an ideal I of $A[T]$ and $a \in A$, $I(a)$ denotes the ideal $\{f(a): f(T) \in I\}$ of A.

Let P be a projective A-module. Then $P[T]$ denotes the projective $A[T]$-module $P \otimes_A A[T]$ and $P(T)$ denotes the projective $A(T)$-module $P[T] \otimes_{A[T]} A(T)$.

Let B be a ring and P a projective B-module. Given an element $\varphi \in P^*$ and an element $p \in P$, we define an endomorphism φ_p as the composite $P \xrightarrow{\varphi} B \xrightarrow{p} P$.

If $\varphi(p) = 0$, then ${\varphi_p}^2 = 0$ and hence $1 + \varphi_p$ is a unipotent automorphism of P.

DEFINITION 2.1. By a 'transvection', we mean an automorphism of P of the form $1 + \varphi_p$, where $\varphi(p) = 0$ and either φ is unimodular in P^* or p is unimodular in P. We denote by $E(P)$ the subgroup of $\mathrm{Aut}\,(P)$ generated by all transvections of P. Note that $E(P)$ is a normal subgroup of $\mathrm{Aut}\,(P)$.

DEFINITION 2.2. Let B be a ring and let P be a projective B-module. An automorphism σ of P is said to be 'isotopic to identity', if there exists an automorphism $\Phi(W)$ of the projective $B[W]$-module $P[W] = P \otimes B[W]$ such that $\Phi(0)$ is the identity automorphism of P and $\Phi(1) = \sigma$.

DEFINITION 2.3. Let B be a ring and P a projective B-module. Elements p_1, $p_2 \in P$ are said to be 'isotopically connected' if there exists an automorphism σ of P such that σ is isotopic to identity and $\sigma(p_1) = p_2$.

Remark 2.4. Let B be a ring and P a projective B-module. Let σ be an automorphism of P and let σ^* be the induced automorphism of P^* defined by $\sigma^*(\alpha) = \alpha\sigma$ for $\alpha \in P^*$.

If $\sigma \in E(P)$ then $\sigma^* \in E(P^*)$. If σ is isotopic to identity then so also is σ^*.

If σ is unipotent then it is isotopic to identity. Therefore any element of $E(P)$ is also isotopic to identity.

Now suppose that $B = A[T]$ and $P = Q[T] = Q \otimes_A A[T]$. Then, since $\mathrm{End}_B(P) = \mathrm{End}_A(Q)[T]$, we regard σ as polynomial in T with coefficients in

$\text{End}_A(Q)$ say $\sigma = \theta(T)$. If $\theta(0)$ is the identity automorphism of Q, then, since $\Phi(W) = \theta(WT)$ is an automorphism of $Q[T, W] = Q \otimes_A A[T, W] = P \otimes_B B[W]$, it follows that σ is isotopic to identity.

The following lemma follows from the well known Quillen splitting lemma ([17], Lemma 1) and its proof is essentially contained in ([17], Theorem 1).

LEMMA 2.5. *Let B be a ring and let P be a projective B-module. Let $a, b \in B$ be such that $Ba + Bb = B$. Let σ be a B_{ab}-automorphism of P_{ab} which is isotopic to identity. Then $\sigma = \tau_a \theta_b$, where τ is a B_b-automorphism of P_b such that $\tau = Id$ modulo the ideal Ba and θ is a B_a-automorphism of P_a such that $\theta = Id$ modulo the ideal Bb.*

The following result is proved in ([3], Proposition 4.1).

PROPOSITION 2.6. *Let B be a ring, I an ideal of B and P a projective B-module. Then any transvection of P/IP can be lifted to an automorphism of P.*

The following result is a consequence of a theorem of Eisenbud–Evans as stated in ([16], p. 1420).

THEOREM 2.7. *Let R be a ring and let P be a projective R-module of rank r. Let $(\alpha, a) \in (P^* \oplus R)$. Then, there exists an element $\beta \in P^*$ such that* $\text{ht}\, I_a \geqslant r$, *where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geqslant r$, then* $\text{ht}\, I \geqslant r$. *Further, if $(\alpha(P), a)$ is an ideal of height $\geqslant r$ and I is a proper ideal of R, then* $\text{ht}\, I = r$.

The following result is due to Lindel ([11], Theorem 2.6).

THEOREM 2.8. *Let B be a ring of dimension d and $R = B[T_1, \ldots, T_n]$. Let P be a projective R-module of rank $\geqslant$ max $(2, d+1)$. Then $E(P \oplus R)$ acts transitively on the set of unimodular elements of $P \oplus R$.*

Now we quote a result of Mandal ([13], Theorem 2.1).

THEOREM 2.9. *Let A be a ring and let $I \subset A[T]$ be an ideal containing a monic polynomial. Let P be a projective A-module of rank $n \geqslant \dim A[T]/I + 2$. Let ϕ: $P[T] \twoheadrightarrow I/(I^2T)$ be a surjection. Then ϕ can be lifted to a surjection Φ: $P[T] \twoheadrightarrow I$.*

The following theorem is due to Mandal and Varma ([14], Theorem 4).

THEOREM 2.10. *Let A be a regular k spot, where k is an infinite perfect field. Let $I \subset A[T]$ be an ideal of height $\geqslant 4$ and let n be an integer such that $n \geqslant \dim A[T]/I + 2$. Let $f_1, \ldots, f_n \in I$ be such that $I = (f_1, \ldots, f_n) + (I^2T)$. Assume that*

$I(0)$ is a complete intersection ideal of A of height n or $I(0) = A$. Then $I = (F_1, \ldots, F_n)$ with $F_i - f_i \in (I^2T)$.

The following proposition is a variant of ([2], Proposition 3.1). We give a proof for the sake of completeness.

PROPOSITION 2.11. *Let B be a ring and let $I \subset B$ be an ideal of height n. Let $f \in B$ be such that it is not a zero divisor modulo I. Let $P = P_1 \oplus B$ be a projective B-module of rank n. Let α: $P \to I$ be a linear map such that the induced map α_f: $P_f \twoheadrightarrow I_f$ is a surjection. Then, there exists $\Psi \in E(P_f{}^*)$ such that* (1) *$\beta = \Psi(\alpha) \in P^*$ and* (2) *$\beta(P)$ is an ideal of B of height n contained in I.*

Proof. Note that, since f is not a zero divisor modulo I and $\alpha_f(P_f) = I_f$, if Δ is an automorphism of $P_f{}^*$ such that $\delta = \Delta(\alpha) \in P^*$, then $\delta(P) \subset I$.

Let $\mathcal{S}$ be the set $\{\ \Gamma \in E(P_f{}^*)$: $\Gamma(\alpha) \in P^*\ \}$. Then $\mathcal{S} \neq \varnothing$, since the identity automorphism of $P_f{}^*$ is an element of $\mathcal{S}$. For $\Gamma \in \mathcal{S}$, let $N(\Gamma)$ denote height of the ideal $\Gamma(\alpha)(P)$. Then, in view of the above observation, it is enough to prove that there exists $\Psi \in \mathcal{S}$ such that $N(\Psi) = n$. This is proved by showing that for any $\Gamma \in \mathcal{S}$ with $N(\Gamma) < n$, there exists $\Gamma_1 \in \mathcal{S}$ such that $N(\Gamma) < N(\Gamma_1)$.

Since $P = P_1 \oplus B$, we write $\alpha = (\theta, a)$, where $\theta \in P_1{}^*$ and $a \in B$. Let $\Gamma \in \mathcal{S}$ be such that $N(\Gamma) < n$. Let $\Gamma((\theta, a)) = (\beta, b) \in P_1{}^* \oplus B$. By (2.7), there exists $\phi \in P_1{}^*$ such that $\operatorname{ht} L_b \geqslant n - 1$, where $L = (\beta + b\phi)(P_1)$. It is easy to see that the automorphism Λ of $P_1{}^* \oplus B$ defined by $\Lambda((\delta, c)) = (\delta + c\phi, c)$ is a transvection of $P_1{}^* \oplus B$ and $\Lambda(\beta, b) = (\beta + b\phi, b)$. Hence, $\Lambda\,\Gamma \in \mathcal{S}$ and moreover $N(\Gamma) = N(\Lambda\,\Gamma)$. Therefore, if necessary, we can replace Γ by $\Lambda\,\Gamma$ and assume that if a prime ideal $\mathfrak{p}$ of B contains $\beta(P_1)$ and does not contain b, then we have $\operatorname{ht}\mathfrak{p} \geqslant n - 1$. Now we claim that $N(\Gamma) = \operatorname{ht}\beta(P_1)$.

We have $N(\Gamma) \leqslant n - 1$. Since $N(\Gamma) = \operatorname{ht}(\beta(P_1), b)$, we have $\operatorname{ht}\beta(P_1) \leqslant N(\Gamma) \leqslant n - 1$. Let $\mathfrak{p}$ be a minimal prime ideal of $\beta(P_1)$ such that $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\beta(P_1)$. If $b \notin \mathfrak{p}$ then $\operatorname{ht}\mathfrak{p} \geqslant n - 1$. Hence, we have the inequalities $n - 1 \leqslant \operatorname{ht}\beta(P_1) \leqslant N(\Gamma) \leqslant n - 1$. This implies that $N(\Gamma) = \operatorname{ht}\beta(P_1) = n - 1$. If $b \in \mathfrak{p}$ then $\operatorname{ht}\beta(P_1) = \operatorname{ht}\mathfrak{p} \geqslant \operatorname{ht}(\beta(P_1), b) = N(\Gamma) \geqslant \operatorname{ht}\beta(P_1)$. This proves the claim.

Let $\mathcal{K}$ denote the set of minimal prime ideals of $\beta(P_1)$. Since P_1 is a projective B-module of rank $n - 1$, if $\mathfrak{p} \in \mathcal{K}$ then $\operatorname{ht}\mathfrak{p} \leqslant n - 1$.

Let $\mathcal{K}_1 = \{\mathfrak{p} \in \mathcal{K}$: $b \in \mathfrak{p}\}$ and let $\mathcal{K}_2 = \mathcal{K} - \mathcal{K}_1$. Note that, since $\operatorname{ht}\beta(P_1) = \operatorname{ht}(\beta(P_1), b)$, $\mathcal{K}_1 \neq \varnothing$. Moreover, every member $\mathfrak{p}$ of $\mathcal{K}_1$ is a prime ideal of height $< n$ which contains $I_1 = (\beta(P_1), b)$. Therefore, since $(I_1)_f = I_f$ and $\operatorname{ht} I = n$, it follows that $f \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{K}_1$.

Since $\bigcap_{\mathfrak{p} \in \mathcal{K}_2} \mathfrak{p} \not\subset \bigcup_{\mathfrak{p} \in \mathcal{K}_1} \mathfrak{p}$, there exists $x \in \bigcap_{\mathfrak{p} \in \mathcal{K}_2} \mathfrak{p}$ such that $x \notin \bigcup_{\mathfrak{p} \in \mathcal{K}_1} \mathfrak{p}$. Since $f \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{K}_1$, we have $xf \in \bigcap_{\mathfrak{p} \in \mathcal{K}} \mathfrak{p}$. This implies that $(xf)^r \in \beta(P_1)$ for some positive integer r.

Let $(xf)^r = \beta(q)$. As before, it is easy to see that the automorphism Φ of $P_1{}^* \oplus B$ defined by $\Phi((\tau, d)) = (\tau, d + \tau(q))$ is a transvection of $P_1{}^* \oplus B$. Let Δ be an automorphism of $(P_1)_f{}^* \oplus B_f$ defined by $\Delta(\eta, c) = (\eta, f^r c)$. Then, since

$E((P_1)_f{}^* \oplus B_f)$ is a normal subgroup of $GL((P_1)_f{}^* \oplus B_f)$, $\Phi_1 = \Delta^{-1}\,\Phi\,\Delta$ is an element of $E((P_1)_f{}^* \oplus B_f)$. Moreover, $\Phi_1((\beta, b)) = (\beta, b + x^r)$.

Let $\Gamma_1 = \Phi_1\,\Gamma$. Then $\Gamma_1(\alpha) = \Gamma_1((\theta, a)) = \Phi_1((\beta, b)) = (\beta, b + x^r)$. Therefore $\Gamma_1 \in \mathcal{S}$. Moreover, since $b + x^r$ does not belong to any minimal prime ideal of $\beta(P_1)$, we have $N(\Gamma) = \text{ht}\,\beta(P_1) < N(\Gamma_1)$. This proves the result. □

3. Subtraction Principle

We begin with the following lemma which is easy to prove.

LEMMA 3.1. *Let B be a ring and let I be an ideal of B. Let $K \subset I$ be an ideal such that $I = K + I^2$. Then $I = K$ if and only if any maximal ideal of B containing K contains I.*

The proof of the next lemma is given in ([5], Lemma 2.11).

LEMMA 3.2. *Let B be a ring and let $I \subset B$ be an ideal. Let I_1 and I_2 be ideals of B contained in I such that $I_2 \subset I^2$ and $I_1 + I_2 = I$. Then $I = I_1 + (e)$ for some $e \in I_2$ and $I_1 = I \cap I'$, where $I_2 + I' = B$.*

LEMMA 3.3. *Let B be a ring and let $I = (c_1, c_2)$ be an ideal of B. Let $b \in B$ be such that $I + (b) = B$ and let r be a positive even integer. Then $I = (e_1, e_2)$ with $c_1 - e_1 \in I^2$ and $b^r c_2 - e_2 \in I^2$.*

Proof. Replacing b by $b^{r/2}$, we can assume that $r = 2$. Since b is a unit modulo $I = (c_1, c_2)$, it is unit modulo $({c_1}^2, {c_2}^2)$. Let $1 - bz = x'{c_1}^2 + y'{c_2}^2 = xc_1 + yc_2$, where $x = x'c_1 \in I$ and $y = y'c_2 \in I$. The unimodular row (z^2, c_1, c_2) has the following Krusemeyer completion ([10]) to an invertible matrix Γ given by

$$\begin{pmatrix} z^2 & c_1 & c_2 \\ -c_1 - 2zy & y^2 & b - xy \\ -c_2 + 2zx & -b - xy & x^2 \end{pmatrix}$$

Let Θ: $B^3 \twoheadrightarrow I$ be a surjective map defined by $\Theta(1, 0, 0) = 0$, $\Theta(0, 1, 0) = -c_2$ and $\Theta(0, 0, 1) = c_1$. Then, since Γ is invertible and $\Theta\,(z^2, c_1, c_2) = 0$, it follows that $I = (d_1, d_2)$, where $d_1 = -y^2 c_2 + c_1(b - xy)$ and $d_2 = c_2(b + xy) + c_1 x^2$. From the construction of elements d_1 and d_2, it follows that $d_1 - c_1 b \in I^2$ and $d_2 - c_2 b \in I^2$. Let $\Delta = \text{diag}\,(z, b) \in M_2(B)$. Since diagonal matrices of determinant 1 are elementary, $\Delta \otimes B/I \in E_2(B/I)$. Since the canonical map $E_2(B) \to E_2(B/I)$ is surjective, there exists $\Phi \in E_2(B)$ such that $\Delta \otimes B/I = \Phi \otimes B/I$. Let $[d_1, d_2]\,\Phi = [e_1, e_2]$. From the construction of Φ, it follows that $I = (e_1, e_2)$ with $e_1 - c_1 \in I^2$ and $e_2 - b^2 c_2 \in I^2$. This proves the lemma. □

LEMMA 3.4. *Let R be a ring and let I be an ideal of R. Let $s \in R$ be such that $I + (s) = R$. Let Q be a projective R-module such that Q/IQ is free and let $P = Q \oplus R^2$. Let $\Phi\colon P \twoheadrightarrow I$ be a surjection. Let r be a positive integer. Then the map $\Phi' = s^r\Phi\colon P \to I$ induces a surjection $\Phi' \otimes R/I\colon P/IP \twoheadrightarrow I/I^2$. Moreover if r is even, then the surjection $\Phi' \otimes R/I$ can be lifted to a surjection $\Psi\colon P \twoheadrightarrow I$.*

Proof. Since $I + (s) = R$ and $\Phi\colon P \twoheadrightarrow I$ is a surjection, it is easy to see that $\Phi' \otimes R/I$ is a surjection from P/IP to I/I^2. Now we assume that $r = 2l$.

Since $P = Q \oplus R^2$, we write $\Phi = (\phi, f_1, f_2)$. Let rank $Q/IQ = n - 2$. Let 'tilde' denote reduction modulo I. Then, since $\widetilde{Q/IQ}$ is free of rank $n - 2$, fixing a basis of Q/IQ, we can write $\widetilde{\Phi} = (\widetilde{k_1, \ldots, k_{n-2}}, \widetilde{f_1}, \widetilde{f_2})$. Let $\beta = \operatorname{diag}(s^r, \ldots, s^r)$. Then $\widetilde{\beta} \in \operatorname{Aut}(P/IP)$ and $\widetilde{\Phi}' = \widetilde{\Phi}\,\widetilde{\beta}$. Since diagonal matrices of determinant 1 are elementary, we get $\widetilde{\beta} = \operatorname{diag}(1, \ldots, 1, \widetilde{s^{nr}})\,\widetilde{\beta}'$, where $\widetilde{\beta}' \in E(P/IP)$. By (2.6), $\widetilde{\beta}'$ can be lifted to an automorphism of P. Therefore, to prove the lemma, it is enough to show that the surjection $(\phi, f_1, s^{nr} f_2) \otimes R/I\colon P/IP \twoheadrightarrow I/I^2$ can be lifted to a surjection $(\phi, g_1, g_2)\colon P \twoheadrightarrow I$. Since nr is even, $s^{nr} = {s_1}^2$. Therefore, replacing s by s_1, we can assume that $nr = 2$.

Let $K = \phi(Q)$ and let 'bar' denote reduction modulo K. Then $\overline{I} = (\overline{f}_1, \overline{f}_2)$. Applying (3.3), we get $\overline{I} = (\overline{h}_1, \overline{h}_2)$ with $\overline{f}_1 - \overline{h}_1 \in \overline{I^2}$ and $\overline{s^2}\overline{f}_2 - \overline{h}_2 \in \overline{I^2}$. Therefore, $I = (h_1, h_2) + K$, where $f_1 - h_1 = f_1' + h_1'$ and $s^2 f_2 - h_2 = f_2' + h_2'$ for some $f_1', f_2' \in I^2$ and $h_1', h_2' \in K$. Let $g_i = h_i + h_i'$ for $i = 1, 2$. Then, we have $I = (g_1, g_2) + K$ with $f_1 - g_1 \in I^2$ and $s^2 f_2 - g_2 \in I^2$. This proves the result. □

LEMMA 3.5. *Let B be a ring and let $s, t \in B$ be such that $Bs + Bt = B$. Let I, L be ideals of B such that $L \subset I^2$. Let P be a projective B-module and let $\phi\colon P \twoheadrightarrow I/L$ be a surjection. If $\phi \otimes B_t$ can be lifted to a surjection $\Phi\colon P_t \twoheadrightarrow I_t$. Then ϕ can be lifted to a surjection $\Psi\colon P \twoheadrightarrow I/(sL)$.*

Proof. Without loss of generality, we can assume that $t = 1$ modulo the ideal (s). Let l be a positive integer such that $t^l\Phi(P) \subset I$. Let $\Phi'\colon P \to I$ be a lift of ϕ. Then, since Φ is a lift of ϕ_t, there exists an integer $r \geqslant l$ such that $(t^r\Phi - t^r\Phi')(P) \subset L$. Let $\Gamma = t^r\Phi$ and $K = \Gamma(P)$. Then, since $r \geqslant l$, $K \subset I$ and $K_t = I_t$. Since $1 - t \in (s)$, we have $K + sI = I$. Let $t^r = 1 - sa$ and let $\Theta = \Gamma + sa\Phi'$. Then $\Theta - \Phi' = \Gamma - t^r\Phi'$. Therefore $(\Theta - \Phi')(P) \subset L$ and, hence, Θ is also lift of ϕ. Moreover, $\Theta(P) + sI = \Gamma(P) + sI = I$. Therefore, by (2.1), $\Theta(P) + sL = I$. If $\Gamma'\colon I \twoheadrightarrow I/sL$ is a canonical surjection, then putting $\Psi = \Gamma'\Theta$, we are through. □

LEMMA 3.6. *Let B be a ring and let I_1, I_2 be two comaximal ideals of B. Let $P = P_1 \oplus B$ be a projective B-module of rank n. Let $\Phi\colon P \twoheadrightarrow I_1$ and $\Psi\colon P \twoheadrightarrow I_1 \cap I_2$ be two surjections such that $\Phi \otimes B/I_1 = \Psi \otimes B/I_1$. Assume that*

(1) $a = \Phi(0, 1)$ *is a non zero divisor modulo the ideal* $\sqrt{\Phi(P_1)}$.
(2) $n - 1 > \dim \overline{B}/\mathcal{J}(\overline{B})$, *where* $\overline{B} = B/(\Phi(P_1))$.

Let $L \subset I_2{}^2$ be an ideal such that $\Phi(P_1) + L = B$. Then, the surjection Ψ: $P \twoheadrightarrow I_1 \cap I_2$ induces a surjection $\overline{\Psi}$: $P \twoheadrightarrow I_2/L$. Moreover, $\overline{\Psi}$ can be lifted to a surjection Λ: $P \twoheadrightarrow I_2$.

Proof. Since $L + I_1 = B$ (in fact $L + \Phi(P_1) = B$) , it is easy to see that Ψ induces a surjection $\overline{\Psi}$: $P \twoheadrightarrow I_2/L$.

Let $K = \Phi(P_1)$ and $S = 1+K$. Then $S \cap L \neq \varnothing$. Therefore, we have surjections Φ_S and Ψ_S from P_S to $(I_1)_S$. □

CLAIM. *There exists an automorphism Δ of P_S such that $\Delta^*(\Psi_S) = \Psi_S\,\Delta = \Phi_S$, where Δ^* is an automorphism of $P_S{}^*$ induced from Δ.*

Assume the claim. Then, there exists $s = 1 + t \in S,\ t \in K$ such that $\Delta \in \mathrm{Aut}\,(P_s)$ and $\Psi_s\,\Delta = \Phi_s$. Since $S \cap L \neq \varnothing$, we can assume that $s \in S \cap L$.

With respect to the decomposition $P = P_1 \oplus B$, we write $\Phi \in P^*$ as (Φ_1, a), where $\Phi_1 \in P_1{}^*$ and $a \in B$. Similarly, we write $\Psi = (\Psi_1, b)$, where $\Psi_1 \in P_1{}^*$ and $b \in B$. Let pr: $P_1 \oplus B (= P) \twoheadrightarrow B$ be the map defined by $pr(p_1, b) = b$, where $p_1 \in P_1$ and $b \in B$.

Since $s \in L$, $(I_2)_s = B_s$ and, therefore, we can regard pr_s as a surjection from $(P_1)_s \oplus B_s$ to $(I_2)_s$. Since $t \in K = \Phi_1(P_1)$, the element $(\Phi_1)_t \in (P_1)_t{}^*$ is a unimodular element. Hence, there exists an element $\Gamma \in E((P_1)_{st} \oplus B_{st})$ such that $\Gamma^*((\Phi_1, a)_{st}) = pr_{st}$, i.e. $(\Phi_t)_s\,\Gamma = (pr_s)_t$. Note that Ψ_t is a surjection from P_t to $(I_2)_t$.

We also have $\Psi_s\,\Delta = \Phi_s$. Hence $(\Psi_s\,\Delta)_t\,\Gamma = (pr_s)_t$. Let $\widetilde{\Delta} = \Delta_t\,\Gamma\,\Delta_t{}^{-1}$. Then we have $(\Psi_s)_t\,\widetilde{\Delta} = (\Psi_t)_s\,\widetilde{\Delta} = (pr_s)_t\,\Delta_t{}^{-1}$. Since Γ is an element of $E(P_{st})$ which is a normal subgroup of $\mathrm{Aut}\,(P_{st})$, $\widetilde{\Delta} \in E(P_{st})$ and hence is isotopic to identity. Therefore, by (2.5), $\widetilde{\Delta} = \Delta''_s\,\Delta'_t$, where Δ' is an automorphism of P_s such that $\Delta' = \mathrm{Id}$ modulo (t) and Δ'' is an automorphism of P_t such that $\Delta'' = \mathrm{Id}$ modulo (s).

Thus we have surjections $(\Psi_t\,\Delta'')$: $P_t \twoheadrightarrow (I_2)_t$ and $(pr_s\,\Delta^{-1}\,(\Delta')^{-1})$: $P_s \twoheadrightarrow (I_2)_s$ such that $(\Psi_t\,\Delta'')_s = (pr_s\,\Delta^{-1}\,(\Delta')^{-1})_t$. Therefore, they patch up to yield a surjection Λ: $P \twoheadrightarrow I_2$. Since $s = 1 + t \in L$, the map $B \to B/(s)$ factors through B_t. Since $\Delta'' = \mathrm{Id}$ modulo (s), we have $\Lambda \otimes B/L = \Psi \otimes B/L$.

Proof of the claim. To simplify the notation, we denote B_S by B, $(P_1)_S$ by P_1 and $(I_1)_S$ by I. Then we have two surjections $\Phi = (\Phi_1, a)$ and $\Psi = (\Psi_1, b)$ from $P_1 \oplus B$ to I such that $\Phi \otimes B/I = \Psi \otimes B/I$. Moreover, $\Phi_1(P_1) = K \subset \mathcal{J}(B)$ and $n - 1$ (rank P_1) $> \dim \overline{B}/\mathcal{J}(\overline{B})$, where $\overline{B} = B/K$. Our aim is to show that there exists an automorphism Δ of $P = P_1 \oplus B$ such that $\Psi\,\Delta = \Phi$.

Hence onward, we write an element $\sigma \in \mathrm{End}\,(P_1 \oplus B)$ in the following matrix form

$$\sigma = \begin{pmatrix} \alpha & p \\ \eta & d \end{pmatrix}, \qquad \text{where } \alpha \in \mathrm{End}\,(P_1),\ p \in P_1,\ \eta \in P_1{}^* \text{ and } d \in B.$$

Note that, with this presentation of $\sigma \in \operatorname{End}(P)$, if $\Theta = (\Theta_1, e) \in {P_1}^* \oplus B$, then $\sigma^*(\Theta) = \Theta\,\sigma = (\Theta_1\,\alpha + e\eta, \Theta_1(p) + ed)$. Moreover, if $\sigma' \in \operatorname{End}(P)$ has a matrix representation $\sigma' = \left(\begin{smallmatrix} \beta & p_1 \\ \mu & f \end{smallmatrix}\right)$, then the endomorphism $\sigma'\sigma$ has the matrix representation

$$\sigma'\sigma = \begin{pmatrix} \beta\,\alpha + \eta_{p_1} & \beta(p) + dp_1 \\ \mu\alpha + f\eta & \mu(p) + fd \end{pmatrix},$$

where $\eta_{p_1} \in \operatorname{End}(P_1)$ is the composite map $P_1 \xrightarrow{\eta} B \xrightarrow{p_1} P_1$.

Since $\Phi \otimes B/I = \Psi \otimes B/I$, there exist $\Gamma, \Gamma' \in \operatorname{End}(P)$ which are identity modulo the ideal I and (1) $\Phi\,\Gamma = \Psi$, (2) $\Psi\,\Gamma' = \Phi$. Let

$$\Gamma = \begin{pmatrix} \gamma & q \\ \zeta & c \end{pmatrix}, \qquad \Gamma' = \begin{pmatrix} \gamma' & q' \\ \zeta' & c' \end{pmatrix}$$

be the matrix representation of Γ and Γ', where $\gamma, \gamma' \in \operatorname{End}(P_1)$, $q, q' \in P_1$, $\zeta, \zeta' \in P_1^*$ and $c, c' \in B$. Then

$$\Gamma\,\Gamma' = \begin{pmatrix} \gamma\gamma' + \zeta'_q & \gamma(q') + c'q \\ \zeta\gamma' + c\zeta' & \zeta(q') + cc' \end{pmatrix}.$$

Since $\Phi\,\Gamma\,\Gamma' = \Phi$, we get $\Phi_1(\gamma(q') + c'q) + a(\zeta(q') + cc') = a$. Hence $a(1 - \zeta(q') - cc') \in K$. Since, by hypothesis, no minimal prime ideal of K contains a, we have $(1 - \zeta(q') - cc') \in \sqrt{K}$, i.e. $(\zeta(q') + cc') + \sqrt{K} = B$. But $K \subset \mathcal{J}(B)$ and hence $(\zeta(q') + cc') = B$, i.e. the element $\zeta(q') + cc' \in B^*$. Therefore $(\zeta, c) \in P^*$ is a unimodular element. Note that, since Γ is an endomorphism of P which is identity modulo I, $(\zeta, c) = (0, 1)$ modulo I. Now, we show that there exists an automorphism Δ_1 of P such that (1) $(\zeta, c)\,\Delta_1 = (0, 1)$ and (2) Δ_1 is an identity automorphism of P modulo I.

Let 'bar' denote reduction modulo K. Since $\dim \overline{B}/\mathcal{J}(\overline{B}) < n-1$, by a classical result of Bass ([1]), there exists $\zeta_1 \in {P_1}^*$ such that $(\overline{\zeta + c\,\zeta_1})$ is a unimodular element of $\overline{P_1{}^*}$. But then, since $K \subset \mathcal{J}(B)$, $\zeta + c\,\zeta_1$ is a unimodular element of ${P_1}^*$. Let $q_1 \in P_1$ be such that $(\zeta + c\,\zeta_1)(q_1) = 1$. Let

$$\varphi_1 = \begin{pmatrix} 1 & 0 \\ \zeta_1 & 1 \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} 1 & (1-c)\,q_1 \\ 0 & 1 \end{pmatrix}, \qquad \varphi_3 = \begin{pmatrix} 1 & 0 \\ -(\zeta + c\zeta_1) & 1 \end{pmatrix}.$$

Let $\Delta_1 = \varphi_1\,\varphi_2\,\varphi_3$. Since $(\zeta, c) = (0, 1)$ modulo I, from the construction, it follows that Δ_1 is an automorphism of $P = P_1 \oplus B$ which is identity modulo I. Moreover, it is easy to see that $(\zeta, c)\Delta_1 = (0, 1)$. Therefore, we have $\Gamma\,\Delta_1 = \left(\begin{smallmatrix} \gamma_1 & q_2 \\ 0 & 1 \end{smallmatrix}\right)$. Since both Γ and Δ_1 are identity modulo I, γ_1 is an endomorphism of P_1 which is identity modulo I and $q_2 \in I P_1$. Therefore, $\Delta_2 = \left(\begin{smallmatrix} 1 & -q_2 \\ 0 & 1 \end{smallmatrix}\right)$ is an automorphism of $P_1 \oplus B$ which is identity modulo I. Moreover,

$$\Delta = \Delta_2\,\Gamma\,\Delta_1 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\widetilde{a} = \Phi_1(q_2) + a$. Then $\Phi\,\Delta_2^{-1} = (\Phi_1, \widetilde{a})$ and, hence, $K + (\widetilde{a}) = I$. Moreover, $(\Phi_1, \widetilde{a})\,\Delta = (\Phi_1\,\gamma_1, \widetilde{a}) = \Psi\,\Delta_1$. Let $\widetilde{\Psi}_1 = \Phi_1\,\gamma_1$. Therefore, to complete the proof (of the claim), it is enough to show that the surjections $\widetilde{\Phi} = (\Phi_1, \widetilde{a})$ and $\widetilde{\Psi} = (\widetilde{\Psi}_1, \widetilde{a})$ from P to I are connected by an automorphism of P.

Since $\gamma_1 \in \operatorname{End}(P_1)$ is identity modulo I, $(1-\gamma_1)(P_1) \subset I P_1$. Since P_1 is a projective B-module, we have $\operatorname{Hom}(P_1, I P_1) = I \operatorname{Hom}(P_1, P_1)$. Hence $1 - \gamma_1 = \sum b_i\,\beta_i$, where $\beta_i \in \operatorname{End}(P_1)$ and $b_i \in I$. Let $b_i = c_i + d_i\,\widetilde{a}$, where $c_i \in K$ and $d_i \in B$. Then $1 - \gamma_1 = \sum c_i\,\beta_i + \widetilde{a}\sum d_i\,\beta_i$. Hence $\gamma_1 = \theta + \widetilde{a}\,\theta'$, where $\theta = 1 - \sum c_i\,\beta_i$ and $\theta' = -\sum d_i\,\beta_i$. Since $\det(\theta) = 1 + x$ for some $x \in K \subset \mathcal{J}(B)$, θ is an automorphism of P_1.

We have $\widetilde{\Psi}_1 = \Phi_1\,\gamma_1 = \Phi_1\,\theta + \widetilde{a}\,\Phi_1\,\theta'$. Let $\Lambda = \begin{pmatrix} \theta & 0 \\ \Phi_1\theta' & 1 \end{pmatrix}$. Then $(\Phi_1, \widetilde{a})\,\Lambda = (\widetilde{\Psi}_1, \widetilde{a})$ and Λ is an automorphism of P. This proves the result. □

THEOREM 3.7 (Subtraction Principle). *Let B be a ring of dimension d and let $I_1, I_2 \subset B$ be two comaximal ideals of height n, where $2n \geqslant d+3$. Let $P = P_1 \oplus B$ be a projective B-module of rank n. Let $\Gamma\colon P \twoheadrightarrow I_1$ and $\Theta\colon P \twoheadrightarrow I_1 \cap I_2$ be two surjections such that $\Gamma \otimes B/I_1 = \Theta \otimes B/I_1$. Then there exists a surjection $\Psi\colon P \twoheadrightarrow I_2$ such that $\Psi \otimes B/I_2 = \Theta \otimes B/I_2$.*

Proof. Let $\Gamma = (\Gamma_1, a)$. Let 'bar' denote reduction modulo I_2. Then $\overline{\Gamma} = (\overline{\Gamma}_1, \overline{a})$ is a unimodular element of $\overline{P^*}$. Since $\dim B/I_2 < \operatorname{rank} \overline{P}_1$, by ([1]), there exists $\Theta_1 \in P_1^*$ such that $\overline{\Gamma}_1 + \overline{a}^2\,\overline{\Theta}_1$ is a unimodular element of $\overline{P}_1{}^*$. Therefore, replacing Γ_1 by $\Gamma_1 + a^2\,\Theta_1$, we can assume that $\Gamma_1(P_1) = K$ is comaximal with I_2. Moreover, using similar arguments, one can assume that height of K is $n-1$ and therefore, $n - 1 > \dim B/K$. Since K is a surjective image of P_1 (a projective B-module of rank $n-1$), every minimal prime ideals of K has height $n-1$. Hence, since $I_1 = K + (a)$ is an ideal of height n, a is a nonzero divisor modulo the ideal $\sqrt{K}$. Therefore, by (3.6), there exists a surjection $\Psi\colon P \twoheadrightarrow I_2$ which is a lift of $\Theta \otimes B/I_2$. This proves the result. □

Remark 3.8. The above theorem has been already proved in ([7], Proposition 3.2) in the case P is free and in ([5], Theorem 3.3) for arbitrary P but $n = d$. Our approach is different from that of [5, 7] and we believe is of some independent interest.

4. Main Theorem

In this section, we prove the main theorem. We begin with a lemma which is proved in ([4], Lemma 3.1).

LEMMA 4.1. *Let A be a ring of dimension d. Suppose $K \subset A[T]$ is an ideal such that $K + \mathcal{J}(A)A[T] = A[T]$ (recall $\mathcal{J}(A)$ denotes the Jacobson radical of A). Then any maximal ideal of $A[T]$ containing K has height $\leqslant d$.*

LEMMA 4.2. *Let A be a ring of dimension d and let n be an integer such that $2n \geqslant d+3$. Let I be an ideal of $A[T]$ of height n and let $J = I \cap A$. Let $\widetilde{P}$ be a projective $A[T]$-module of rank n and $f \in A[T]$. Suppose $\phi: \widetilde{P} \twoheadrightarrow I/(I^2 f)$ be a surjection. Then we can find a lift $\Phi' \in \mathrm{Hom}_{A[T]}(\widetilde{P}, I)$ of ϕ such that the ideal $\Phi'(\widetilde{P}) = I''$ satisfies the following properties*:

(i) $I = I'' + (J^2 f)$.
(ii) $I'' = I \cap I'$, *where* $\mathrm{ht}\, I' \geqslant n$.
(iii) $I' + (J^2 f) = A[T]$.

Remark 4.3. The above lemma has been proved in ([4], Lemma 3.6) in the case A is an affine algebra over a field and $f = T$. Since the same proof works, we omit the proof.

LEMMA 4.4. *Let C be a ring with $\dim C/\mathcal{J}(C) = r$ and let P be a projective C-module of rank $m \geqslant r+1$. Let I and L be ideals of C such that $L \subset I^2$. Let $\phi: P \twoheadrightarrow I/L$ be a surjection. Then ϕ can be lifted to a surjection $\Psi: P \twoheadrightarrow I$.*

Proof. Let $\Psi: P \to I$ be a lift of ϕ. Then $\Psi(P) + L = I$. Since $L \subset I^2$, by (3.2), there exists $e \in L$ such that $\Psi(P) + (e) = I$.

Let the 'tilde' denote reduction modulo $\mathcal{J}(C)$. Then $\widetilde{\Psi}(\widetilde{P}) + (\widetilde{e}) = \widetilde{I}$. Applying (2.7) to the element $(\widetilde{\Psi}, \widetilde{e})$ of $\widetilde{P^*} \oplus \widetilde{C}$, we see that there exists $\Theta \in P^*$ such that if $K = (\Psi + e\Theta)(P)$, then $\mathrm{ht}\, \widetilde{K}_{\widetilde{e}} \geqslant m$. As $\dim \widetilde{C} = r \leqslant m-1$, we have $\widetilde{K}_{\widetilde{e}} = \widetilde{C}_{\widetilde{e}}$. Hence $\widetilde{e}^l \in \widetilde{K}$ for some positive integer l. Since $\widetilde{K} + (\widetilde{e}) = \widetilde{I}$ and $e \in L \subset I^2$, by (3.1), $\widetilde{K} = \widetilde{I}$. Since $e \in L$, the element $\Psi + e\Theta$ is also a lift of ϕ. Hence, replacing Ψ by $\Psi + e\Theta$, we can assume that $\widetilde{\Psi(P)} = \widetilde{I}$ i.e. $\widetilde{\Psi}: \widetilde{P} \twoheadrightarrow \widetilde{I}$ is a surjection. Therefore, since $\widetilde{I} = (I + \mathcal{J}(C))/\mathcal{J}(C) = I/(I \cap \mathcal{J}(C))$, we have $\Psi(P) + (I \cap \mathcal{J}(C)) = I$. We also have $\Psi(P) + L = I$. Therefore, since $L \subset I^2$, by (3.1), $\Psi(P) = I$. □

As a consequence, we have the following result.

LEMMA 4.5. *Let A be a ring with $\dim A/\mathcal{J}(A) = r$. Let I and L be ideals of $A[T]$ such that $L \subset I^2$ and L contains a monic polynomial. Let P' be a projective $A[T]$-module of rank $m \geqslant r+1$. Let $\phi: P' \oplus A[T] \twoheadrightarrow I/L$ be a surjection. Then we can lift ϕ to a surjection $\Phi: P' \oplus A[T] \twoheadrightarrow I$ with $\Phi(0,1)$ a monic polynomial.*

Proof. Let $\Phi' = (\Theta, g(T))$ be a lift of ϕ. Let $f(T) \in L$ be a monic polynomial. By adding some large power of $f(T)$ to $g(T)$, we can assume that the lift $\Phi' = (\Theta, g(T))$ of ϕ is such that $g(T)$ is a monic polynomial. Let $C = A[T]/(g(T))$. Since $A \hookrightarrow C$ is an integral extension, we have $\mathcal{J}(A) = \mathcal{J}(C) \cap A$ and, hence, $A/\mathcal{J}(A) \hookrightarrow C/\mathcal{J}(C)$ is also an integral extension. Therefore, $\dim C/\mathcal{J}(C) = r$.

Let 'bar' denote reduction modulo $(g(T))$. Then, Θ induces a surjection $\alpha: \overline{P'} \twoheadrightarrow \overline{I}/\overline{L}$, which, by (4.4), can be lifted to a surjection from $\overline{P'}$ to $\overline{I}$. Therefore, there exists a map $\Gamma\colon P' \to I$ such that $\Gamma(P') + (g(T)) = I$ and $(\Theta - \Gamma)(P') = K \subset L + (g(T))$. Hence, $\Theta - \Gamma \in KP'^*$. This shows that $\Theta - \Gamma = \Theta_1 + g(T)\,\Gamma_1$, where $\Theta_1 \in LP'^*$ and $\Gamma_1 \in P'^*$.

Let $\Phi_1 = \Gamma + g(T)\,\Gamma_1$ and let $\Phi = (\Phi_1, g(T))$. Then, $\Phi(P' \oplus A[T]) = \Phi_1(P') + (g(T)) = \Gamma(P') + (g(T)) = I$. Thus $\Phi\colon P' \oplus A[T] \twoheadrightarrow I$ is a surjection. Moreover, $\Phi(0, 1) = g(T)$ is a monic polynomial. Since $\Phi - \Phi' = (\Phi_1 - \Theta, 0)$, $\Phi_1 - \Theta \in LP'^*$ and Φ' is a lift of ϕ, we see that Φ is a (surjective) lift of ϕ. □

In the case A is semi-local, the following lemma has been proved in ([9], Lemma 3.6) for $n = d \geqslant 3$.

LEMMA 4.6. *Let A be a ring of dimension d and let n be an integer such that $2n \geqslant d + 3$. Let I be an ideal of $A[T]$ of height n such that $I + \mathcal{J}(A)A[T] = A[T]$, where $\mathcal{J}(A)$ denotes the Jacobson radical of A. Assume that* $\operatorname{ht}\mathcal{J}(A) \geqslant n - 1$. *Let P be a projective A-module of rank n and let $\phi\colon P[T] \twoheadrightarrow I/I^2$ be a surjection. If the surjection $\phi \otimes A(T)\colon P(T) \twoheadrightarrow IA(T)/I^2A(T)$ can be lifted to a surjection from $P(T)$ to $IA(T)$, then ϕ can be lifted to a surjection $\Phi\colon P[T] \twoheadrightarrow I$.*

Proof. It is easy to see that, under the hypothesis of the lemma, there exists a monic polynomial $f(T) \in A[T]$ and a surjection $\Phi'\colon P[T]_f \twoheadrightarrow I_f$ such that Φ' is a lift of ϕ_f. Since $I + \mathcal{J}(A)A[T] = A[T]$, I is not contained in any maximal ideal of $A[T]$ which contains a monic polynomial and, hence, $f(T)$ is a unit modulo I.

Since $\dim A/\mathcal{J}(A) \leqslant d - n + 1 \leqslant n - 2$, P has a free direct summand of rank 2, i.e. $P = Q \oplus A^2$.

For the sake of simplicity of notation, we write R for $A[T]$, $\widetilde{Q}$ for $Q[T]$ and $\widetilde{P}$ for $P[T]$. Since $\Phi' \in \operatorname{Hom}_{R_f}(\widetilde{P}_f, I_f)$, there exists a positive even integer N such that $\Phi'' = f^N\Phi' \in \operatorname{Hom}_R(\widetilde{P}, I)$. It is easy to see, by the very construction of Φ'', that the induced map Φ''_f from $\widetilde{P}_f$ to I_f is a surjection. Since f is a unit modulo I, the canonical map $R/I \to R_f/I_f$ is an isomorphism and, hence, $I/I^2 = I_f/{I_f}^2$. Putting these facts together, we see that $\phi'' = \Phi'' \otimes R/I\colon \widetilde{P} \twoheadrightarrow I/I^2$ is surjective. Moreover, $\phi'' = f^N\phi$. □

CLAIM. $\phi''\colon \widetilde{P} \twoheadrightarrow I/I^2$ *can be lifted to a surjection from $\widetilde{P}$ to I.*

Proof. We first note that if Δ is an automorphism of $\widetilde{P}$ and if the surjection $\phi''\Delta\colon \widetilde{P} \twoheadrightarrow I/I^2$ has a surjective lift from $\widetilde{P}$ to I, then so also ϕ''. We also note that, by (2.6), any element of $E(\widetilde{P}/I\widetilde{P})$ can be lifted to an automorphism of $\widetilde{P}$. Keeping these facts in mind, we proceed to prove the claim.

By (2.11), there exists $\Delta_1 \in E(\widetilde{P}_f)$ such that (1) $\Psi = {\Delta_1}^*(\Phi'') \in \operatorname{Hom}_R(\widetilde{P}, I)$ and (2) $\Psi(\widetilde{P})$ is an ideal of R of height n, where ${\Delta_1}^*$ is an element of $E(\widetilde{P}_f^*)$ induced from Δ_1.

Since $\Psi_f(\widetilde{P}_f) = I_f$ and f is a unit modulo I, we have $I = \Psi(\widetilde{P}) + I^2$. Hence, by (3.2), $\Psi(\widetilde{P}) = I_1 = I \cap I'$, where $I' + I = R$. Since $(I_1)_f = I_f$, $I'_f = R_f$ and hence I' contains a monic polynomial f^r for some positive integer r.

Since $\Delta_1 \in E(\widetilde{P}_f)$, $\overline{\Delta} = \Delta_1 \otimes R_f/I_f \in E(\widetilde{P}_f/I_f\widetilde{P}_f)$. Since $\widetilde{P}/I\widetilde{P} = \widetilde{P}_f/I_f\widetilde{P}_f$, we can regard $\overline{\Delta}$ as an element of $E(\widetilde{P}/I\widetilde{P})$. By (2.6), $\overline{\Delta}$ can be lifted to an automorphism Δ of $\widetilde{P}$.

The map $\Psi\colon \widetilde{P} \twoheadrightarrow I \cap I'$ induces a surjection $\psi\colon \widetilde{P} \twoheadrightarrow I/I^2$ and it is easy to see that $\psi = \phi''\Delta$. Therefore, to prove the claim, it is enough to show that ψ can be lifted to a surjection from $\widetilde{P}$ to I. If $I' = R$, then obviously Ψ is a required surjective lift of ψ. Hence, we assume that I' is an ideal of height n.

The map $\Psi\colon \widetilde{P} \twoheadrightarrow I \cap I'$ induces a surjection $\psi'\colon \widetilde{P} \twoheadrightarrow I'/I'^2$. Recall that $\widetilde{P} = \widetilde{Q} \oplus R^2$ and $\widetilde{Q} = Q[T]$. Therefore, since I' contains f^r; a monic polynomial, by (4.5), ψ' can be lifted to a surjection $\Psi'(= (\Gamma, h_1, h_2))\colon \widetilde{P} \twoheadrightarrow I'$, where $\Gamma \in \widetilde{Q}^*$, $h_1, h_2 \in R = A[T]$ and h_1 is monic. Moreover, if necessary, by (2.7), we can replace Γ by $\Gamma + {h_2}^2\,\Gamma_1$ for suitable $\Gamma_1 \in \widetilde{Q}^*$ and assume that ht $K = n-1$, where $K = \Gamma(\widetilde{Q}) + Rh_1$. Let $\overline{R} = R/K$ and $\overline{A} = A/(K \cap A)$. Then $\overline{A} \hookrightarrow \overline{R}$ is an integral extension and, hence, $\dim \overline{R}/\mathcal{J}(\overline{R}) = \dim \overline{A}/\mathcal{J}(\overline{A}) \leqslant \dim A/\mathcal{J}(A) \leqslant d-n+1 < n-1$.

Let $P_1 = \widetilde{Q} \oplus R$. Then $\widetilde{P} = P_1 \oplus R$ and $K = \Psi'(P_1)$. Since K contains a monic polynomial h_1, $K + I^2 = R$. Moreover, surjections $\Psi\colon \widetilde{P} \twoheadrightarrow I \cap I'$ and $\Psi'\colon \widetilde{P} \twoheadrightarrow I'$ are such that $\Psi \otimes R/I' = \Psi' \otimes R/I'$. Therefore, since $\overline{R} = R/K$ and $\dim \overline{R}/\mathcal{J}(\overline{R}) < n-1$, by (3.6), there exists a surjection $\Lambda_1\colon \widetilde{P} \twoheadrightarrow I$ with $\Lambda_1 \otimes R/I = \Psi \otimes R/I = \psi$. Therefore, $\Lambda = \Lambda_1\,\Delta^{-1}\colon \widetilde{P} \twoheadrightarrow I$ is a lift of ϕ''. Thus the proof of the claim is complete. □

Let L denote the ideal of $R = A[T]$ generated by $\mathcal{J}(A)f(T)$ and let $D = R/L$. Since $L + I = R$ and $\Lambda(\widetilde{P}) = I$, $\Lambda \otimes D$ is a unimodular element of $\widetilde{P}^* \otimes D$. Let $\Lambda = (\lambda, d_1, d_2)$, where $\lambda \in \operatorname{Hom}_R(\widetilde{Q}, R)$ and $d_1, d_2 \in R$.

Since $f(T)$ is monic, $D/\mathcal{J}(D) = A/\mathcal{J}(A)[T]$. Moreover, $\dim A/\mathcal{J}(A) \leqslant d + 1 - n \leqslant n-2$. Therefore, in view of (2.8), the unimodular element $(\lambda, d_1, d_2) \otimes D$ can be taken to $(0, 0, 1)$ by an element of $E(\widetilde{P}^* \otimes D)$. By (2.6), every element of $E(\widetilde{P}^* \otimes D)$ can be lifted to an automorphism of $\widetilde{P}^*$. Moreover, since $I + (f) = R$, a lift can be chosen to be an automorphism of $\widetilde{P}^*$ which is identity modulo I.

The upshot of the above discussion is that there exists an automorphism Ω of $\widetilde{P}$ such that Ω is identity modulo I and $\Omega^*(\Lambda) = \Lambda\,\Omega = (0, 0, 1)$ modulo L. Therefore, replacing Λ by $\Lambda\,\Omega$, we can assume that $\Lambda = (\lambda, d_1, d_2)$ with $1 - d_2 \in L$.

Recall that our aim is to lift the surjection $\phi\colon \widetilde{P} \twoheadrightarrow I/I^2$ to a surjection $\Phi\colon \widetilde{P} \twoheadrightarrow I$. Recall also that the surjection $\Lambda\colon \widetilde{P} \twoheadrightarrow I$ is a lift of $f^N\phi\colon \widetilde{P} \twoheadrightarrow I/I^2$.

Let $g \in R$ be such that $fg = 1$ modulo (d_2) and, hence, modulo I. Let $\mathfrak{a} = (g^N d_1, d_2)$. Then, since N is even, by (3.3), $\mathfrak{a} = (e_1, e_2)$ with $e_1 - g^N d_1 \in \mathfrak{a}^2$ and $e_2 - g^N d_2 \in \mathfrak{a}^2$. Since $\Lambda = (\lambda, d_1, d_2)$, $\Lambda(\widetilde{P}) = I$ and $Rg + Rd_2 = R$, we see that

$$I = \lambda(\widetilde{Q}) + (d_1, d_2) = g^N\lambda(\widetilde{Q}) + (g^N d_1, d_2) = g^N\lambda(\widetilde{Q}) + (e_1, e_2).$$

Let $\Phi = (g^N\lambda, e_1, e_2) \in \operatorname{Hom}_R(\widetilde{P}, I)$. From the above equality, we see that $\Phi\colon \widetilde{P} \twoheadrightarrow I$ is a surjection. Moreover, since $1 - fg \in I$, $\Phi \otimes R/I = g^N\Lambda \otimes R/I$

and $\Lambda \otimes R/I = f^N\phi \otimes R/I$, Φ is a (surjective) lift of ϕ. This proves the lemma. □

LEMMA 4.7. *Let A be a ring of dimension d and let $I, I_1 \subset A[T]$ be two comaximal ideals of height n, where $2n \geqslant d+3$. Let $P = P_1 \oplus A$ be a projective A-module of rank n. Assume $J = I \cap A \subset \mathcal{J}(A)$ and $I_1 + (J^2T) = A[T]$. Let Φ: $P[T] \twoheadrightarrow I \cap I_1$ and Ψ: $P[T] \twoheadrightarrow I_1$ be two surjections with $\Phi \otimes A[T]/I_1 = \Psi \otimes A[T]/I_1$. Then we get a surjection Λ: $P[T] \twoheadrightarrow I$ such that $(\Phi - \Lambda)(P[T]) \subset (I^2T)$.*

Proof. We first note that, to prove the lemma, we can replace Φ and Ψ by $\Phi\,\Delta$ and $\Psi\,\Delta$, where Δ is an automorphism of $P[T]$.

Let $\Psi = (\Psi_1, f)$. Let 'bar' denote reduction modulo (J^2T) and let $D = \overline{A[T]/(J^2T)}$. Since $I_1 + (J^2T) = A[T]$, it follows that $(\overline{\Psi}_1, \overline{f}) \in Um\,(\overline{P_1[T]^*} \oplus D)$. Since $J \subset \mathcal{J}(A)$, $JD \subset \mathcal{J}(D)$. Moreover, $D/JD = A/J[T]$ and $\dim A/J \leqslant d+1-n \leqslant n-2$. Therefore, since rank $P_1 = n-1$, by ([16], Corollary 2, p. 1429), $\overline{P_1[T]}$ has a unimodular element. By (2.8), $E(\overline{P_1[T]^*} \oplus D)$ acts transitively on the set of unimodular elements of $\overline{P_1[T]^*} \oplus D$ and by (2.6), any element of $E(\overline{P_1[T]^*} \oplus D)$ can be lifted to an automorphism of $P_1[T] \oplus A[T]$. Putting above facts together, we can assume, replacing (Ψ_1, f) by $(\Psi_1, f)\Delta$ (Δ: suitable automorphism of $P[T]$) if necessary, that $\Psi_1(P_1[T]) + (J^2T)A[T] = A[T]$ and $f \in (J^2T)$. Moreover, applying (2.7), we can assume, that $\operatorname{ht} \Psi_1(P_1[T]) = n-1$.

Since $J \subset \mathcal{J}(A)$ and $\Psi_1(P_1[T]) + (J^2T) = A[T]$, we have $\Psi_1(P_1[T]) + \mathcal{J}(A)A[T] = A[T]$ and therefore, by (4.1), $\dim A[T]/(\Psi_1(P_1[T])) \leqslant d-n+1 \leqslant n-2$. Hence, applying (3.6), we get a surjection Λ: $P[T] \twoheadrightarrow I$ such that $(\Phi - \Lambda)(P[T]) \subset (I^2T)$. □

The following result is due to Bhatwadekar and Raja Sridharan ([4], Lemma 3.5).

LEMMA 4.8. *Let A be a regular domain containing a field k, $I \subset A[T]$ an ideal, $J = A \cap I$ and $B = A_{1+J}$. Let P be a projective A-module and let $\overline{\phi}$: $P[T] \twoheadrightarrow I/(I^2T)$ be a surjective map. Suppose there exists a surjection θ: $P_{1+J}[T] \twoheadrightarrow I_{1+J}$ such that θ is a lift of $\overline{\phi} \otimes B$. Then there exists a surjection Φ: $P[T] \twoheadrightarrow I$ such that Φ is a lift of $\overline{\phi}$.*

PROPOSITION 4.9. *Let A be a regular domain of dimension d containing a field k and let n be an integer such that $2n \geqslant d+3$. Let I be an ideal of $A[T]$ of height n. Let P be a projective A-module of rank n and let ψ: $P[T] \twoheadrightarrow I/(I^2T)$ be a surjection. If there exists a surjection Ψ': $P[T] \otimes A(T) \twoheadrightarrow IA(T)$ which is a lift of $\psi \otimes A(T)$. Then we can lift ψ to a surjection Ψ: $P[T] \twoheadrightarrow I$.*

Proof. In view of (4.8), we can assume that $J = I \cap A \subset \mathcal{J}(A)$. Hence, $\operatorname{ht} \mathcal{J}(A) \geqslant n-1$ and $n > \dim A/\mathcal{J}(A)$. Therefore, we can assume that P has a unimodular element i.e. $P = P_1 \oplus A$.

Applying (4.2) for the surjection $\psi\colon P[T] \twoheadrightarrow I/(I^2T)$, we get a lift $\Theta \in \mathrm{Hom}_{A[T]}(P[T], I)$ of ψ such that the ideal $\Theta(P[T]) = I''$ satisfies the following properties:

(i) $I = I'' + (J^2T)$.
(ii) $I'' = I \cap I'$, where I' is an ideal of height n.
(iii) $I' + (J^2T) = A[T]$.

The surjection $\Theta\colon P[T] \twoheadrightarrow I \cap I'$ induces a surjection $\Theta \otimes A(T)\colon P(T) \twoheadrightarrow (I \cap I')A(T)$ such that $\Psi' \otimes A(T)/IA(T) = (\Theta \otimes A(T)) \otimes A(T)/IA(T)$. Since $\dim A(T) = d$ and I, I' are two comaximal ideals of height n, where $2n \geqslant d + 3$, applying (3.7) to surjections Ψ' and $\Theta \otimes A(T)$, we get a surjection $\Phi'\colon P(T) \twoheadrightarrow I'A(T)$ such that $\Phi' \otimes A(T)/I'A(T) = (\Theta \otimes A(T)) \otimes A(T)/I'A(T)$.

The map $\Theta\colon P[T] \twoheadrightarrow I \cap I'$ induces a surjection $\phi\,(= \Theta \otimes A[T]/I')\colon P[T]/I'P[T] \twoheadrightarrow I'/I'^2$. Since $I' + \mathcal{J}(A) = A[T]$ and $\phi \otimes A(T)$ has a surjective lift, namely, $\Phi'\colon P(T) \twoheadrightarrow I'A(T)$, by (4.6), there exists a surjection $\Phi\colon P[T] \twoheadrightarrow I'$ which is a lift of ϕ.

Thus, we have surjections $\Phi\colon P[T] \twoheadrightarrow I'$ and $\Theta\colon P[T] \twoheadrightarrow I \cap I'$ such that $\Phi \otimes A[T]/I' = \phi = \Theta \otimes A[T]/I'$. Hence, as $I' + (J^2T) = A[T]$ and $J \subset \mathcal{J}(A)$, by (4.7), there exists a surjection $\Psi\colon P[T] \twoheadrightarrow I$ such that $(\Psi - \Theta)(P[T]) \subset (I^2T)$. Since Θ is a lift of ψ, we are through.

Thus the proposition is proved. □

Remark 4.10. For $n = d$, the above proposition has been already proved in ([9], Theorem 4.7) in the case A is an arbitrary ring containing a field of characteristic 0. As an application of (4.9), we prove the following result.

COROLLARY 4.11 (Subtraction Principle). *Let A be a regular domain of dimension d containing an infinite field k and let n be an integer such that $2n \geqslant d + 3$. Let $P = P_1 \oplus A$ be a projective A-module of rank n and let $I, I' \subset A[T]$ be two comaximal ideals of height n. Assume that we have surjections $\Gamma\colon P[T] \twoheadrightarrow I$ and $\Theta\colon P[T] \twoheadrightarrow I \cap I'$ such that $\Gamma \otimes A[T]/I = \Theta \otimes A[T]/I$. Then, we have a surjection $\Psi\colon P[T] \twoheadrightarrow I'$ such that $\Psi \otimes A[T]/I' = \Theta \otimes A[T]/I'$.*

Remark 4.12. Since $\dim A[T] = d + 1$, if $2n \geqslant d + 4$, then we can appeal to (3.7) for the proof. So, we need to prove the result only in the case $2n = d + 3$. However, the proof given below in this case works equally well for $2n > d + 3$ and, hence, allows us to give a unified treatment.

Proof. Let $K = I \cap I'$. Then, since k is infinite, there exists a $\lambda \in k$ such that $K(\lambda) = A$ or $K(\lambda)$ has height n. Therefore, replacing T by $T - \lambda$, if necessary, we assume that $K(0) = A$ or $\mathrm{ht}\, K(0) = n$.

Note that Θ induces a surjection $\overline{\theta}\colon P[T] \twoheadrightarrow I'/I'^2$. We first show that $\overline{\theta}$ can be lifted to a surjection from $P[T]$ to $I'/(I'^2T)$.

If $I'(0) = A$, then, since $P = P_1 \oplus A$, we can lift $\overline{\theta}$ to a surjection ϕ: $P[T] \twoheadrightarrow I'/(I'^2T)$. Now we assume that $\operatorname{ht} I'(0) = n$. The map Θ induces a surjection $\Theta(0)$: $P \twoheadrightarrow K(0)(= I(0) \cap I'(0))$. If $I(0) = A$, then $K(0) = I'(0)$ and therefore it is easy to see that $\Theta(0)$ and $\overline{\theta}$ will patch up to give a surjection ψ: $P[T] \twoheadrightarrow I'/(I'^2T)$ which is a lift of $\overline{\theta}$. If $\operatorname{ht} I(0) = n$, then, since $\Gamma \otimes A[T]/I = \Theta \otimes A[T]/I$, we can apply the subtraction principle (3.7) to the surjections $\Gamma(0)$: $P \twoheadrightarrow I(0)$ and $\Theta(0)$: $P \twoheadrightarrow I(0) \cap I'(0)$ to conclude that there is a surjection φ: $P \twoheadrightarrow I'(0)$ such that $\varphi \otimes A/I'(0) = \Theta(0) \otimes A/I'(0)$. Hence, as before, we see that $\overline{\theta}$ and φ will patch up to give a surjection ψ: $P[T] \twoheadrightarrow I'/(I'^2T)$ which is a lift of $\overline{\theta}$.

In view of (4.9), to show that there exists a surjection Ψ: $P[T] \twoheadrightarrow I'$ such that $\Psi \otimes A[T]/I' = \overline{\theta} = \Theta \otimes A[T]/I'$, it is enough to show that $\psi \otimes A(T)$ has a surjective lift from $P(T)$ to $I'A(T)$.

The surjections Γ, Θ induces surjections

$$\Gamma \otimes A(T)\colon P(T) \twoheadrightarrow IA(T), \qquad \Theta \otimes A(T)\colon P(T) \twoheadrightarrow (I \cap I')A(T),$$

respectively, with the property

$$(\Gamma \otimes A(T)) \otimes A(T)/IA(T) = (\Theta \otimes A(T)) \otimes A(T)/IA(T).$$

Therefore, by (3.7), there exists a surjection Ψ': $P(T) \twoheadrightarrow I'A(T)$ with the property

$$\Psi' \otimes A(T)/I'A(T) = (\Theta \otimes A(T)) \otimes A(T)/I'A(T).$$

Since, $(\Theta \otimes A(T)) \otimes A(T)/I'A(T) = \psi \otimes A(T)$, we are through. □

Let k be a field. Recall that a k-algebra A is said to be 'essentially of finite type over k', if A is a localization of an affine algebra over k.

Now we prove our main theorem.

THEOREM 4.13. *Let k be an infinite perfect field and let A be a regular domain of dimension d which is essentially of finite type over k. Let n be an integer such that $2n \geqslant d + 3$. Let $I \subset A[T]$ be an ideal of height n and let P be a projective A-module of rank n. Assume that we are given a surjection ϕ: $P[T] \twoheadrightarrow I/(I^2T)$. Then there exists a surjection Φ: $P[T] \twoheadrightarrow I$ such that Φ is a lift of ϕ.*

Proof. If I has height $d + 1$, then I contains a monic polynomial in T. Hence, by (2.9), we are through. Therefore, we always assume that $n \leqslant d$ and, hence, the inequality $2n \geqslant d + 3$ would imply that $d \geqslant 3$.

We first assume that A is local. In this case, if $n \geqslant 4$ and $I(0) = A$ or $I(0)$ is a complete intersection ideal of height n, then, by (2.10), we are through. It is easy to see that in the case $I(0) = A$, (2.10) is valid even if $\operatorname{ht} I = \dim A = 3$. To complete the proof in the case A is local we proceed as follows.

Let $J = I \cap A$. By (4.2), the surjection ϕ: $P[T] \twoheadrightarrow I/(I^2T)$ has a lift $\Phi' \in \operatorname{Hom}_{A[T]}(P[T], I)$ such that the ideal $\Phi'(P[T]) = I''$ satisfies the following properties:

(i) $I'' + (J^2T) = I$.
(ii) $I'' = I \cap I'$, where I' is an ideal of height $\geqslant n$.
(iii) $I' + (J^2T) = A[T]$.

Since I' is locally generated by n elements, if $\operatorname{ht} I' > n$, then $I' = A[T]$ and we are through. So assume that $\operatorname{ht} I' = n$. The surjection Φ': $P[T] \twoheadrightarrow I''(= I \cap I')$ induces a surjection ψ': $P[T] \twoheadrightarrow I'/I'^2$. Since $I' + (J^2T) = A[T]$, $I'(0) = A$. Hence, as P is free, ψ' can be lifted to a surjection ψ: $P[T] \twoheadrightarrow I'/(I'^2T)$. Now, as $I'(0) = A$, by (2.10), the surjection ψ can be lifted to a surjection Ψ: $P[T] \twoheadrightarrow I'$. Thus, we have surjections Φ': $P[T] \twoheadrightarrow I \cap I'$ and Ψ: $P[T] \twoheadrightarrow I'$ such that $\Phi' \otimes A[T]/I' = \Psi \otimes A[T]/I'$. Therefore, since $I' + (J^2T) = A[T]$, by (4.7), there exists a surjection Φ: $P[T] \twoheadrightarrow I$ such that $(\Phi - \Phi')(P[T]) \subset (I^2T)$. Since Φ' is a lift of ϕ, we are through.

Now we prove the theorem in the general case. Let

$$S = \{s \in A \mid \exists\ \Lambda\colon P_s[T] \twoheadrightarrow I_s\ ;\ \Lambda \text{ is a lift of } \phi \otimes A_s[T]\ \}.$$

Our aim is to prove that $1 \in S$. Note that if $t \in S$ and $a \in A$, then $at \in S$. Moreover, since the theorem is proved in the local case, it is easy to see that for every maximal ideal $\mathfrak{m}$ of A, there exists $s \in A - \mathfrak{m}$ such that P_s is free and $s \in S$. Hence we can find $s_1, \ldots, s_r \in S$ such that P_{s_i} is free and $s_1 + \cdots + s_r = 1$. Therefore, by inducting on r, it is enough to show that if $s, t \in S$ and P_s is free, then $s + t \in S$. Since, in the ring $B = A_{s+t}$, $x + y = 1$, where $x = s/s + t$ and $y = t/s + t$, replacing A by B if necessary, we are reduced to prove that if $s, 1 - s = t \in S$ and P_s is free, then $1 \in S$.

The rest of the argument is devoted to the proof of this assertion. The proof is given in steps.

Step 1. Let $J = I \cap A$. In view of (4.8), replacing A by A_{1+J} if necessary, we assume that $J \subset \mathcal{J}(A)$. If s or t is a unit in A, then obviously $1 \in S$. So, without loss of generality, we can assume that s and t are not invertible elements of A. Therefore, as $J \subset \mathcal{J}(A)$, $s \notin \sqrt{J}$ and $t \notin \sqrt{J}$.

Since $\operatorname{ht} I = n$, $\operatorname{ht} J \geqslant n - 1$. Therefore

$$\dim A/\mathcal{J}(A) \leqslant \dim A/J \leqslant \dim A - \operatorname{ht} J \leqslant n - 2.$$

Hence, since rank $P = n$, $P \stackrel{\sim}{\to} Q \oplus A^2$.

Let Γ_2: $P_t[T] \twoheadrightarrow I_t$ be a surjection which is a lift of $\phi \otimes A_t[T]$. Since $As + At = A$, applying (3.5) (with $L = (I^2T)$ and $B = A[T]$), we get a surjection γ': $P[T] \twoheadrightarrow I/(I^2Ts)$ which is a lift of ϕ. By (4.2), we can find a lift

$\Gamma' \in \mathrm{Hom}_{A[T]}(P[T], I)$ of γ' such that the ideal $\Gamma'(P[T]) = \widetilde{I}$ satisfies the following properties:

(i) $\widetilde{I} + (J^2Ts) = I$.
(ii) $\widetilde{I} = I \cap I_1$, where $\mathrm{ht}\, I_1 \geqslant n$.
(iii) $I_1 + (J^2Ts) = A[T]$.

As before, if $\mathrm{ht}\, I_1 > n$, then $I_1 = A[T]$ and we are through. So we assume that $\mathrm{ht}\, I_1 = n$. The surjection $\Gamma'\colon P[T] \twoheadrightarrow I \cap I_1$ induces a surjection $\theta\colon P[T] \twoheadrightarrow I_1/I_1^2$. Recall that $J \subset \mathcal{J}(A)$ and hence $P \stackrel{\sim}{\to} Q \oplus A^2$. Moreover, $I_1 + (J^2T) = A[T]$. Therefore, if θ can be lifted to a surjection $\Theta\colon P[T] \twoheadrightarrow I_1$, then, by (4.7), ϕ can be lifted to a surjection $\Phi\colon P[T] \twoheadrightarrow I$.

In subsequent steps, we will show that θ has a surjective lift $\Theta\colon P[T] \twoheadrightarrow I_1$.

Step 2. Let $\Gamma_1\colon P_s[T] \twoheadrightarrow I_s$ be a surjection which is a lift of $\phi \otimes A_s[T]$. Since the map $\Gamma'\colon P[T] \twoheadrightarrow I \cap I_1$ is a lift of ϕ, applying (4.11), we get a surjection $\Theta_1\colon P_s[T] \twoheadrightarrow (I_1)_s$ which is a lift of $\theta \otimes A_s[T]$.

Since $I_1 + (J^2Ts) = A[T]$, there exists an element $g \in A[T]$ such that $1 - sg \in I_1$ and the canonical map $A[T]/I_1 \to A_s[T]/(I_1)_s$ is an isomorphism. Therefore, as $P[T] = Q[T] \oplus A^2[T]$ and $P_s[T]$ is a free $A_s[T]$-module, $Q[T]/I_1Q[T]$ is a stably free $A[T]/I_1$-module of rank $n-2$. Since $J \subset \mathcal{J}(A)$, $I_1 + JA[T] = A[T]$ and $\mathrm{ht}\, I_1 = n$, by (4.1),

$$\dim A[T]/I_1 < \dim A[T] - \mathrm{ht}\, I_1 = d - n + 1 \leqslant n - 2.$$

Hence, by a classical result of Bass ([1]), $Q[T]/I_1Q[T]$ is a free $A[T]/I_1$-module.

Let N be a positive even integer such that $(s^N\Theta_1)(P[T]) \subset I_1$ and let $\widetilde{\Theta} = s^N\Theta_1 \in \mathrm{Hom}_{A[T]}(P[T], I_1)$. Then, as $1 - sg \in I_1$, $\widetilde{\Theta}$ induces a surjection $\widetilde{\theta}\colon P[T] \twoheadrightarrow I_1/I_1^2$. Since N is even, if $\widetilde{\theta}$ can be lifted to a surjection $\Theta_2\colon P[T] \twoheadrightarrow I_1$, then, by (3.4), there would exist a surjection $\Theta\colon P[T] \twoheadrightarrow I_1$ such that $\Theta \otimes A[T]/I_1 = g^N\Theta_2 \otimes A[T]/I_1$. In that case, since $1 - s^N g^N \in I_1$,

$$A[T]/I_1 = A_s[T]/(I_1)_s, \qquad \Theta_2 \otimes A[T]/I_1 = s^N\Theta_1 \otimes A[T]/I_1$$

and Θ_1 is a lift of θ, Θ would be a lift of θ.

Thus, it is enough to show that the surjection $\widetilde{\theta}\colon P[T] \twoheadrightarrow I_1/I_1^2$ can be lifted to a surjection $\Theta_2\colon P[T] \twoheadrightarrow I_1$.

Step 3. Recall that $\Theta_1\colon P_s[T] \twoheadrightarrow (I_1)_s$ is a surjection and $\widetilde{\Theta} = s^N\Theta_1\colon P[T] \to I_1$ is a lift of $\widetilde{\theta}$. Therefore, the induced map $\widetilde{\Theta}_s\colon P_s[T] \twoheadrightarrow (I_1)_s$ is also a surjection. Hence, by (2.11), there exists $\Delta \in E(P_s[T])$ such that if $\Delta^*(\widetilde{\Theta}) = \Lambda$ then (1) $\Lambda \in P[T]^*$ and (2) $\Lambda_1(P[T]) = K \subset I_1$ is an ideal of $A[T]$ of height n, where Δ^* is an element of $E(P[T]^*)$ induced by Δ. Since

$K_s = (I_1)_s$ and $A[T] \cap (I_1)_s = I_1$ (as the ideals I_1 and $sA[T]$ are comaximal), we get $K = I_1 \cap I_2$ with $(I_2)_s = A_s[T]$. Therefore, $s^r \in I_2$ and, hence, $I_1 + I_2 = A[T]$, since $I_1 + (s) = A[T]$. Since K is an ideal of $A[T]$ of height n which is a surjective image of $P[T]$, either $I_2 = A[T]$ or I_2 is an ideal of height n.

Since

$$A[T]/I_1 = A_s[T]/(I_1)_s, \qquad P[T]/I_1P[T] = P_s[T]/I_1P_s[T].$$

Hence, the element Δ of $E(P_s[T])$ gives rise to an element $\overline{\Delta}$ of $E(P[T]/I_1P[T])$. By (2.6), there exists an automorphism Δ_0 of $P[T]$ which is a lift of $\overline{\Delta}$. Let $\widetilde{\theta}\,\overline{\Delta} = \lambda_1\colon P[T]/I_1P[T] \twoheadrightarrow I_1/I_1{}^2$ be a surjection. Then, it is obvious that if λ_1 can be lifted to a surjection $\Lambda_1\colon P[T] \twoheadrightarrow I_1$, then $\widetilde{\theta}$ also has a surjective lift Θ_2: $P[T] \twoheadrightarrow I_1$.

Step 4. Note that $\Lambda\colon P[T] \twoheadrightarrow I_1 \cap I_2$ is a surjection such that $\Lambda \otimes A[T]/I_1 = \lambda_1$. Therefore, if $I_2 = A[T]$, then we are through. Now we assume that I_2 is an ideal of $A[T]$ of height n.

Since $I_1(0) = A$, Λ gives rise to a surjection $\lambda_2\colon P[T] \twoheadrightarrow I_2/(I_2{}^2T)$. If λ_2 has a surjective lift from $P[T]$ to I_2, then, by (4.11), λ_1 would have a surjective lift $\Lambda_1\colon P[T] \twoheadrightarrow I_1$. Therefore, it is enough to show that λ_2 can be lifted to a surjection $\Lambda_2\colon P[T] \twoheadrightarrow I_2$.

Since $s^r \in I_2 \cap A$ and $t = 1 - s$, by (4.8), it is enough to show that $\lambda_2 \otimes A_t[T]$: $P_t[T] \twoheadrightarrow (I_2)_t/(I_2{}^2T)_t$ has a surjective lift. In view of (4.9), it is sufficient to prove that the surjection $\lambda_2 \otimes A_t(T)\colon P_t(T) \twoheadrightarrow I_2A_t(T)/I_2{}^2A_t(T)$ can be lifted to a surjection $\widetilde{\Lambda}_2\colon P_t(T) \twoheadrightarrow I_2A_t(T)$.

Recall that we have a surjection $\Gamma_2\colon P_t[T] \twoheadrightarrow I_t$ which is a lift of $\phi \otimes A_t[T]$. Moreover, we also have surjections $\Gamma'\colon P[T] \twoheadrightarrow I \cap I_1$, $\Lambda\colon P[T] \twoheadrightarrow I_1 \cap I_2$, where I_1 and I_2 are ideals of $A[T]$ of height n and an automorphism Δ_0 of $P[T]$ such that

(1) $\Gamma' \otimes A[T]/I = \phi$.
(2) $I_1 + (J^2Ts) = A[T]$, where $J = I \cap A \subset \mathcal{J}(A)$.
(3) $I_1 + I_2 = A[T]$.
(4) $s^N\,\Gamma' \otimes A[T]/I_1 = \Lambda\,\Delta_0^{-1} \otimes A[T]/I_1$, where N is an even integer.

Let $R_1 = A_t(T)$. Then, by (3.7), there exists a surjection $\Phi_1\colon P[T] \otimes R_1 \twoheadrightarrow I_1R_1$ such that $\Phi_1 \otimes R_1/I_1R_1 = \Gamma' \otimes R_1/I_1R_1$. Since $P[T] = Q[T] \oplus A[T]^2$ and $Q[T]/I_1Q[T]$ is free, by (3.4), there exists a surjection $\Phi_2\colon P[T] \otimes R_1 \twoheadrightarrow I_1R_1$ such that $\Phi_2 \otimes R_1/I_1R_1 = s^N\,\Gamma' \otimes R_1/I_1R_1 = \Lambda\,\Delta_0^{-1} \otimes R_1/I_1R_1$. Since Δ_0 is an automorphism of $P[T]$, there exists a surjection $\Phi_3\colon P[T] \otimes R_1 \twoheadrightarrow I_1R_1$ such that $\Phi_3 \otimes R_1/I_1R_1 = \Lambda \otimes R_1/I_1R_1$. Therefore, by (3.7), there exists a surjection $\widetilde{\Lambda}_2\colon P[T] \otimes R_1 \twoheadrightarrow I_2R_1$ such that $\widetilde{\Lambda}_2 \otimes R_1/I_2R_1 = \lambda_2 \otimes R_1$.

Thus the proof of the theorem is complete. □

5. Some Auxiliary Results

In this section we prove two results. Though these results do not have any direct bearing on the main theorem (proved in the last section), we think that they are interesting offshoots of (4.5) and (3.7) and are of independent interest.

First result gives a partial answer to the following question of Roitman:

QUESTION. Let A be a ring and let P be a projective $A[T]$-module such that $P_{f(T)}$ has a unimodular element for some monic polynomial $f(T)$. Then, does P have a unimodular element?

Roitman in ([18], Lemma 10) answered this question affirmatively in the case A is local. If rank $P > \dim A$, then, by ([16], Theorem 2), P has a unimodular element. In ([6], Theorem 3.4) an affirmative answer is given to the above question in the case rank $P = \dim A$ under the additional assumption that A contains an infinite field. In this section we settle the case (affirmatively): P is extended from A, rank $P \geqslant (\dim A + 3)/2$ and A contains an infinite field.

For the proof we need the following two lemmas which are proved in ([6], Lemma 3.1 and Lemma 3.2 respectively).

LEMMA 5.1. *Let A be a ring containing an infinite field k and let $\widetilde{P}$ be a projective $A[T]$-module of rank n. Suppose $\widetilde{P}_{f(T)}$ has a unimodular element for some monic polynomial $f(T) \in A[T]$. Then, there exists a surjection from $\widetilde{P}$ to I, where $I \subset A[T]$ is an ideal of height $\geqslant n$ containing a monic polynomial.*

LEMMA 5.2. *Let R be a ring and Q a projective R-module. Let $(\alpha(T), f(T))$: $Q[T] \oplus R[T] \twoheadrightarrow R[T]$ be a surjective map with $f(T)$ monic. Let pr_2: $Q[T] \oplus R[T] \twoheadrightarrow R[T]$ be the projection onto the second factor. Then, there exists an automorphism $\sigma(T)$ of $Q[T] \oplus R[T]$ which is isotopic to identity and $pr_2\, \sigma(T) = (\alpha(T), f(T))$.*

THEOREM 5.3. *Let A be a ring of dimension d containing an infinite field k and let $\widetilde{P}$ be a projective $A[T]$-module of rank n which is extended from A, where $2n \geqslant d + 3$. Suppose $\widetilde{P}_{f(T)}$ has a unimodular element for some monic polynomial $f(T) \in A[T]$. Then $\widetilde{P}$ has a unimodular element.*

Proof. By (5.1), we get a surjection Φ: $\widetilde{P} \twoheadrightarrow I$, where I is an ideal of height $\geqslant n$ containing a monic polynomial. If $\operatorname{ht} I > n$, then $I = A[T]$ and, hence, $\widetilde{P}$ has a unimodular element. Hence, we assume that $\operatorname{ht} I = n$.

Since $\widetilde{P}$ is extended from A, we write $\widetilde{P} = P[T]$, where P is a projective A-module of rank n. Then Φ induces a surjection ϕ: $P[T] \twoheadrightarrow I/(I^2T)$ which in its turn induces a surjection $\Phi(0)$: $P \twoheadrightarrow I(0)$.

Let $J = A \cap I$. Since rank $P > \dim A/J$, P_{1+J} has a free direct summand. Let $P_{1+J} = Q \oplus A_{1+J}$. Then by (4.5), there exists a surjection

$$\Psi(= (\psi, h(T))): P_{1+J}[T](= Q[T] \oplus A_{1+J}[T]) \twoheadrightarrow I_{1+J}$$

such that Ψ is a lift of $\phi \otimes A_{1+J}[T]$ and $h(T)$ is a monic polynomial. Hence $\Phi(0) \otimes A_{1+J} = \Psi(0)$.

It is easy to see that there exists $a \in J$ such that if $b = 1 + a$, then there exists a projective A_b-module Q_1 with the properties (i) $Q_1 \otimes A_{1+J} = Q$, (ii) $P_b = Q_1 \oplus A_b$, (iii) Ψ: $P_b[T] \twoheadrightarrow IA_b[T]$ and (iv) $\Phi(0)_b = \Psi(0)$. Let pr_2: $Q_1[T] \oplus A_b[T] \twoheadrightarrow A_b[T]$ be the surjection defined by $pr_2(q, x) = x$ for $q \in Q_1[T]$ and $x \in A_b[T]$.

Since $a \in J$, $I(0)_a = A_a$ and, hence, $\Phi(0)_a$ is a surjection from $P_a[T]$ to $A_a[T]$. Since $\Psi_a = (\psi, h(T))_a$ is a unimodular element of $P_{ab}[T]^*$ with $h(T)$ monic, by (5.2), unimodular elements $(pr_2)_a$ and Ψ_a of $P_{ab}[T]^*$ are isotopically connected. Moreover, since $h(T)$ is monic, kernel of Ψ_a is a projective $A_{ab}[T]$-module which is extended. Therefore, it is easy to see that there exists an automorphism Θ of $P_{ab}[T]$ such that $\Theta(0)$ is identity automorphism of P_{ab} and $\Psi_a\Theta = \Psi(0)_a \otimes A_{ab}[T] = \Phi(0)_{ab} \otimes A_{ab}[T]$. Hence Ψ_a and $\Phi(0)_{ab} \otimes A_{ab}[T]$ are isotopically connected. Thus, unimodular elements $(pr_2)_a$ and $\Phi(0)_{ab} \otimes A_{ab}[T]$ are isotopically connected. Therefore, there exists an automorphism Γ of $P_{ab}[T]$ such that Γ is isotopic to identity and $\Phi(0) \otimes A_{ab}[T]\ \Gamma = (pr_2)_a$.

Applying (2.5), we get $\Gamma = \Omega'_b\, \Omega_a$, where Ω is an $A_b[T]$-automorphism of $P_b[T]$ and Ω' is an $A_a[T]$-automorphism of $P_a[T]$. Hence, we have surjections $\Delta_1 = pr_2\, \Omega^{-1}$: $P_b[T] \twoheadrightarrow A_b[T]$ and $\Delta_2 = \Phi(0) \otimes A_a[T]\ \Omega'$: $P_a[T] \twoheadrightarrow A_a[T]$ such that $(\Delta_1)_a = (\Delta_2)_b$. Therefore, they patch up to yield a surjection Δ: $P[T] \twoheadrightarrow A[T]$. Hence, $\widetilde{P} = P[T]$ has a unimodular element. This proves the result. □

COROLLARY 5.4. *Let A be a regular ring of dimension d containing an infinite field k and let $\widetilde{P}$ be a projective $A[T]$-module of rank n, where $2n \geqslant d+3$. Suppose $\widetilde{P}_{f(T)}$ has a unimodular element for some monic polynomial $f(T) \in A[T]$. Then $\widetilde{P}$ has a unimodular element.*

Now we prove our second result which is a complement of the 'subtraction principle' (3.7) and is labeled as the 'addition principle'. For this result we need the following lemma which is proved in ([5], Corollary 2.14) for $n = d$ and in ([7], Corollary 2.4) in the case P is free. Since the proof is quite similar to the free case, we omit it.

LEMMA 5.5. *Let A be a ring of dimension d and let P be a projective A-module of rank n, where $2n \geqslant d + 1$. Let $J \subset A$ be an ideal of height n and let ϕ: $P/JP \twoheadrightarrow J/J^2$ be a surjection. Then, there exists an ideal $J' \subset A$ of height $\geqslant n$, comaximal with J and a surjection Φ: $P \twoheadrightarrow J \cap J'$ such that $\Phi \otimes A/J = \phi$. Further, given finitely many ideals $J_1, \ldots, J_r$ of height n, J' can be chosen to be comaximal with $\cap_1^r J_i$.*

THEOREM 5.6 (Addition Principle). *Let A be a noetherian ring of dimension d. Let $J_1, J_2 \subset A$ be two comaximal ideals of height n, where $2n \geqslant d + 3$. Let*

$P = Q \oplus A$ be a projective A-module of rank n. Let Φ: $P \twoheadrightarrow J_1$ and Ψ: $P \twoheadrightarrow J_2$ be two surjections. Then, there exists a surjection Θ: $P \twoheadrightarrow J_1 \cap J_2$ such that $\Phi \otimes A/J_1 = \Theta \otimes A/J_1$ and $\Psi \otimes A/J_2 = \Theta \otimes A/J_2$.

Proof. Let $J = J_1 \cap J_2$. Since $J/J^2 = J_1/{J_1}^2 \oplus J_2/J_2^2$, Φ and Ψ induces a surjection γ: $P \twoheadrightarrow J/J^2$ such that $\gamma \otimes A/J_1 = \Phi \otimes A/J_1$ and $\gamma \otimes A/J_2 = \Psi \otimes A/J_2$.

Applying (5.5), we get an ideal K of height n which is comaximal with J and a surjection Γ: $P \twoheadrightarrow J \cap K$ such that $\Gamma \otimes A/J = \gamma \otimes A/J$. Hence,

$$\Gamma \otimes A/J_1 = \Phi \otimes A/J_1 \quad \text{and} \quad \Gamma \otimes A/J_2 = \Psi \otimes A/J_2.$$

Applying (3.7) for the surjections Φ and Γ, we get a surjection Λ: $P \twoheadrightarrow J_2 \cap K$ such that $\Lambda \otimes A/(J_2 \cap K) = \Gamma \otimes A/(J_2 \cap K)$. Hence, $\Lambda \otimes A/J_2 = \Psi \otimes A/J_2$.

Applying (3.7) for the surjections Ψ and Λ, we get a surjection Δ: $P \twoheadrightarrow K$ such that $\Delta \otimes A/K = \Lambda \otimes A/K$. Since $\Lambda \otimes A/K = \Gamma \otimes A/K$, we have $\Delta \otimes A/K = \Gamma \otimes A/K$.

Applying (3.7) for the surjections Δ and Γ, we get a surjection Θ: $P \twoheadrightarrow J$ such that $\Theta \otimes A/J = \Gamma \otimes A/J$. Hence, $\Theta \otimes A/J_1 = \Phi \otimes A/J_1$ and $\Theta \otimes A/J_2 = \Psi \otimes A/J_2$. This proves the result. □

In a similar manner, using (4.11), we have the following 'addition principle' for polynomial algebra.

THEOREM 5.7 (Addition Principle). *Let A be a regular domain of dimension d containing an infinite field k and let n be an integer such that $2n \geqslant d + 3$. Let $P = P_1 \oplus A$ be a projective A-module of rank n and let $I, I' \subset A[T]$ be two comaximal ideals of height n. Assume that we have surjections Γ: $P[T] \twoheadrightarrow I$ and Θ: $P[T] \twoheadrightarrow I'$. Then, we have a surjection Ψ: $P[T] \twoheadrightarrow I \cap I'$ such that $\Psi \otimes A[T]/I = \Gamma \otimes A[T]/I$ and $\Psi \otimes A[T]/I' = \Theta \otimes A[T]/I'$.*

References

1. Bass, H.: *K*-theory and stable algebra, *Inst. Hantes Études Sci.* **22** (1964), 5–60.
2. Bhatwadekar, S. M.: Some results on a question of Quillen, In: *Proc. Internat. Bombay Colloquium on Vector Bundles on Algebraic Varieties*, Oxford Univ. Press, 1987, pp. 107–125.
3. Bhatwadekar, S. M. and Roy, A.: Some theorems about projective modules over polynomial rings, *J. Algebra* (1984), 150–158.
4. Bhatwadekar, S. M. and Sridharan, R.: Projective generation of curves in polynomial extensions of an affine domain and a question of Nori, *Invent. Math.* **133** (1998), 161–192.
5. Bhatwadekar, S. M. and Sridharan, R.: The Euler class group of a Noetherian ring, *Compositio Math.* **122** (2000), 183–222.
6. Bhatwadekar, S. M. and Sridharan, R.: On a question of Roitman, *J. Ramanujan Math. Soc.* **16**(1) (2001), 45–61.
7. Bhatwadekar, S. M. and Sridharan, R.: On Euler classes and stably free projective modules, *Proc. Internat. Colloquium on Algebra, Arithmetic and Geometry* (Mumbai 2000), Narosa Publishing House, 2000, pp. 139–158.

8. Boratynski, M.: When is an ideal generated by a regular sequence? *J. Algebra* **57** (1979), 236–241.
9. Das, M. K.: A question of Nori and the Euler class groups of polynomial algebras, Preprint.
10. Krusemeyer, M.: Skewly completable rows and a theorem of Swan and Towber, *Comm. Algebra* **4**(7) (1975), 657–663.
11. Lindel, H.: Unimodular elements in projective modules, *J. Algebra* **172** (1995), 301–319.
12. Mandal, S.: On efficient generation of ideals, *Invent. Math.* **75** (1984), 59–67.
13. Mandal, S.: Homotopy of sections of projective modules, *J. Algebraic Geometry* **1** (1992), 639–646.
14. Mandal, S. and Varma, P. L. N.: On a question of Nori: the local case, *Comm. Algebra* **25** (1997), 451–457.
15. Mohan Kumar, N.: On two conjectures about polynomial rings, *Invent. Math.* **46** (1978), 225–236.
16. Plumstead, B.: The conjecture of Eisenbud and Evans, *Amer. J. Math* **105** (1983), 1417–1433.
17. Quillen, D.: Projective modules over polynomial rings, *Invent. Math.* **36** (1976), 167–171.
18. Roitman, M.: Projective modules over polynomial rings, *J. Algebra* **58** (1979), 51–63.