A simple constructive approach to quadratic BSDEs with or without delay

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Abstract

This paper provides a simple approach for the consideration of quadratic BSDEs with bounded terminal conditions. Using solely probabilistic arguments, we retrieve the existence and uniqueness result derived via PDE-based methods by Kobylanski [11]. This approach is related to the study of quadratic BSDEs presented by Tevzadze [14]. Our argumentation, as in [14], highly relies on the theory of BMO martingales which was used for the first time for BSDEs in [9]. However, we avoid in our method any fixed point argument and use Malliavin calculus to overcome the difficulty. Our new scheme of proof allows also to extend the class of quadratic BSDEs, for which there exists a unique solution: we incorporate delayed quadratic BSDEs, whose driver depends on the recent past of the Y component of the solution. When the delay vanishes, we verify that the solution of a delayed quadratic BSDE converges to the solution of the corresponding classical non-delayed quadratic BSDE.

Key words: quadratic BSDE, delay, BMO martingales, Malliavin calculus

MSC Classification (2010): 60H30, 60H07

1 Introduction

Since there introduction by Pardoux and Peng [13], BSDEs have attracted a lot of attention due mainly to there connection with stochastic control problems. Solving a BSDE consists in the obtention of an adapted couple process (Y, Z) satisfying a dynamics of the form

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s \,, \qquad 0 \le t \le T \,. \tag{1.1}$$

In there seminal paper [13], Pardoux and Peng provide the existence of a unique solution (Y, Z) to this equation for a given square integrable terminal condition ξ and a lipschitz random driver F. Many extensions have been considered so far: addition of a jump component, constraints on the solution, drivers which are solely integrable, drivers which may depend on all the past of the solution (Y, Z) ... Allowing to treat in particular exponential utility maximization problems, the main innovation on BSDEs since there introduction is probably the consideration of quadratic BSDEs.

Restricting to bounded terminal conditions ξ, Kobylanski [11] derives the existence of a unique solution for the BSDE (1.1) whenever Y is a scalar process but F is quadratic with respect to its Z component. The consideration of scalar Y components allows to use comparison arguments and hereby to order BSDE solutions once the drivers and the terminal conditions are. The approach of Kobylanski follows an exponential change of variable and relies on the adaptation of PDE based approximation methods to the resolution of a probabilistic question. The main objective of this paper is to provide an alternative direct and probabilistic proof of the existence of a unique solution in this framework. In the Lipschitz-quadratic case, Tevzadze [14] obtained via a Picard iteration argument, the existence of a unique solution to quadratic scalar BSDEs, which remarkably extends to multidimensional Y components for small terminal conditions. One of the key point in[14] is the use of the theory of BMO martingales which allows a fixed point argument in this quadratic setting. Let us recall that the theory of BMO martingales was used for the first time in [9] in the context of BSDEs. For classical quadratic BSDEs, Briand and Hu [4, 5] proved the existence of a solution for unbounded terminal conditions ξ with exponential moments, and verified that uniqueness holds under an extra convexity condition on the driver F. Barrieu and El Karoui [2] obtained recently similar results using on the contrary a forward approach for this question. Observe also that Delbaen Hu and Bao [7] exhibited bounded terminal conditions for which a BSDE with superquadratic driver admits no bounded solution.

Our argumentation restricts here to the consideration of bounded terminal conditions ξ and Lipschitz-quadratic generators and relies on the theory BMO martingales, as in [14]. Instead of a fixed point argument, we propose here to construct the solution to our quadratic BSDE by approximation based on Malliavin calculus which leads to a quite short and simple proof. To be a little bit more precise, our scheme of proof is the following. Using linearization arguments, we exhibit a priori estimates on the $\mathcal{S}^\infty$ norm and the BMO norm of the (Y, Z) process, solution of a quadratic BSDE in the spirit of [1, 3]. These estimates provide a control on the distance between two solutions of quadratic BSDEs with similar driver in terms of the distance between the corresponding bounded terminal conditions. This provides in particular the uniqueness of solution for quadratic BSDEs. Our second main observation is that the $\mathcal{S}^\infty$ norm of the Y component for solutions of Lipschitz BSDEs does not depend on the Lipschitz constant of the generator with respect to z. For Malliavin differentiable terminal conditions, this property naturally extends to the Z component of Lipschitz BSDE solutions. Combining this estimate with the approximation of quadratic drivers by Lipschitz ones provides the existence of a solution for quadratic BSDEs associated to bounded Malliavin differentiable terminal conditions. A direct density argument then allows to relieve the Malliavin differentiability restriction.

The second part of the paper is dedicated to the extension of existence and uniqueness results for quadratic BSDES using the probabilistic tools developed in our new approach. We focus on delayed quadratic BSDEs whose driver admits a functional Lipschitz dependence with respect to the recent past of the process Y. Considering Lipschitz drivers depending on all the past of Y, Delong and Imkeller [6] derived the existence of a unique solution whenever the Lipschitz constant of the generator or the maturity T is small enough. In order to consider BSDEs with any maturity, we restrict here to drivers depending at time t on the recent values $(Y_s)_{(t-\delta)^+\leq s\leq t}$ of the process Y, for small time delay $\delta > 0$. Following the methodology developed in the first Section, sharp BMO estimates together with linearization arguments provide the existence of unique solution for time-delayed quadratic BSDEs with bounded terminal conditions, as soon as the delay δ is small enough. We also establish the convergence of the solution to the one associated to undelayed classical BSDE as the delay vanishes to zero.

The rest of the paper is organized as follows. Section 2 is dedicated to the construction of a unique solution for quadratic BSDEs using solely probabilistic arguments. We first present the framework of interest, a priori sharp estimates on the solution and a powerful stability property for quadratic BSDEs. Then, we derive an almost sure upper bound for the (Y, Z) solution of Lipschitz BSDE with uniformly Malliavin differentiable bounded terminal conditions. Since this upper bound does not depend on the z-Lipschitz constant of the driver, the previous stability result together with a density argument provides the existence of a unique solution for quadratic BSDEs. Section 3 focuses on delayed quadratic BSDEs and provides successively the existence of a unique solution for small time delay and the convergence of the solution when the delay vanishes. The corresponding technical results are reported to Section 3.4.

Notations. Throughout this paper, we are given a finite horizon T and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a d-dimensional standard Brownian motion $W = (W_t)_{t\geq 0}$. Any element $x \in \mathbb{R}^d$ will be identified to a column vector with i-th component x^i and Euclidian norm $|x|$. $\mathcal{C}_T$ denotes the set $C([0,T],\mathbb{R})$ of continuous functions from $[0,T]$ to $\mathbb{R}$. For $p > 1$, we denote by:

- $\mathcal{S}^p$ the set of real valued $\mathcal{F}$-adapted continuous processes Y on $[0,T]$ such that $\|Y\|_{\mathcal{S}^p} := \mathbb{E}\left[\sup_{0\leq r\leq T}|Y_r|^p\right]^{\frac{1}{p}} < \infty$,
- $\mathcal{S}^\infty$ the set of real valued $\mathcal{F}$-adapted continuous processes Y on $[0,T]$ such that $\|Y\|_{\mathcal{S}^\infty} := \sup_{\omega\in\Omega}\sup_{0\leq r\leq T}|Y_r(\omega)| < \infty$,
- $\mathbf{L}^p$ the set of predictable $\mathbb{R}^d$-valued processes Z s.t. $\|Z\|_{\mathbf{L}^p} := \mathbb{E}\left[\left(\int_0^T |Z_r|^p dr\right)\right]^{\frac{1}{p}} < \infty$,
- $\mathcal{H}^p$ the set of predictable $\mathbb{R}^d$-valued processes Z s.t. $\|Z\|_{\mathcal{H}^p} := \mathbb{E}\left[\left(\int_0^T |Z_r|^2 dr\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} < \infty$,
- $\mathcal{H}^2_{BMO}$ denotes the subset of $\mathcal{H}^2$ of processes Z such that $\int_0^\cdot Z_s\cdot dB_s$ is a Bounded in Mean Oscillation (BMO for short) martingale, i.e such that, there exists a nonnegative constant C such that, for each stopping time $\tau \leq T$,

  $$\mathbb{E}\left[\int_\tau^T |Z_s|^2\, ds \mid \mathcal{F}_\tau\right] \leq C^2.$$

  The $\|\int_0^\cdot Z_s\cdot dB_s\|_{BMO}$ is defined as the best nonnegative constant C for which the above inequality is satisfied. We refer to the book by N. Kazamaki [10] for further details on BMO martingales.

2 Quadratic BSDEs revisited

2.1 The framework for quadratic BSDE

Here is the quadratic BSDE of interest:

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s\cdot dB_s\,, \qquad 0\leq t\leq T\,, \tag{2.1}$$

with $\xi \in \mathcal{S}^\infty$ and $F : [0,T]\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ a given deterministic map. All along the paper we shall work under the following assumptions.

$(\mathbf{H_0})$ The random variables ξ and $\|F(.,0,0)\|_\infty$ are almost surely bounded by $K_0 > 0$.

$(\mathbf{H_q})$ There exist two constants $L_y > 0$ and $K_z > 0$ such that:

$$|F(t,y,z) - F(t,y',z')| \leq L_y|y-y'| + K_z(1+|z|+|z'|)|z-z'|\,,$$

for any $(t,y,z,y',z') \in [0,T]\times[\mathbb{R}\times\mathbb{R}^d]^2$.

Remark 2.1 Observe for later use that $(\mathbf{H_q})$ together with $(\mathbf{H_0})$ implies the following useful estimate:

$$\begin{aligned} |F(t,y,z)| &\leq |F(t,0,0)| + |F(t,y,0) - F(t,0,0)| + |F(t,y,z) - F(t,y,0)| \\ &\leq \|F(.,0,0)\|_\infty + L_y|y| + K_z(1+|z|)|z| \\ &\leq K_0 + \frac{K_z}{2} + L_y|y| + \frac{3K_z}{2}|z|^2 \,, \end{aligned} \tag{2.2}$$

for any $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$.

Remark 2.2 We restrict to the consideration of deterministic driver g in order to avoid unnecessary technicalities, induced by the Malliavin regularization method presented in Section 2.5. We favor a spotless presentation of an innovative approach for the treatment of quadratic BSDEs.

Theorem 2.1 ***[Kobylanski [11]]*** *If* $(\mathbf{H_0})$ *and* $(\mathbf{H_q})$ *hold, then the BSDE* (2.1) *has a unique solution* $(Y,Z) \in \mathcal{S}^\infty \times \mathcal{H}^2$.

In this section, we retrieve the result of Kobylanski using a very different approach, which relies only on probabilistic arguments. Here is the line of proof. First, we derive in Section 2.2 precise a priori upper bounds on the $\mathcal{S}^\infty$ norm of Y and the BMO norm of $\int_0^\cdot Z_s dB_s$ for any solution (Y,Z) of the BSDE (2.1). These bounds together with nice properties of BMO martingales allow to derive in Proposition 2.3 a stability result for the BSDE (2.1). This powerful property directly provides in Corollary 2.2 the uniqueness of solution for the quadratic BSDE (2.1), and also permits to restrict the search of solution for (2.1) to the particular case where the terminal condition ξ has bounded Malliavin derivatives, see Theorem 2.2. This is the purpose of Lemma 2.1, whose main argument relies on the property observed in Proposition 2.4: the Z component of a Lipschitz BSDE with such a terminal condition admits an a.s. upper bound, which does not depend on the Lipschitz constant of the driver with respect to z.

Remark 2.3 *Let us point out that this approach, based on BMO martingales, leads directly to a comparison result using almost the same computations as in the proof of the stability result Proposition 2.3.*

2.2 A priori estimates for quadratic BSDEs

We first observe that any solution $(Y,Z) \in \mathcal{S}^\infty \times \mathcal{H}^2$ of the quadratic BSDE provides a BMO martingale $\int_0^\cdot Z_s \cdot dB_s$, for which we exhibit an explicit control on its BMO norm.

Proposition 2.1 *Suppose that* $(\mathbf{H_0})$ *and* $(\mathbf{H_q})$ *are in force. Then, for any solution* $(Y,Z)\in\mathcal{S}^\infty\times\mathcal{H}^2$ *of the BSDE* (2.1)*, the process* $\left(\int_0^t Z_s\cdot dB_s\right)_{0\le t\le T}$ *is a BMO martingale which satisfies:*

$$\left\|\int_0^\cdot Z_s\cdot dB_s\right\|_{BMO} \;\le\; \frac{1\vee T}{3}\left(1+\frac{4K_0}{K_z}+\frac{2L_y}{K_z}\|Y\|_{\mathcal{S}^\infty}\right)e^{3K_z\|Y\|_{\mathcal{S}^\infty}}\;. \tag{2.3}$$

Proof. The main argumentation of this proof follows the idea of [3], see p. 831 for details. Let introduce the C^2 function $\varphi:\mathbb{R}\mapsto(0,\infty)$ defined by

$$\varphi: x\mapsto\frac{e^{3K_z|x|}-1-3K_z|x|}{|3K_z|^2}\;,\qquad\text{so that}\qquad \varphi''(.)-3K_z|\varphi'(.)|\;=\;1\;.$$

We pick a stopping time τ and applying Ito's formula to the regular function φ, we compute

$$\begin{aligned}\varphi(Y_\tau) &= \varphi(\xi)+\int_\tau^T\left(\varphi'(Y_s)F(s,Y_s,Z_s)-\frac{\varphi''(Y_s)Z_s^2}{2}\right)ds-\int_\tau^T\varphi'(Y_s)Z_sdB_s\\ &\le \varphi(\xi)+\int_\tau^T\left(\frac{3K_z}{2}|\varphi'(Y_s)|-\frac{\varphi''(Y_s)}{2}\right)Z_s^2ds+\int_\tau^T|\varphi'(Y_s)|\left(K_0+\frac{K_z}{2}+L_y|Y_s|\right)ds\\ &\quad-\int_\tau^T\varphi'(Y_s)Z_sdB_s\;,\end{aligned}$$

where the last inequality follows from (2.2). Since $\varphi''(.)-3K_z|\varphi'(.)|=1$ and $\varphi\ge0$, taking the conditional expectation with respect to $\mathcal{F}_\tau$, we compute

$$\frac{1}{2}\mathbb{E}\left[\int_\tau^T Z_s^2ds\mid\mathcal{F}_\tau\right]\;\le\;\mathbb{E}\left[\varphi(\xi)+\int_\tau^T|\varphi'(Y_s)|\left(K_0+\frac{K_z}{2}+L_y|Y_s|\right)ds\mid\mathcal{F}_\tau\right]\;.$$

Because Y is bounded and $|\varphi'(x)|\le(3K_z)^{-1}e^{3K_z\|Y\|_{\mathcal{S}^\infty}}$ whenever $|x|\le\|Y\|_{\mathcal{S}^\infty}$, the previous estimate together with $\varphi(0)=0$ implies

$$\begin{aligned}\mathbb{E}\left[\left(\int_\tau^T Z_s\cdot dB_s\right)^2\mid\mathcal{F}_\tau\right] &\le \frac{2e^{3K_z\|Y\|_{\mathcal{S}^\infty}}}{3K_z}\mathbb{E}\left[|\xi|+T\left(K_0+\frac{K_z}{2}+L_y\|Y\|_{\mathcal{S}^\infty}\right)\mid\mathcal{F}_\tau\right]\\ &\le \frac{1\vee T}{3}\left(1+\frac{4K_0}{K_z}+\frac{2L_y}{K_z}\|Y\|_{\mathcal{S}^\infty}\right)e^{3K_z\|Y\|_{\mathcal{S}^\infty}}\;.\end{aligned}$$

The arbitrariness of the stopping time τ together with the definition of the BMO norm concludes the proof. □

For any solution $(Y,Z)\in\mathcal{S}^2\times\mathcal{H}^2_{BMO}$ of the quadratic BSDE (2.1), the next proposition shows that $Y\in\mathcal{S}^\infty$ and provides an explicit upper bound for $Y\in\mathcal{S}^\infty$. Once again we derive this classical upper bound (see [11] or [4]) with simple arguments.

Proposition 2.2 *Suppose that* $(\mathbf{H_0})$ *and* $(\mathbf{H_q})$ *hold and let* $(Y,Z)\in\mathcal{S}^2\times\mathcal{H}^2_{BMO}$ *be a solution of the quadratic BSDE* (2.1)*. Then* $Y\in\mathcal{S}^\infty$ *and* Y *is controlled by an upper-bound which does not depend on* K_z*:*

$$\|Y\|_{\mathcal{S}^\infty}\;\le\;e^{L_yT}(\|\xi\|_\infty+\|F(.,0,0)\|_\infty T)\;. \tag{2.4}$$

Proof. The proof relies on a classical linearization argument together with the BMO property of Z. Observe first that the BSDE (2.1) may rewrite as a linear one

$$Y_t = \xi + \int_t^T (F(s,0,0) + a_s Y_s + b_s \cdot Z_s) ds - \int_t^T Z_s \cdot dB_s , \qquad 0 \leq t \leq T ,$$

where the processes a and b are respectively defined by

$$a_s = \frac{F(s,Y_s,Z_s) - F(s,0,Z_s)}{Y_s} \mathbf{1}_{|Y_s|>0} \qquad \text{and} \qquad b_s = \frac{F(s,0,Z_s) - F(s,0,0)}{|Z_s|^2} Z_s \mathbf{1}_{|Z_s|>0} ,$$

for $0 \leq s \leq T$. Observe that Assumption $(\mathbf{H_q})$ directly implies

$$|a_t| \leq L_y \quad \text{and} \quad |b_t| \leq K_z(1 + |Z_t|) , \quad 0 \leq t \leq T . \tag{2.5}$$

Since $Z \in \mathcal{H}^2_{BMO}$, we deduce that $\left(\int_0^t b_s \cdot dBs\right)_{0 \leq t \leq T}$ is a BMO martingale. Therefore, Girsanov theorem ensures that the process $B^b := (B_t + \int_0^t b_s ds)_{0 \leq t \leq T}$ is a Brownian motion under a new equivalent probability $\mathbb{P}^b$. A direct application of Ito's formula provides

$$e^{\int_0^t a_u du} Y_t = e^{\int_0^T a_u du} \xi + \int_t^T e^{\int_0^s a_u du} F(s,0,0) ds - \int_t^T Z_s \cdot dB^b_s , \qquad 0 \leq t \leq T ,$$

which directly leads to

$$Y_t = \mathbb{E}^{\mathbb{P}^b} \left[e^{\int_t^T a_u du} \xi + \int_t^T e^{\int_t^s a_u du} F(s,0,0) ds \mid \mathcal{F}_t \right] , \qquad 0 \leq t \leq T .$$

Plugging (2.5) in this expression, we get

$$|Y_t| \leq e^{L_y T} (\|\xi\|_\infty + \|F(.,0,0)\|_\infty T) , \qquad 0 \leq t \leq T ,$$

and the arbitrariness of $t \in [0,T]$ completes the proof. □

Corollary 2.1 *Let* $(\mathbf{H_0})$ *and* $(\mathbf{H_q})$ *be in force and consider any solution* (Y,Z) *of the quadratic BSDE* (2.1). *Then if* (Y,Z) *is valued in* $\mathcal{S}^\infty \times \mathcal{H}^2$ *or* $\mathcal{S}^2 \times \mathcal{H}^2_{BMO}$*, it in fact belongs to* $\mathcal{S}^\infty \times \mathcal{H}^2_{BMO}$ *and satisfies*

$$\|Y\|_{\mathcal{S}^\infty} + \left\| \int_0^\cdot Z_s dB_s \right\|_{BMO} \leq C_0 , \tag{2.6}$$

where C_0 *is a constant that only depends on* T, L_y, K_z *and* K_0.

Proof. Proposition 2.1 and Proposition 2.2 ensure that $(Y,Z) \in \mathcal{S}^\infty \times \mathcal{H}^2_{BMO}$ whenever it belongs to $\mathcal{S}^\infty \times \mathcal{H}^2$ or $\mathcal{S}^2 \times \mathcal{H}^2_{BMO}$. Assumption $(\mathbf{H_0})$ together with estimates (2.4) and (2.3) provide directly (2.6). □

2.3 Stability and uniqueness result for quadratic BSDEs

The next proposition details a powerful stability property for quadratic BSDEs, whose proof mainly relies on properties of BMO martingales in particular the reverse Hölder inequality (see [10] or [3]).

Proposition 2.3 *For $i = 1, 2$, let (Y^i, Z^i) valued in $\mathcal{S}^\infty \times \mathcal{H}^2$ or $\mathcal{S}^2 \times \mathcal{H}^2_{BMO}$ be solution of the BSDE*

$$Y^i_t \;=\; \xi^i + \int_t^T F(s, Y^i_s, Z^i_s)ds - \int_t^T Z^i_s \cdot dB_s \;, \qquad 0 \le t \le T \;,$$

where ξ^1, ξ^2 and F satisfy Assumptions $(\mathbf{H_0})$ and $(\mathbf{H_q})$. Then, we have

$$\|Y^1 - Y^2\|_{\mathcal{S}^{2p}} + \|Z^1 - Z^2\|_{\mathcal{H}^p} \;\le\; C_p \|\xi^1 - \xi^2\|_{\mathbf{L}^{2p}} \;, \quad \textit{for any } p > p_0 \;, \tag{2.7}$$

where $p_0 > 1$ and $(C_p)_{p>p_0}$ are suitable constants which depend on T, L_y, K_z and K_0.

Proof. Let denote $\Delta Y := Y^1 - Y^2$, $\Delta Z := Z^1 - Z^2$, $\Delta\xi := \xi^1 - \xi^2$ and $\Delta F_t := F(t, Y^1_t, Z^1_t) - F(t, Y^2_t, Z^2_t)$, for $t \in [0, T]$. All along the proof, for a given parameter p, C_p denotes a generic constant whose value may change from line to line and which only depends on T, L_y, K_z and K_0. We split the proof in two steps, where we demonstrate respectively the controls on the ΔY and the ΔZ terms.

Step 1. Control of $\|\Delta Y\|_{\mathcal{S}^p}$
As in the proof of Proposition 2.2, we use a linearization argument and rewrite

$$\Delta Y_t \;=\; \xi + \int_t^T (\bar{a}_s \Delta Y_s + \bar{b}_s \cdot \Delta Z_s) ds - \int_t^T \Delta Z_s \cdot dB_s \;,$$

where the processes $\bar{a}$ and $\bar{b}$ are respectively defined by

$$\bar{a}_s = \frac{F(s, Y^1_s, Z^1_s) - F(s, Y^2_s, Z^1_s)}{\Delta Y_s} \mathbf{1}_{|\Delta Y_s| > 0} \;, \quad \bar{b}_s = \frac{F(s, Y^2_s, Z^1_s) - F(s, Y^2_s, Z^2_s)}{|\Delta Z_s|^2} \Delta Z_s \mathbf{1}_{|\Delta Z_s| > 0} \;,$$

for $0 \le s \le T$. Observe that Assumption $(\mathbf{H_q})$ directly implies that

$$|\bar{a}_t| \;\le\; L_y \quad \text{and} \quad |\bar{b}_t| \;\le\; K_z (1 + |Z^1_t| + |Z^2_t|) \;, \qquad 0 \le t \le T \;. \tag{2.8}$$

According to Proposition 2.1, Z^1 and Z^2 belong to $\mathcal{H}^2_{BMO}$, so that $\left(\int_0^t \bar{b}_s \cdot dBs\right)_{0 \le t \le T}$ is a BMO martingale. Furthermore, Corollary 2.1 ensures the existence of a constant $C_0 > 0$, which only depends on T, L_y, K_z and K_0 such that

$$\left\| \int_0^\cdot \bar{b}_s dBs \right\|_{BMO} + \left\| \int_0^\cdot Z^1_s dBs \right\|_{BMO} + \left\| \int_0^\cdot Z^2_s dBs \right\|_{BMO} \;\le\; C_0 \;. \tag{2.9}$$

Besides, we deduce from Girsanov theorem the existence of an equivalent probability measure $\mathbb{P}^{\bar{b}}$ under which the process $B^{\bar{b}} := (B+\int_0^\cdot \bar{b}_s ds)$ is a Brownian motion, and we compute

$$e^{\int_0^t \bar{a}_u du}\Delta Y_t \;=\; e^{\int_0^T \bar{a}_u du}\Delta\xi - \int_t^T e^{\int_0^s \bar{a}_u du}\Delta Z_s \cdot dB_s^{\bar{b}}\,, \qquad 0\le t\le T\,.$$

Denoting $(\mathcal{E}_t^{\bar{b}})_{0\le t\le T}$ the Doleans-Dade exponential of $\bar{b}$ and plugging (2.8) in the previous expression, we get

$$|\Delta Y_t| \;\le\; \mathbb{E}^{\mathbb{P}^{\bar{b}}}\left[e^{\int_t^T \bar{a}_s ds}|\Delta\xi| \mid \mathcal{F}_t\right] \;\le\; e^{L_y T}\left(\mathcal{E}_t^{\bar{b}}\right)^{-1}\mathbb{E}\left[|\Delta\xi|\mathcal{E}_T^{\bar{b}} \mid \mathcal{F}_t\right]\,, \quad 0\le t\le T\,. \tag{2.10}$$

Since $\int_0^\cdot \bar{b}_s dBs$ is a BMO martingale, the reverse HŽlder inequality implies

$$\mathbb{E}\left[\left(\mathcal{E}_T^{\bar{b}}\right)^q \mid \mathcal{F}_t\right] \;\le\; C_q^*\left(\mathcal{E}_t^{\bar{b}}\right)^q\,, \qquad 0\le t\le T\,, \qquad 1<q<q^*\,, \tag{2.11}$$

where the constants $q^*>1$ and $(C_q)_{1<q<q^*}$ are given by

$$\begin{aligned} q^* &:= \phi^{-1}\left(\left\|\int_0^\cdot \bar{b}_s dBs\right\|_{BMO}\right)\,, \qquad \text{with}\quad \phi\,:\,q\mapsto\left(1+\frac{1}{q^2}\log\frac{2q-1}{2q-2}\right)^{1/2}-1\,,\\ C_q^* &:= 2\left(1-\frac{2q-2}{2q-1}\exp\left\{q^2\left(\left\|\int_0^\cdot \bar{b}_s dBs\right\|_{BMO}^2+2\left\|\int_0^\cdot \bar{b}_s dBs\right\|_{BMO}\right)\right\}\right)^{-1}\,, \end{aligned}$$

see the proof of Theorem 3.1 in [10] for details. Since q^* and $(C_q^*)_{1<q<q^*}$ are respectively non-increasing and non-decreasing with respect to $\|\int_0^\cdot \bar{b}_s dBs\|_{BMO}$, estimate (2.9) allows to suppose without loss of generality that they only depend on T, L_y, K_z, K_0. Denoting by p the conjugate of a given $q\in(1,q^*)$ and combining the conditional Cauchy-Schwartz inequality together with (2.10) and (2.11), we derive

$$|\Delta Y_t|^p \;\le\; e^{pL_yT}(\mathcal{E}_t^{\bar{b}})^{-p}\mathbb{E}\left[\left(\mathcal{E}_T^{\bar{b}}\right)^q \mid \mathcal{F}_t\right]^{\frac{p}{q}}\mathbb{E}\left[|\Delta\xi|^p\mid\mathcal{F}_t\right] \;\le\; e^{pL_yT}\left|C_q^*\right|^{\frac{p}{q}}\mathbb{E}\left[|\Delta\xi|^p\mid\mathcal{F}_t\right]\,,$$

for any $0\le t\le T$. Introducing the constant $p^* := q^*/(q^*-1)$ which only depends on T, L_y, K_z and K_0, we deduce from Doob's maximal inequality that

$$\|\Delta Y\|_{\mathcal{S}^p} \;\le\; C_p\,\|\Delta\xi\|_{\mathbf{L}^p}\,, \quad \text{for any } p>p^*\,. \tag{2.12}$$

Finally, choosing any $p_0>p^*$ provides the expected control on ΔY.

Step 2. Control of $\|\Delta Z\|_{\mathcal{H}^p}$

We now turn to the obtention on the control of the ΔZ term and fix $p>p^*$. A direct application of Ito's formula provides

$$|\Delta Y_0|^2+\int_0^T|\Delta Z_r|^2dr \;=\; |\Delta\xi|^2-2\int_0^T(\Delta Y_r)\Delta Z_r\cdot dB_r+2\int_0^T\Delta Y_r\Delta F_r dr\,. \tag{2.13}$$

Observe that $(\mathbf{H_q})$ also implies

$$
\begin{aligned}
2\int_0^T \Delta Y_s \Delta F_s ds &\le 2L_y \int_0^T |\Delta Y_s|^2 ds + 2K_z \int_0^T (1+|Z_s^1|+|Z_s^2|)|\Delta Y_s||\Delta Z_s| ds \\
&\le \sup_{0\le t\le T} |\Delta Y_t|^2 \left(L_y T + 2K_z^2 \int_0^T (1+|Z_r^1|+|Z_r^2|)^2 dr \right) + \frac{1}{2}\int_0^T |\Delta Z_r|^2 dr \, .
\end{aligned}
$$

Plugging this expression into (2.13), the application of Burkholder-Davis-Gundy and Cauchy-Schwartz inequalities provides

$$
\begin{aligned}
\|\Delta Z\|_{\mathcal{H}^p}^p &\le C_p \left(\|\Delta \xi\|_{\mathbf{L}_p}^p + \mathbb{E}\left[\left(\int_0^T (\Delta Y_r)\Delta Z_r \cdot dB_r \right)^{\frac{p}{2}} \right] + \|\Delta Y_r\|_{\mathcal{S}^p}^p \right) \\
&+ C_p \, \mathbb{E}\left[\sup_{0\le t\le T} |\Delta Y_t|^p \left(\int_0^T (1+|Z_r^1|+|Z_r^2|)^2 dr \right)^{\frac{p}{2}} \right] \\
&\le C_p \left(\|\Delta \xi\|_{\mathbf{L}_p}^p + \mathbb{E}\left[\left(\int_0^T |\Delta Z_r|^2 dr \right)^{\frac{p}{2}} \right]^{\frac{1}{2}} \mathbb{E}\left[\sup_{t\le T} |\Delta Y_t|^p \right]^{\frac{1}{2}} + \|\Delta Y_r\|_{\mathcal{S}^p}^p \right) \\
&+ C_p \, \mathbb{E}\left[\left(\int_0^T (1+|Z_r^1|+|Z_r^2|)^2 dr \right)^p \right]^{\frac{1}{2}} \mathbb{E}\left[\sup_{0\le t\le T} |\Delta Y_t|^{2p} \right]^{\frac{1}{2}} \, .
\end{aligned}
$$

Besides, since Z^1 and Z^2 belong to $\mathcal{H}^2_{BMO}$, the energy inequality for BMO martingales, see Section 2.1 in [10] for e.g., together with Corollary (2.1) provide

$$
\mathbb{E}\left[\left(\int_0^T |Z_s^1|^2 ds \right)^p + \left(\int_0^T |Z_s^2|^2 ds \right)^p \right] \le p! \left(\left\| \int_0^{\cdot} Z_s^1 dB_s \right\|_{BMO}^{2p} + \left\| \int_0^{\cdot} Z_s^2 dB_s \right\|_{BMO}^{2p} \right) \le C_p \, .
$$

Combining the two previous expressions together with the inequality $ab \le 2a^2 + b^2/2$ yields

$$
\frac{1}{2}\|\Delta Z\|_{\mathcal{H}^p}^p \le C_p \left(\|\Delta \xi\|_{\mathbf{L}_p}^p + \|\Delta Y\|_{\mathcal{S}^p}^p + \|\Delta Y\|_{\mathcal{S}^{2p}}^p \right) \, .
$$

Hence, the arbitrariness of $p > p^*$ together with (2.12) leads to

$$
\|\Delta Z\|_{\mathcal{H}^p} \le C_p \|\Delta \xi\|_{\mathbf{L}_{2p}} \, , \qquad \text{for any } p > p^* \, .
$$

Therefore, picking $p_0 := p^*$ concludes the proof. □

Choosing $\xi^1 = \xi^2 = \xi \in \mathcal{S}^\infty$, the uniqueness property of solution for the quadratic BSDE (2.1) under $(\mathbf{H_0})$-$(\mathbf{H_q})$ is a direct consequence of Proposition 2.3. This is the point of the following corollary.

Corollary 2.2 *If* $(\mathbf{H_0})$ *and* $(\mathbf{H_q})$ *hold, the quadratic BSDE* (2.1) *admits at most one solution in* $\mathcal{S}^\infty \times \mathcal{H}^2$ *and in* $\mathcal{S}^2 \times \mathcal{H}^2_{BMO}$.

2.4 An auxiliary fine property for Lipschitz BSDEs

This subsection is dedicated to the obtention of tractable nice upper bounds for the solution (Y, Z) of the BSDE (2.1), whenever the driver F is Lipschitz. For this purpose, we suppose all along this subsection that F satisfies the following classical Lipschitz condition:

$(\mathbf{H_{lip}})$ There exist two constants $L_y > 0$ and $L_z > 0$ such that:

$$|F(t,y,z) - F(t,y',z')| \quad \leq \quad L_y|y-y'| + L_z|z-z'| \ , \quad (t,y,z,y',z') \in [0,T] \times [\mathbb{R} \times \mathbb{R}^d]^2 \ .$$

The existence of a unique solution $(Y, Z) \in \mathcal{S}^2 \times \mathcal{H}^2$ for the BSDE (2.1) under $(\mathbf{H_0})$-$(\mathbf{H_{lip}})$ is a well known result due to the seminal paper of Pardoux and Peng [13]. We observe in the next proposition that the process Y, which lies in $\mathcal{S}^\infty$, is controlled by an upper bound which does not depend on L_z. More remarkably, Z satisfies a similar property whenever the terminal condition has uniformly bounded Malliavin derivatives.

Proposition 2.4 *If* $(\mathbf{H_0})$ *and* $(\mathbf{H_{lip}})$ *hold, then the unique solution* $(Y, Z) \in \mathcal{S}^2 \times \mathcal{H}^2$ *of the Lipschitz BSDE* (2.1) *satisfies the following properties.*

(i) $Y \in \mathcal{S}^\infty$ *and* $\|Y\|_{\mathcal{S}^\infty}$ *admits an upper-bound which does not depend of* L_z*:*

$$\|Y\|_{\mathcal{S}^\infty} \quad \leq \quad e^{L_y T} \left(\|\xi\|_\infty + \|F(.,0,0)\|_\infty T \right) . \tag{2.14}$$

(ii) If in addition ξ *is Malliavin differentiable and* $\|D\xi\|_{\mathcal{S}^\infty} = \sup_{0 \leq t \leq T} \|D_t \xi\|_\infty < \infty$*, then* $Z \in \mathcal{S}^\infty$ *and* $\|Z\|_{\mathcal{S}^\infty}$ *admits an upper-bound which does not depend of* L_z*:*

$$\|Z\|_{\mathcal{S}^\infty} \quad \leq \quad \|D\xi\|_{\mathcal{S}^\infty} \, e^{L_y T} \ . \tag{2.15}$$

Proof. We prove each assertion separately.

(i) Although a direct comparison argument is available in this lipschitz context, observe that (2.14) can be proved following the line of proof of Proposition 2.2. In this Lipschitz context, the proof is simpler since no BMO martingale arguments are necessary.

(ii) We now turn to the proof of (2.15) and first consider the case where F is a C^1 function with respect to the y and z components. Since ξ is Malliavin differentiable, it follows from [8] that (Y, Z) is also Malliavin differentiable and its derivative satisfies

$$D_\theta Y_t = D_\theta \xi + \int_t^T [\partial_y F(s, Y_s, Z_s) D_\theta Y_s - \partial_z F(s, Y_s, Z_s) D_\theta Z_s] ds - \int_t^T D_\theta Z_s dB_s \ ,$$

for $0 \leq \theta \leq t \leq T$. Besides, Assumption $(\mathbf{H_{lip}})$ implies that $|\partial_y F| \leq L_y$ and $|\partial_z F| \leq L_z$, so that, applying the exact same reasoning as above leads to

$$|D_\theta Y_t| \quad \leq \quad \|D\xi\|_{\mathcal{S}^\infty} \, e^{L_y T} \ , \qquad 0 \leq \theta \leq t \leq T \ .$$

Since $(D_tY_t)_{0\leq t\leq T}$ is a version of $(Z_t)_{0\leq t\leq T}$, see Proposition 5.3 in [8], we directly get (2.15).

The general case, where F is only Lipschitz continuous, can be treated by standard regularization arguments. We skip the details. □

2.5 Existence of solution for quadratic BSDEs

With the help of the estimates for Lipschitz BSDEs derived in the previous Section, we first prove the existence of a solution for the quadratic BSDE (2.1) whenever the terminal condition ξ has bounded Malliavin derivatives.

Lemma 2.1 *Let* $(\mathbf{H_0})$ *and* $(\mathbf{H_q})$ *hold and suppose that* ξ *is Malliavin differentiable and satisfies*

$$\|D\xi\|_{\mathcal{S}^\infty} \;=\; \sup_{0\leq t\leq T}\|D_t\xi\| \;<\; \infty \quad \mathbb{P}-a.s.$$

Then, the quadratic BSDE (2.1) *admits a solution in* $\mathcal{S}^\infty\times\mathcal{S}^\infty$.

Proof. Let consider the sequence $(\bar{F}_k)_{k\in\mathbb{N}}$ of driver functions defined on $[0,T]\times\mathbb{R}\times\mathbb{R}^d$ by

$$\bar{F}_k : (t,y,z)\mapsto F\left(t,y,\frac{|z|\wedge k}{|z|}z\right)\ , \qquad k\in\mathbb{N}\ .$$

Since F satisfies $(\mathbf{H_q})$, we deduce that

$$|\bar{F}_k(t,y,z)-\bar{F}_k(t,y',z')| \;\leq\; L_y|y-y'|+K_z(1+2k)|z-z'|\ , \qquad k\in\mathbb{N}\ ,$$

for any $(t,y,z,y',z')\in[0,T]\times[\mathbb{R}\times\mathbb{R}^d]^2$. Hence, for any $k\in\mathbb{N}$, $\bar{F}_k$ is a Lipschitz function and $(\mathbf{H_0})$ provides the existence of a unique solution $(Y^k,Z^k)\in\mathcal{S}^2\times\mathcal{H}^2$ to the BSDE

$$Y_t^k \;=\; \xi+\int_t^T\bar{F}_k(s,Y_s^k,Z_s^k)ds-\int_t^T Z_s^k\cdot dB_s\ , \qquad 0\leq t\leq T\ .$$

Observe that Proposition 2.4 ensures that $(Y^k,Z^k)\in\mathcal{S}^\infty\times\mathcal{S}^\infty$, for any $k\in\mathbb{N}$. More importantly, $\|Y^k\|_{\mathcal{S}^\infty}$ and $\|Z^k\|_{\mathcal{S}^\infty}$ are upper-bounded by a constant which does not depend on the Lipschitz constant with respect to z of F_k, and therefore does not depend on k, for any $k\in\mathbb{N}$. Hence, we can fix $k^*>\sup_{k\in\mathbb{N}}\|Z^k\|_{\mathcal{S}^\infty}$ and we observe that

$$F_{k^*}(s,Y_s^{k^*},Z_s^{k^*}) \;=\; F(s,Y_s^{k^*},Z_s^{k^*})\ , \qquad 0\leq s\leq T\ ,$$

since $k^*>\|Z^{k^*}\|_{\mathcal{S}^\infty}$. Therefore (Y^{k^*},Z^{k^*}) is in fact solution of the BSDE (2.1), which thereby admits a solution in $\mathcal{S}^\infty\times\mathcal{S}^\infty$. □

We now extend this result and prove that the quadratic BSDE (2.1) admits a solution for any bounded terminal condition ξ. The line of proof follows from a density argument together with the stability property for quadratic BSDEs derived in Proposition 2.3.

Theorem 2.2 *Under Assumptions* $(\mathbf{H_0})$ *and* $(\mathbf{H_q})$*, the quadratic BSDE* (2.1) *admits a solution in* $\mathcal{S}^\infty \times \mathcal{H}^2_{BMO}$.

Proof. First observe that the terminal condition ξ can be approximated by a sequence of random variables of the form $\xi^n = \Phi^n(B_{t_1}, \ldots, B_{t_n})$, where $(\Phi_n)_n$ is valued in C_b^∞ and p_n goes to infinity, as n goes to infinity. Since ξ is bounded, the sequence $(\xi^n)_n$ can be built in such a way that $(\xi^n)_n$ converges to ξ in probability and in any $\mathbf{L}^p$ space, $p \geq 1$, see [12]. Up to replacing Φ^n by $\Phi^n \wedge \|\xi\|_\infty$, we can suppose $\|\xi^n\|_\infty \leq \|\xi\|_\infty \leq K_0$ so that ξ^n and F satisfy $(\mathbf{H_0})$, for any $n \in \mathbb{N}$. The regularity of ξ^n together with Lemma 2.1 ensures the existence of a solution $(Y^n, Z^n) \in \mathcal{S}^\infty \times \mathcal{S}^\infty$ to the quadratic BSDE

$$Y_t^n \;=\; \xi^n + \int_t^T F(s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n \cdot dB_s\,, \qquad 0 \leq t \leq T\,, \quad n \in \mathbb{N}\,. \tag{2.16}$$

Furthermore, a direct application of the stability property in Proposition 2.3 provides

$$\|Y^n - Y^m\|_{\mathcal{S}^{2p}} + \|Z^n - Z^m\|_{\mathcal{H}^p} \;\leq\; C_p \|\xi^n - \xi^m\|_{\mathbf{L}^{2p}}\,, \qquad n, m \in \mathbb{N}\,,$$

where $p > 1$ and $C_p > 0$ are suitable constants which do not depend neither on n nor on m, since they only rely on T, L_y, K_z and K_0. Since it converges, the sequence $(\xi^n)_n$ is Cauchy in $\mathbf{L}^{2p}$, and we deduce that $(Y^n, Z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}^{2p} \times \mathcal{H}^p$. But the sequence $(Y^n, Z^n)_{n\in\mathbb{N}}$ is valued in the Banach space $\mathcal{S}^\infty \times \mathcal{H}^2_{BMO}$ and therefore admits a limit $(Y, Z) \in \mathcal{S}^\infty \times \mathcal{H}^2_{BMO}$. Letting n go to infinity in (2.16), we observe that (Y, Z) solves indeed (2.1) and conclude the proof. □

3 Quadratic BSDEs with delayed Y variable

We consider now BSDEs, whose driver depends on the recent past of the Y component of the solution. As recently observed in [6], there exists a unique solution for functional Lipschitz BSDEs with drivers depending on the past of solution, as soon as the maturity or the Lipschitz constant of the driver is small enough. We choose here to focus instead on BSDEs with driver depending only on the recent past of the Y component of the solution, i.e. $(Y_{.+u})_{-\delta<u<0}$ where $\delta > 0$ is a given delay. Adapting the line of argument presented in the previous section, we verify hereafter that there exists also in this framework a unique solution for quadratic BSDEs (with respect to the Z-component of the solution), as long as the time delay δ remains small enough. This framework provides hereby a nice application of the innovative approach presented in the previous section. We verify also that the solution of the delayed quadratic BSDE converges to the classical solution of the corresponding un-delayed quadratic BSDE as the delay vanishes to zero.

3.1 The framework of interest

For a given delay parameter $\delta > 0$, we now consider a delayed BSDE, whose dynamics are given by

$$Y_t \;=\; \xi + \int_t^T F_\delta\left(s, G_s^\delta(Y), Z_s\right) ds - \int_t^T Z_s \cdot dB_s \;, \qquad 0 \le t \le T \;, \tag{3.1}$$

where the parameters ξ and F_δ satisfy the assumptions $(\mathbf{H_0})$ and $(\mathbf{H_q})$ given above and, for any $0 \le t \le T$, $G_t^\delta : \mathcal{C}_T \to \mathbb{R}$ is a function. We suppose that the sequence of functions G satisfies the following assumption.

$(\mathbf{H}_\delta)$ For any $0 \le t \le T$, $\delta > 0$ and $y, \bar{y} \in \mathcal{C}_T$, we have $G_t^\delta(0) = 0$ and

$$G_t^\delta(y) = G_t^\delta\left(\{y_{s\wedge t}\}_{0\le s\le T}\right) \;, \qquad \lim_{\delta\to 0^+} G_t^\delta(y) = y_t \;, \qquad \left|G_t^\delta(y) - G_t^\delta(\bar{y})\right| \le \sup_{(t-\delta)^+\le u\le t} |y_u - \bar{y}_u| \;.$$

Remark 3.1 *Observe for later use that, under assumption* $(\mathbf{H}_\delta)$*, the sequence of functions* $(G^\delta)_{\delta>0}$ *satisfies*

$$|G_t^\delta(y)| \;\le\; \sup_{(t-\delta)^+\le u\le t} |y_u| \;\le\; \|y\|_\infty \;, \qquad y \in \mathcal{C}_T \;, \qquad 0 \le t \le T \;, \quad \delta > 0 \;. \tag{3.2}$$

Remark 3.2 *Under* $(\mathbf{H}_\delta)$*, for a given* $y \in \mathcal{C}_T$*,* $\delta > 0$ *and* $0 \le t \le T$*,* $G_t^\delta(y)$ *roughly speaking only depends on the past of* y *on the time interval* $[(t-\delta)^+, t]$ *and converges to the identity function as* δ *vanishes to* 0*. Typical examples of interest are cases where the dynamics of the BSDE depend on :*

- *the delayed value of the solution,* $G_t^\delta : y \mapsto y_{(t-\delta)^+}$ *;*
- *the recent maximum of the solution,* $G_t^\delta : y \mapsto \sup_{(t-\delta)^+\le u\le t} y_u$ *;*
- *the averaged recent value of the solution,* $G_t^\delta : y \mapsto \frac{1}{\delta}\int_{(t-\delta)^+}^t y_u$ *.*

To our knowledge, no result of existence or uniqueness of solution for quadratic BSDEs with delayed generator is available in the literature. Using ideas similar to the one developed in the previous section, the purpose of the next section is to fill this gap.

3.2 Existence and Uniqueness property

In order to derive the existence of a unique solution for the delayed quadratic BSDE, we require hereafter that the driver depends only on Z and the recent past of the solution Y. Namely, we consider only delays δ smaller than a benchmark delay δ_0 defined by

$$\delta_0 \;:=\; \frac{1}{L_y^2 e T} \;. \tag{3.3}$$

Theorem 3.1 *Let the parameters f, ξ and G satisfy assumptions* $(\mathbf{H_0})$ $(\mathbf{H_q})$*and* $(\mathbf{H}_\delta)$. *Then, for $\delta < \delta_0$, there exists a unique solution in $\mathcal{S}^\infty \times \mathcal{H}^2_{BMO}$ to the BSDE* (3.1).

Proof. Let introduce the function $\phi : \mathcal{S}^\infty \mapsto \mathcal{S}^\infty$, such that, for any $U \in \mathcal{S}^\infty$, $\phi(U)$ is the first component of the solution of the following BSDE

$$Y_t = \xi + \int_t^T F_\delta\left(s, G_s^\delta(U), Z_s\right) ds - \int_t^T Z_s \cdot dB_s\,, \qquad 0 \le t \le T\,, \tag{3.4}$$

for some fixed delay $\delta < \delta_0$. We intend to show that ϕ is a contraction.

We consider $(U, U') \in \mathcal{S}^\infty \times \mathcal{S}^\infty$ and denote by $(\phi(U), V)$ and $(\phi(U'), V')$ the solutions to the associated BSDEs of the form (3.4). Observe that the BSDE (3.4) does not exactly fit in the framework developed in the previous Section since the corresponding driver function is random. Nevertheless, the existence of uniqueness of solution is ensured by [11] and one easily verifies that the arguments developed in Proposition 2.1 apply as such, so that V and V' are BMO martingales satisfying

$$\left\|\int_0^\cdot V_s \cdot dB_s\right\|_{BMO} + \left\|\int_0^\cdot V'_s \cdot dB_s\right\|_{BMO} \le C_0\,, \tag{3.5}$$

for some constant C_0.

In the following, we denote $\Delta\phi(U) := \phi(U') - \phi(U)$, $\Delta U := U' - U$ and $\Delta V := V' - V$. Introducing $\lambda := 1/\delta$ and applying Ito's lemma to $e^{\lambda\cdot}|\Delta\phi(U)|^2$, we compute

$$\begin{aligned} & e^{\lambda t}|\Delta\phi(U_t)|^2 + \int_t^T e^{\lambda s}|\Delta V_s|^2 ds + \lambda\int_t^T e^{\lambda s}|\Delta\phi(U_s)|^2 ds \\ = & \int_t^T 2e^{\lambda s}\Delta\phi(U_s)\left(F_\delta\left(s, G_s^\delta(U), V_s\right) - F_\delta\left(s, G_s^\delta(U'), V'_s\right)\right) ds - \int_t^T 2e^{\lambda s}\Delta\phi(U_s)\Delta V_s dB_s\,, \end{aligned}$$

for $0 \le t \le T$. Observe that $(\mathbf{H}_\delta)$ together with (3.5) ensure that the process

$$\hat{b}_s := \frac{F_\delta\left(s, G_s^\delta(U'), V_s\right) - F_\delta\left(s, G_s^\delta(U'), V'_s\right)}{|V_s - V'_s|^2}(V_s - V'_s)\mathbf{1}_{|V_s - V'_s| > 0}\,, \qquad 0 \le s \le T\,,$$

is a BMO martingale, so that $B^{\hat{b}} := B + \int_0^\cdot \hat{b}_s ds$ is a Brownian Motion under the new probability $\mathbb{P}^{\hat{b}}$. Hence, we deduce that

$$\begin{aligned} & e^{\lambda t}|\Delta\phi(U_t)|^2 + \lambda\int_t^T e^{\lambda s}|\Delta\phi(U_s)|^2 ds \\ \le & \int_t^T 2e^{\lambda s}\Delta\phi(U_s)\left(F_\delta\left(s, G_s^\delta(U), V_s\right) - F_\delta\left(s, G_s^\delta(U'), V_s\right)\right) ds - \int_t^T 2e^{\lambda s}\Delta\phi(U_s)\Delta V_s dB_s^{\hat{b}}\,, \end{aligned}$$

for $0 \le t \le T$. Since $2xy \le \lambda x^2 + y^2/\lambda$, we deduce from $(\mathbf{H}_\delta)$ and $(\mathbf{H_q})$that

$$\begin{aligned} e^{\lambda t}|\Delta\phi(U_t)|^2 & \le \frac{1}{\lambda}\mathbb{E}^{\mathbb{P}^{\hat{b}}}\left[\int_t^T e^{\lambda s}\left|F_\delta\left(s, G_s^\delta(U), V_s\right) - F_\delta\left(s, G_s^\delta(U'), V_s\right)\right|^2 ds \quad \Big| \quad \mathcal{F}_t\right] \\ & \le \frac{L_y^2}{\lambda}\mathbb{E}^{\mathbb{P}^{\hat{b}}}\left[\int_t^T e^{\lambda s}\sup_{(s-\delta)^+ \le r \le s}|\Delta U_r|^2 ds \quad \Big| \quad \mathcal{F}_t\right]\,, \end{aligned} \tag{3.6}$$

for $0 \leq t \leq T$. Since $\delta > 0$, we compute

$$e^{\lambda s} \sup_{(s-\delta)^+ \leq r \leq s} |\Delta U_r|^2 \quad \leq \quad e^{\lambda \delta} \sup_{(s-\delta)^+ \leq r \leq s} e^{\lambda r} |\Delta U_r|^2 \ \leq \ e^{\lambda \delta} \| e^{\lambda \cdot} |\Delta U|^2 \|_{\mathcal{S}^\infty} \ ,$$

for $0 \leq s \leq T$. Plugging this estimate in (3.6) and taking the maximum over $t \in [0, T]$ leads to

$$\| e^{\lambda \cdot} |\Delta \phi(U)|^2 \|_{\mathcal{S}^\infty} \quad \leq \quad \frac{L_y^2 T e^{\lambda \delta}}{\lambda} \| e^{\lambda \cdot} |\Delta U|^2 \|_{\mathcal{S}^\infty} \ .$$

Since $\lambda = 1/\delta$, observe that $L_y^2 T e^{\lambda \delta} < \lambda$ since δ is smaller than δ_0 given in (3.3). Thus, ϕ is a contraction on $\mathcal{S}^\infty$ so that there exists a unique solution in $\mathcal{S}^\infty \times \mathcal{H}^2_{BMO}$ to the delayed quadratic BSDE (3.1). □

3.3 BSDE asymptotics for vanishing delay

We now intend to identify the limit of the solution $(Y^\delta, Z^\delta)_{\delta > 0}$ of the BSDE (3.1) as the delay vanishes to 0. For this purpose, we introduce $(Y^0, Z^0) \in \mathcal{S}^\infty \times \mathcal{H}^2_{BMO}$ the solution of the following classical quadratic BSDE

$$Y^0_t \quad = \quad \xi + \int_t^T F_0\left(s, Y^0_s, Z^0_s\right) ds - \int_t^T Z^0_s \cdot dB_s \ , \qquad 0 \leq t \leq T \ . \tag{3.7}$$

In order to verify that $(Y^\delta, Z^\delta)_{\delta > 0}$ converges to (Y^0, Z^0) as δ goes to 0 whenever F_δ converges to F_0, we first observe the following stability property allowing to control the distance between the solutions of the BSDEs (3.1) and (3.7).

Proposition 3.1 *Let the sequence of parameters $(F_\delta)_{\delta \geq 0}$, ξ and $(G^\delta)_{\delta \geq 0}$ satisfy Assumptions* $(\mathbf{H_0})$ $(\mathbf{H_q})$ *and* $(\mathbf{H}_\delta)$. *Then, for $\delta < \delta_0/4$, we have*

$$\left\| e^{\beta \cdot} |Y^\delta - Y^0| \right\|_{\mathcal{S}^\infty} + \left\| e^{\beta \cdot} |Z^\delta - Z^0| \right\|_{\mathcal{H}^2} \quad \leq \quad \bar{C} \left\| e^{\beta \cdot} \left[F_\delta \left(., G^\delta(Y^0), Z^0 \right) - F_0(., Y^0, Z^0) \right] \right\|_{\mathcal{H}^{\bar{p}}}$$

with $\beta := 2L_y^2 e T$ and $(\bar{C}, \bar{p})$ constants which do no depend on δ.

Before providing the proof of the proposition in the next Subsection 3.4, let first present a direct corollary ensuring the convergence of the solutions of the BSDE (3.1) to (Y^0, Z^0) as the delay vanishes.

Corollary 3.1 *Let the sequence of parameters $(F_\delta)_{\delta \geq 0}$, ξ and $(G^\delta)_{\delta \geq 0}$ satisfy Assumptions* $(\mathbf{H_0})$ $(\mathbf{H_q})$*and* $(\mathbf{H}_\delta)$. *If $(F_\delta)_{\delta > 0}$ converges uniformly to F_0 as δ goes to 0, then $(Y^\delta, Z^\delta)_{\delta > 0}$ converges to (Y^0, Z^0) in $\mathcal{S}^\infty \times \mathcal{H}^2$.*

Proof. Observe from Proposition 3.1 that, since $\beta > 0$, we have

$$\left\|Y^\delta - Y^0\right\|_{\mathcal{S}^\infty} + \quad \left\|Z^\delta - Z^0\right\|_{\mathcal{H}^2} \leq \bar{C}\,\mathbb{E}\left[\left(\int_0^T e^{\beta s}\left|F_\delta\left(s, G_s^\delta(Y^0), Z_s^0\right) - F_0(s, Y_s^0, Z_s^0)\right|^2 ds\right)^{\frac{\bar{p}}{2}}\right]^{\frac{1}{\bar{p}}},$$

for δ smaller than $\delta_0/4$. Therefore, the uniform convergence of $(F_\delta)_{\delta>0}$ to F_0 ensures that all the terms of the previous expression go to 0 as δ vanishes. □

3.4 Uniform estimates and stability property

This section is dedicated to the proof of Proposition 3.1. For this purpose, we first need to establish uniform estimates with respect to δ on the solution (Y^δ, Z^δ) of the BSDE (3.1).

Lemma 3.1 *Let the sequence of parameters $(F_\delta)_{\delta\geq 0}$, ξ and $(G^\delta)_{\delta\geq 0}$ satisfy assumptions* $(\mathbf{H_0})$ $(\mathbf{H_q})$*and* $(\mathbf{H}_\delta)$*. Then, there exists a constant C_0 such that*

$$\left\|Y^\delta\right\|_{\mathcal{S}^\infty} + \left\|\int_0^\cdot Z_s^\delta dB_s\right\|_{BMO} \leq C_0\,, \qquad 0 \leq \delta < \delta_0/2\,.$$

Proof. We omit the immediate case where $\delta = 0$ and pick δ in $(0, \delta_0/2)$. Theorem 3.1 ensures the existence of a unique solution (Y^δ, Z^δ) to the BSDE (3.1) and control the norms on Y^δ and Z^δ separately.

1. Control on $\|Y^\delta\|_{\mathcal{S}^\infty}$

The proof follows mainly from similar arguments as the one presented in the proof of Theorem 3.1 above. We pick $\beta = 2/\delta_0$ and apply Ito's formula to $(e^{\beta t}|Y_t|^2)_t$ in order to derive

$$e^{\beta t}|Y_t^\delta|^2 + \beta\int_t^T e^{\beta s}|Y_s^\delta|^2 ds \leq e^{\beta T}|\xi|^2 + \int_t^T e^{\beta s}\Delta Y_s^\delta F_\delta\left(s, G_s^\delta(Y^\delta), 0\right) ds - \int_t^T e^{\beta s}\Delta Y_s^\delta \Delta Z_s^\delta \cdot dB_s^{\tilde{b}}\,,$$

where $B^{\tilde{b}} := B - \int_0^\cdot \tilde{b}_s ds$ and $\tilde{b}$ is given by

$$\tilde{b}_s \;:=\; \frac{F_\delta\left(s, G_s^\delta(Y^\delta), Z_s^\delta\right) - F_\delta\left(s, G_s^\delta(Y^\delta), 0\right)}{|Z_s^\delta|^2} Z_s^\delta \mathbf{1}_{|Z_s^\delta|>0} \;\leq\; K_z(1 + |Z_s^\delta|)\,,$$

for $0 \leq s \leq T$. Since $Z^\delta \in \mathcal{H}^2_{BMO}$, $\int_0^\cdot \tilde{b}_s dB_s$ is a BMO martingale and therefore $B^{\tilde{b}}$ is a Brownian motion under the new probability $\mathbb{P}^{\tilde{b}}$. Hence $(\mathbf{H_q})$ together with the inequality $2xy \leq \beta y^2 + x^2/\beta$ implies

$$\begin{aligned} e^{\beta t}|Y_t^\delta|^2 &\leq e^{\beta T}\|\xi\|_\infty^2 + \frac{1}{\beta}\int_t^T e^{\beta s}|F_\delta(s,0,0)|^2 ds + \frac{L_y^2}{\beta}\int_t^T e^{\beta s}\mathbb{E}^{\mathbb{P}^{\hat{b}}}\left[\sup_{(s-\delta)^+\leq r\leq s}|Y_r^\delta|^2 \mid \mathcal{F}_t\right] ds \\ &\leq e^{\beta T}\left(\|\xi\|_\infty^2 + \frac{T}{\beta}\|F_\delta(.,0,0)\|_\infty^2\right) + \frac{L_y^2 T e^{\beta\delta}}{\beta}\|e^{\beta\cdot}|Y^\delta|^2\|_{\mathcal{S}^\infty}\,, \end{aligned}$$

for any $t \in [0,T]$. Since $\beta = 2/\delta_0 = 2L_y^2 eT$ and $\delta < \delta_0/2$, combining $(\mathbf{H_0})$ and the previous estimate leads to

$$\frac{1}{2}\|Y^\delta\|^2_{\mathcal{S}^\infty} \quad \leq \quad \frac{1}{2}\|e^\beta|Y^\delta|^2\|_{\mathcal{S}^\infty} \ \leq \ e^{2T/\delta^0}\left(\|\xi\|^2_\infty + + \frac{T\delta_0}{2}K_0^2\right) . \tag{3.8}$$

Hence $\|Y^\delta\|_{\mathcal{S}^\infty}$ admits an upper-bound which does not depend on δ.

2. Control on $\left\|\int_0^\cdot Z_s^\delta dB_s\right\|_{BMO}$
We simply observe from (2.2) together with (3.2) that $(\mathbf{H_0})$, $(\mathbf{H_q})$ and $(\mathbf{H}_\delta)$ imply

$$|F_\delta(s, G_s^\delta(Y^\delta), Z_s^\delta)| \quad \leq \quad K_0 + \frac{K_z}{2} + L_y\|Y^\delta\|_{\mathcal{S}^\infty} + \frac{3K_z}{2}|Z_s^\delta|^2 ,$$

for $0 \leq s \leq T$. Following the lines of the proof of Proposition 2.1, this estimate together with (3.8) provides the expected control on $\left\|\int_0^\cdot Z_s^\delta dB_s\right\|_{BMO}$. □

With these sharper estimates in hand, we are finally in position to turn to the proof of Proposition 3.1.

Proof of Proposition 3.1.
We fix $\delta < \delta_0/2$ and introduce the notation $\Delta Y := Y^\delta - Y^0$, $\Delta Z := Z^\delta - Z^0$ together with

$$\Delta F_s \quad := \quad F_\delta\left(s, G_s^\delta(Y^0), Z_s^0\right) - F_0\left(s, Y_s^0, Z_s^0\right) , \qquad 0 \leq s \leq T ,$$

so that the estimate of interest rewrites

$$\left\|e^{\beta\cdot}\Delta Y\right\|_{\mathcal{S}^\infty} + \left\|e^{\beta\cdot}\Delta Z\right\|_{\mathcal{H}^2} \quad \leq \quad \bar{C}\left\|e^{\beta\cdot}\Delta F\right\|_{\mathcal{H}^{\bar{p}}} , \tag{3.9}$$

with $\beta = 2L_y^2 eT$ and $(\bar{C}, \bar{p})$ constants which do no depend on δ. We control hereafter each term separately.

1. Control of $\left\|e^{\beta\cdot}|\Delta Y|\right\|_{\mathcal{S}^\infty}$.
Ito's formula yields

$$\begin{aligned} e^{2\beta t}|\Delta Y_t|^2 \quad &+ \quad \int_t^T e^{2\beta s}|\Delta Z_s|^2 ds + 2\beta\int_t^T e^{2\beta s}|\Delta Y_s|^2 ds + 2\int_t^T e^{2\beta s}\Delta Y_s \Delta Z_s \cdot dB_s \\ &= \quad 2\int_t^T e^{2\beta s}\Delta Y_s\left[F_\delta\left(s, G_s^\delta(Y^\delta), Z_s^\delta\right) - F_0\left(s, Y_s^0, Z_s^0\right)\right] ds , \end{aligned} \tag{3.10}$$

for $0 \leq t \leq T$. Observe from Assumption $(\mathbf{H_q})$ and $(\mathbf{H}_\delta)$ that

$$F_\delta\left(s, G_s^\delta(Y^\delta), Z_s^\delta\right) - F_0\left(s, Y_s^0, Z_s^0\right) \quad \leq \quad b_s^\delta\,\Delta Z_s + L_y \sup_{(s-\delta)^+ \leq r \leq s}|\Delta Y_r| + \Delta F_s ,$$

for $0 \leq s \leq T$, where b_s^δ is given by

$$b_s^\delta \ := \ \frac{F_\delta\left(s, G_s^\delta(Y^\delta), Z_s^\delta\right) - F_\delta\left(s, G_s^\delta(Y^\delta), Z_s^0\right)}{|\Delta Z_s|^2}\Delta Z_s \mathbf{1}_{|\Delta Z_s|>0} \ \leq \ K_z\left(1 + \left|Z_s^0\right| + \left|Z_s^\delta\right|\right) .$$

Since Z^0 and Z^δ belong to $\mathcal{H}^2_{BMO}$, $\int_0^\cdot b^\delta_s dB_s$ is a BMO martingale, and Lemma 3.1 provides a uniform in δ upper-bound for its BMO norm. In particular the process $B^\delta := B - \int_0^\cdot b^\delta_s ds$ is a Brownian motion under the new probability $\mathbb{P}^\delta$. Hence, plugging the previous estimate in (3.10) leads to

$$\begin{aligned}
&|e^{\beta t}\Delta Y_t|^2 + \int_t^T |e^{\beta s}\Delta Z_s|^2 ds + 2\beta\int_0^T |e^{\beta s}\Delta Y_s|^2 ds + 2\int_t^T e^{2\beta s}\Delta Y_s \Delta Z_s \cdot dB^\delta_s \\
\leq\ & 2L_y \int_t^T e^{2\beta s}\Delta Y_s \sup_{(s-\delta)^+\leq r\leq s}|\Delta Y_r| ds + 2\int_t^T e^{2\beta s}\Delta Y_s \Delta F_s ds \\
\leq\ & 2\beta\int_t^T |e^{\beta s}\Delta Y_s|^2 ds + \frac{L_y^2}{\beta}e^{2\beta\delta}\int_t^T \sup_{(s-\delta)^+\leq r\leq s}|e^{\beta r}\Delta Y_r|^2 ds + \frac{1}{\beta}\int_t^T |e^{\beta s}\Delta F_s|^2 ds\ ,
\end{aligned}$$

for $0 \leq t \leq T$. Observing that $\beta = 2L_y^2 eT$ and $\beta\delta < 1$, this implies directly

$$\begin{aligned}
|e^{\beta t}\Delta Y_t|^2 + \int_t^T |e^{\beta s}\Delta Z_s|^2 ds \ &+\ 2\int_t^T e^{2\beta s}\Delta Y_s \Delta Z_s \cdot dB^\delta_s \\
&\leq\ \frac{1}{2}\left\| e^{\beta\cdot}|\Delta Y|\right\|^2_{\mathcal{S}^\infty} + \frac{1}{\beta}\int_0^T |e^{\beta s}\Delta F_s|^2 ds\ , \qquad 0\leq t\leq T\ .
\end{aligned} \tag{3.11}$$

Since B^δ is a Brownian Motion under $\mathbb{P}^\delta$, we deduce

$$\left\| e^{\beta\cdot}\Delta Y\right\|^2_{\mathcal{S}^\infty} \ \leq\ C \sup_{0\leq t\leq T}\mathbb{E}^{\mathbb{P}^\delta}\left[\int_0^T |e^{\beta s}\Delta F_s|^2 ds \mid \mathcal{F}_t\right]\ .$$

Arguing as in Step 1 of the proof of Proposition 2.3 and observing that the BMO norm of b^δ does not depend on δ, we deduce the existence of $q > 0$ and $C_q > 0$ which do not depend on δ such that the Doleans-Dade exponential of b^δ denoted $\mathcal{E}^\delta$ satisfies

$$\mathbb{E}\left[\left(\mathcal{E}^\delta_T\right)^q \mid \mathcal{F}_t\right]\ \leq\ C_q\left(\mathcal{E}^\delta_t\right)^q\ , \qquad 0\leq t\leq T\ .$$

Combining Cauchy-Schwartz inequality together with this estimate and Doob's maximal inequality, we derive

$$\left\| e^{\beta\cdot}\Delta Y\right\|^2_{\mathcal{S}^\infty} \leq C\, C_q\, \mathbb{E}\left[\sup_{0\leq t\leq T}\mathbb{E}\left[\left(\int_0^T |e^{\beta s}\Delta F_s|^2 ds\right)^p \mid \mathcal{F}_t\right]^{\frac{1}{p}}\right] \leq C\ \left\| e^{\beta\cdot}\Delta F\right\|^2_{\mathcal{H}^{2p}}\ ,$$

where p denotes the conjugate of q. Picking $\bar{p} := 2p$ directly leads to the expected control

$$\left\| e^{\beta\cdot}\Delta Y\right\|_{\mathcal{S}^\infty}\ \leq\ C\ \left\| e^{\beta\cdot}\Delta F\right\|_{\mathcal{H}^{\bar{p}}}\ . \tag{3.12}$$

2. Control of $\left\| e^{\beta\cdot}|\Delta Z|\right\|_{\mathcal{H}^2}$.
The estimate (3.11) computed at time $t = 0$ provides

$$\mathbb{E}\int_0^T |e^{\beta s}\Delta Z_s|^2 ds \leq \frac{1}{2}\left\| e^{\beta\cdot}\Delta Y\right\|^2_{\mathcal{S}^\infty} + \frac{1}{\beta}\mathbb{E}\int_0^T |e^{\beta s}\Delta F_s|^2 ds + 2\mathbb{E}\int_0^T e^{2\beta s}\Delta Y_s \Delta Z_s \cdot b^\delta_s ds\ .$$

Using the relation $2|xy| < 2y^2 + x^2/2$, we deduce

$$\frac{1}{2}\mathbb{E}\int_0^T |e^{\beta s}\Delta Z_s|^2 ds \quad \leq \quad \left(\frac{1}{2} + 2\mathbb{E}\int_0^T |b_s^\delta|^2 ds\right) \left\|e^{\beta\cdot}\Delta Y\right\|_{\mathcal{S}^\infty}^2 + \frac{1}{\beta}\left\|e^{\beta\cdot}|\Delta F|\right\|_{\mathcal{H}^2}^2 .$$

Since the $\mathbf{L}^2$ norm of $\int_0^T b_s^\delta dB_s$ is dominated by the BMO norm of $\int_0^\cdot b_s^\delta dB_s$, it is upper-bounded by a constant which does not depend on δ. Hence, plugging (3.12) in the previous estimate provides

$$\mathbb{E}\int_0^T |e^{\beta s}\Delta Z_s|^2 ds \quad \leq \quad C\left(\left\|e^{\beta\cdot}\Delta F\right\|_{\mathcal{H}^{\bar{p}}}^2 + \left\|e^{\beta\cdot}\Delta F\right\|_{\mathcal{H}^2}^2\right) .$$

Classical norm inequality leads to the expected control on $\left\|e^{\beta\cdot}\Delta Z\right\|_{\mathcal{H}^2}$ which, together with (3.12), provides (3.9) and concludes the proof. □

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