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Maxwell's theory on a post-Riemannian spacetime and the equivalence principle

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Abstract. The form of Maxwell's theory is well known in the framework of general relativity, a fact that is related to the applicability of the principle of equivalence to electromagnetic phenomena. We pose the question whether this form changes if torsion and/or non-metricity fields are allowed for in spacetime. Starting from the conservation laws of electric charge and magnetic flux, we recognize that the Maxwell equations themselves remain the same, but the constitutive law must depend on the metric and, additionally, may depend on quantities related to torsion and/or non-metricity. We illustrate our results by putting an electric charge on top of a spherically symmetric exact solution of the metric-affine gauge theory of gravity (which indicates torsion and non-metricity). All this is compared to the recent results of Vandyck.

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1. Introduction

It was Minkowski, in 1908, who formulated Maxwell's theory in a four-dimensional flat pseudo-Euclidean spacetime, Minkowski's special-relativistic 'world'. The next step, generalizing Maxwell's theory if *gravity* can no longer be neglected, was performed by Einstein and Grossmann in 1913. They 'lifted' Maxwell's theory to a four-dimensional pseudo-Riemannian spacetime. This amounted to a successful application of the equivalence principle to Maxwell's theory. Not too much later, after the creation of general relativity theory, Einstein [1] reformulated Maxwell's theory such that it became apparent that the basic structure of Maxwell's theory, namely the field equations, remains intact even when a metric is not used. Later Kottler [2], E Cartan [3], van Dantzig [4] and others put forward the so-called metric-free formulation of electrodynamics; see Post [5, 6] and Schouten [7]. It was recognized in this general framework that Maxwell's equations can be understood as arising from the conservation laws of *electric charge* and *magnetic flux*; see Truesdell and Toupin [8].

These conservation laws can be reduced to *counting statements*, since electric charge comes in quantized portions of elementary charges (or rather as one thirds of them) and magnetic flux can also exist, in superconducting media, in a quantized form, the flux quantum or fluxoid $h/(2e)$, as was already predicted (up to a factor 2) by F London [9]. Thus it is obvious that the formulation of these laws only requires a four-dimensional *differentiable manifold* and the possibility of a foliation of it into three-dimensional hypersurfaces. The *constitutive laws* do need a metric, in contrast to the Maxwellian field equations themselves,

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a point of view which has been repeatedly stressed by Post [6, 10]; see also Bamberg and Sternberg [11]. Any *post*-Riemannian geometry, that is, any spacetime geometry which has more geometrical field variables ('gravitational fields') than the metric, is irrelevant to the Maxwell equations. In particular, neither torsion nor non-metricity couple to it, a point which has already been made by Benn *et al* [12]. Only the constitutive law may depend in a very restricted way on torsion and non-metricity structures.

In the Einstein–Cartan theory of gravity, spacetime carries an additional *torsion*, and, still more generally, in the framework of metric-affine gravity [13], a *non-metricity* enters the geometrical arena of spacetime. What can we predict about Maxwell's theory under these more exotic circumstances? Can we again apply the equivalence principle? Should we write down Maxwell's equations in the Minkowski world in Cartesian coordinates and replace the partial derivatives by covariant ones, or should we do that in the Lagrangian? What type of coupling to gravity should we assume?

In particular, the application of the 'comma goes to semicolon rule' (see Misner *et al* [14]) creates difficulties for charge conservation if applied to the inhomogeneous Maxwell equation in post-Riemannian spacetimes. Some test theory for the coupling of the Maxwell equations to non-metric structures of spacetime has been investigated by Coley [15].

Vandyck has addressed these questions in a recent article [16]. We find his answers not totally convincing. Therefore we will try to argue that the axiomatic formulation of Maxwell's theory, alluded to above, is sufficient for formulating Maxwell's theory in such post-Riemannian spacetimes, including possibly torsion and non-metricity.

As a formalism we use exterior calculus, for our conventions see [13].

2. Electric charge conservation

Let us be given the odd (or twisted, see [17]) electric current 3-form J. We assume that a $(1+3)$ foliation of spacetime holds locally. The different three-dimensional hypersurfaces are labelled by a parameter τ. We introduce a normal vector n such that $n \rfloor \mathrm{d}\tau = 1$. Then we can decompose the current 3-form according to

$$J = \rho - j \wedge \mathrm{d}\tau \,, \tag{2.1}$$

where ρ is the charge density 3-form and j the electric current 2-form. As axiom 1 we assume electric charge conservation (d denotes the four-dimensional exterior derivative):

Axiom 1:
$$\oint_{\partial V_4} J = \int_{V_4} \mathrm{d}J = 0 \,. \tag{2.2}$$

Here V_4 is an arbitrary four-dimensional volume and ∂V_4 its three-dimensional boundary. If this is assumed to be valid for all three-cycles $c_3 = \partial V_4$, then J is exact [10, 18, 19]:

$$\boxed{\mathrm{d}G = J \,.} \tag{2.3}$$

The electromagnetic *excitation* G is an odd 2-form which decomposes as

$$G = D - H \wedge \mathrm{d}\tau \,. \tag{2.4}$$

Therefore (2.3) is equivalent to the inhomogeneous Maxwell equations

$$\underline{\mathrm{d}} D = \rho \qquad \text{(Gauss law)}, \tag{2.5}$$

$$\underline{\mathrm{d}} H - \dot{D} = j \qquad \text{(Oersted–Ampère law)}; \tag{2.6}$$

here $\underline{\mathrm{d}} := \mathrm{d} - \mathrm{d}\tau\, (n \rfloor \mathrm{d}) \wedge$ is the three-dimensional exterior derivative, and $\dot{D} := \pounds_n D$ is defined via the Lie derivative, for details see [20] and references therein. Note that, up to now, only the differential structure of the spacetime was needed. The electric excitation D can be measured by means of Maxwellian double plates (see Pohl [21]) as charge per unit area, the magnetic excitation by means of a small test coil, which compensates the H-field to be measured, as current per unit length. In other words, the extensive quantities D and H have an operational significance provided we know the characterizing properties of an ideal conductor.

3. Lorentz force

From mechanics we take the notion of an even covector-valued force density 4-form f_α. In the conventional manner, we *define* the electromagnetic *field strength* F via axiom 2:

Axiom 2: $$f_\alpha = \left(e_\alpha \rfloor F\right) \wedge J \,. \tag{3.1}$$

From mechanics originates the notion of f_α, from axiom 1 the current J, the e_α's denote the frame. The even 2-form F can be decomposed as

$$F = B + E \wedge \mathrm{d}\tau \,, \tag{3.2}$$

that is, for $a, b = 1, 2, 3$,

$$\mathfrak{f}_a = -\rho\left(e_a \rfloor E\right) - j \wedge \left(e_a \rfloor B\right) \qquad \text{with} \qquad f_\alpha = \mathfrak{f}_\alpha \wedge \mathrm{d}\tau \,. \tag{3.3}$$

Therefore the Lorentz force (3.3), via (3.2), yields an operational definition of the electromagnetic field strength F as a force field—and hence as an intensive quantity. Again no metric nor connection is necessary for formulating axiom 2.

An alternative way of introducing F—again independent of the metric etc—is provided by quantum interference measurements of Aharonov–Bohm type, yielding an observable phase shift $\delta\varphi = (e/\hbar)\int_{V_2} F$.

Now we have to impose some conditions on the newly defined field strength F.

4. Magnetic flux conservation

The field strength F is a 2-form. Thus we can postulate the conservation of magnetic flux as axiom 3:

Axiom 3: $$\oint_{\partial V_3} F = \int_{V_3} \mathrm{d}F = 0 \,. \tag{4.1}$$

By Stokes' theorem and the arbitrariness of the two-cycles $c_2 = \partial V_3$, we have

$$\boxed{\mathrm{d}F = 0\,.} \tag{4.2}$$

In the $(1+3)$ decomposition this reads

$$\underline{\mathrm{d}}B = 0 \qquad \text{(magnetic field closed)}\,, \tag{4.3}$$

$$\underline{\mathrm{d}}E + \dot{B} = 0 \qquad \text{(Faraday law)}\,. \tag{4.4}$$

Maxwell's equations are represented by (2.3) and (4.2) or, equivalently, by (2.5), (2.6), (4.3) and (4.4). In this form, they are generally covariant, i.e. valid in arbitrary frames and arbitrary coordinates. Moreover, neither the metric nor the torsion nor non-metricity take part in this set-up. Therefore, if one starts from a four-dimensional differential manifold, which admits a $(1+3)$ foliation, and introduces a metric and a connection, then the structure of the Maxwell equations (2.3) and (4.2) is insensitive to it and does *not* change.

We can argue similarly in the complementary situation: in a Minkowski space, we formulate the Maxwell equations as above. Then they keep their form with respect to an *accelerated* frame. Consequently, switching on *gravity* and requiring the equivalence principle to be valid, the Maxwell equations must not change either. In spite of the 'deformation' of spacetime by means of gravity, the Maxwell equations remain 'stable'. Therefore, in this framework of deriving Maxwell's theory from electric charge and magnetic flux conservation, the Maxwell equations stay the same in a Minkowskian, Riemannian or post-Riemannian spacetime. No additional effort is needed in order to adapt the Maxwell equations if a spacetime is considered with additional geometrical attributes. This is the most straightforward application of the equivalence principle one can think of.

5. Constitutive law

So far, the Maxwell equations (2.3) and (4.2) represent an underdetermined system of evolution equations for G and F. In order to reduce the number of independent variables, we have to set up a relation between G and F:

$$G = G(F)\,. \tag{5.1}$$

Special cases of this constitutive law are:

(i) *Vacuum.* The standard constitutive law for a vacuum is

$$G = {}^\star F\,. \tag{5.2}$$

On the right-hand side of (5.2), the factor $(\varepsilon_0/\mu_0)^{1/2}$ has been absorbed for simplicity. Here, by means of the Hodge star, the metric enters the Maxwell theory for the first time. The appearance of the metric is necessary from a physical point of view in order to get the *light cone* as the characteristic surface of the evolution equations for the Maxwellian field strength F. The law (5.2) is valid in Minkowksi, Riemannian and post-Riemannian spacetimes.

(ii) *Axion.* The constitutive law of the vacuum (5.2) relates the ordinary 2-form F, via the Hodge star (which is twisted), to the twisted 2-form G. If we had a twisted 0-form θ ('pseudo-scalar') at our disposal, then we could supplement the right-hand side of the vacuum law by the twisted term θF:

$$G = {}^\star F + \theta F = ({}^\star + \theta) F\,. \tag{5.3}$$

The exterior derivative of this equation, because of $\mathrm{d}F = 0$, turns out to be

$$\mathrm{d}G = \mathrm{d}\,{}^\star F + \mathrm{d}\theta \wedge F\,. \tag{5.4}$$

Thus the inhomogeneous Maxwell equation, in terms of F, reads

$$\big(\mathrm{d}\,{}^\star + (\mathrm{d}\theta)\wedge\big) F = J\,, \qquad \text{with} \qquad \mathrm{d}J = 0\,. \tag{5.5}$$

The 'pseudo-scalar' field θ is known in the literature as the hypothetical axion field, see [22, 23]. Its possible implications for cosmology are discussed in [24]. The axion-Maxwell interaction Lagrangian turns out to be $\sim \theta\, F \wedge F = \theta\, \mathrm{d}(F \wedge A)$.

We can relate the axion field to the torsion of spacetime. The torsion T^α is an ordinary 2-form. Its axial piece is proportional to the ordinary 3-form $T^\alpha \wedge \vartheta_\alpha$. The dual of it is a twisted 1-form ${}^\star(T^\alpha \wedge \vartheta_\alpha)$. Therefore, with some constant c, we can make the identification

$$\mathrm{d}\theta = c\,{}^\star(T^\alpha \wedge \vartheta_\alpha)\,, \tag{5.6}$$

which yields (Gasperini and de Sabbata [25], see also [26])

$$\left(\mathrm{d}^\star + c\,^\star(T^\alpha \wedge \vartheta_\alpha)\wedge\right) F = J\,. \tag{5.7}$$

This equation describes the coupling of the inhomogeneous Maxwell equation to an axial piece of the torsion. However, this interpretation is not compulsory. Incidentally, (5.7) seems to represent the most general post-Riemannian coupling linear in F, which is compatible with charge conservation [27]; a piece with, e.g. the Weyl covector, is excluded since it is an ordinary, not a twisted, form.

We recognize also in this example that there does not seem to exist a chance to introduce other post-Riemannian structures in the axion–Maxwell equation (5.7) in an *ad hoc* way. We would like to stress that (5.7) is valid in a spacetime with arbitrary metric and connection.

(iii) *Born–Infeld.* The nonlinear Born–Infeld theory [28] represents a classical generalization of Maxwell's theory for accommodating stable solutions for the description of 'electrons'. Its constitutive law reads (with a dimensionful parameter f, the so-called maximal field strength, see also [29]):

$$G = \frac{^\star F - (1/2f^2)\ \ ^\star(F \wedge F)\, F}{\sqrt{1 + (1/f^2)\ \ ^\star(F \wedge {}^\star F) - (1/4f^4)[^\star(F \wedge F)]^2}}\,. \tag{5.8}$$

It leads to a nonlinear equation for the dynamical evolution of the field strength F. As a consequence, the characteristic surface, the light cone, depends on the field strength, and the superposition principle for the electromagnetic field no longer holds.

(iv) *Heisenberg–Euler.* Quantum electrodynamical vacuum corrections to Maxwell's theory can be accounted for by an effective constitutive law constructed by Heisenberg and Euler [30]. To second order in the fine structure constant α, it is given by (see also [31])

$$G = \left[1 + \frac{4\,\alpha^2}{45\,m^4}\,^\star\!\left(F \wedge {}^\star F\right)\right]{}^\star F + \frac{7\,\alpha^2}{45\,m^4}\,^\star\!\left(F \wedge F\right)F\,, \tag{5.9}$$

where m is the mass of the electron. Again, post-Riemannian structures do not interfere here.

6. Energy–momentum current of the electromagnetic field

For quantifying the gravitational effect of the electromagnetic field, we need its energy–momentum current. The Lagrangian 4-form of Maxwell's field reads

$$L_{\mathrm{Max}} = -\tfrac{1}{2} F \wedge G\,. \tag{6.1}$$

The canonical energy–momentum current is computed from the Lagrangian 4-form (6.1) and can be represented by the odd covector-valued 3-form

$$\Sigma_\alpha^{\mathrm{Max}} = e_\alpha \rfloor L_{\mathrm{Max}} + (e_\alpha \rfloor F) \wedge G = \tfrac{1}{2}\left[(e_\alpha \rfloor F) \wedge G - (e_\alpha \rfloor G) \wedge F\right]. \tag{6.2}$$

This energy–momentum current will enter the right-hand side of the first field equation, as we will see below.

7. An electric charge in Einstein–dilation–shear gravity

As a non-trivial example, let us consider the electromagnetic field in the framework of the metric-affine gauge theory (MAG) of gravity [13], in particular its effect on an exact solution of this theory [32], see also [33]. Similar solutions have been found by Tucker and Wang, see [34, 35]. The geometrical ingredients of MAG are the curvature 2-form $R_\alpha{}^\beta = \frac{1}{2} R_{ij\alpha}{}^\beta \, dx^i \wedge dx^j$, and, as post-Riemannian structures, the non-metricity 1-form $Q_{\alpha\beta} = Q_{i\alpha\beta} \, dx^i$ and the torsion 2-form $T^\alpha = \frac{1}{2} T_{ij}{}^\alpha \, dx^i \wedge dx^j$. The simple toy model that we want to consider is specified by a gravitational gauge Lagrangian, quadratic in curvature, torsion and non-metricity, see [32],

$$V_{\text{dil-sh}} = -\frac{1}{2\kappa}\left(R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda\eta + \beta\, Q \wedge {}^\star Q + \gamma\, T \wedge {}^\star T\right) - \tfrac{1}{8}\alpha\, R_\alpha{}^\alpha \wedge {}^\star R_\beta{}^\beta \,, \tag{7.1}$$

coupled to the Maxwell Lagrangian (6.1) according to $L_{\text{tot}} = V_{\text{dil-sh}} + L_{\text{Max}}$. In (7.1) we have introduced the Weyl covector $Q := Q_\gamma{}^\gamma/4$ and the covector piece of the torsion $T := e_\alpha \rfloor T^\alpha$. Einstein's gravitational constant is denoted by $\kappa = \ell^2/(\hbar c)$ (with the Planck length ℓ), and λ is the cosmological constant. The coupling constants α, β, and γ are dimensionless.

Varying the coframe and the connection, we find the two relevant field equations of MAG [13],

$$\text{D}H_\alpha - E_\alpha = \Sigma_\alpha \,, \tag{7.2}$$

$$\text{D}H^\alpha{}_\beta - E^\alpha{}_\beta = \Delta^\alpha{}_\beta \,, \tag{7.3}$$

referred to as the *first* and the *second* field equation, respectively, with D as the covariant exterior derivative. In (7.2) and (7.3) we have the canonical energy–momentum and hypermomentum currents of matter Σ_α and $\Delta^\alpha{}_\beta$, the gravitational gauge field momenta

$$H_\alpha := -\frac{\partial V_{\text{dil-sh}}}{\partial T^\alpha} \qquad H^\alpha{}_\beta := -\frac{\partial V_{\text{dil-sh}}}{\partial R_\alpha{}^\beta} \,, \tag{7.4}$$

and the canonical energy–momentum and hypermomentum currents of the gauge fields

$$E_\alpha = e_\alpha \rfloor V_{\text{dil-sh}} + (e_\alpha \rfloor T^\beta) \wedge H_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge H^\beta{}_\gamma + \tfrac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma}) M^{\beta\gamma} \,, \tag{7.5}$$

$$E^\alpha{}_\beta = -\vartheta^\alpha \wedge H_\beta - M^\alpha{}_\beta \,. \tag{7.6}$$

The gravitational gauge field momentum $M^{\alpha\beta}$ is coupled to the non-metricity:

$$M^{\alpha\beta} := -2\,\frac{\partial V_{\text{dil-sh}}}{\partial Q_{\alpha\beta}} \,. \tag{7.7}$$

We study only the behaviour of the electromagnetic field in the metric-affine framework. Thus, for the matter currents in (7.2) and (7.3), we have $\Sigma_\alpha = \Sigma_\alpha^{\text{Max}}$, cf (6.2), and $\Delta^\alpha{}_\beta = 0$.

The formalism of MAG, as outlined in the present section, is not limited to the simple and very restricted Lagrangian (7.1); more general choices for the Lagrangian are possible. It is our intention, however, not to look at MAG for its own interest, but to investigate the behaviour of Maxwell's theory within a non-Riemannian spacetime. This was our motivation for making the simplest possible choice of the metric-affine part of the Lagrangian that still allows for propagating torsion and non-metricity, see Obukhov *et al* [32].

8. Exact solution with spherical symmetry

The field equations (7.2) and (7.3), together with Maxwell's equations (2.3) and (4.2)—assuming the constitutive law (5.2)—are approached as follows. The spherically symmetric coframe

$$\vartheta^{\hat{0}} = f\, dt \,, \qquad \vartheta^{\hat{1}} = \frac{1}{f}\, dr \,, \qquad \vartheta^{\hat{2}} = r\, d\theta \,, \qquad \vartheta^{\hat{3}} = r\, \sin\theta\, d\phi \,, \tag{8.1}$$

contains the 0-form $f = f(r)$ and is assumed to be orthonormal, i.e. the metric reads

$$ds^2 = o_{\alpha\beta}\,\vartheta^\alpha \otimes \vartheta^\beta = -f^2\,dt^2 + \frac{1}{f^2}\,dr^2 + r^2\big(d\theta^2 + \sin^2\theta\,d\phi^2\big)\,. \tag{8.2}$$

The non-metricity 1-form is taken to contain only two irreducible pieces (see [13, appendix B.1]),

$$Q_{\alpha\beta} = {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta}\,, \tag{8.3}$$

namely the dilation (or Weyl) piece ${}^{(4)}Q_{\alpha\beta} = Q g_{\alpha\beta}$ and a proper shear piece

$${}^{(3)}Q_{\alpha\beta} = \tfrac{4}{9}\big(\vartheta_{(\alpha}e_{\beta)} \rfloor \Lambda - \tfrac{1}{4}g_{\alpha\beta}\Lambda\big)\,, \qquad \text{with} \qquad \Lambda := \vartheta^\alpha e^\beta \rfloor {\not{Q}}_{\alpha\beta}\,. \tag{8.4}$$

Furthermore, we allow only the covector piece ${}^{(2)}T^\alpha$ in the torsion 2-form:

$$T^\alpha = {}^{(2)}T^\alpha = \tfrac{1}{3}\,\vartheta^\alpha \wedge T\,. \tag{8.5}$$

Finally, we use a spherically symmetric electric (Coulomb) charge at the origin of the spatial coordinates with the corresponding field strength

$$F = \frac{q}{r^2}\,\vartheta^{\hat{1}} \wedge \vartheta^{\hat{0}}\,, \tag{8.6}$$

and we impose the constitutive law (5.2).

With these prescriptions and the ansatz

$$Q = u(r)\,\vartheta^{\hat{0}}\,, \qquad \Lambda = v(r)\,\vartheta^{\hat{0}}\,, \qquad T = \tau(r)\,\vartheta^{\hat{0}} \tag{8.7}$$

for the *1-form triplet* (Q, Λ, T), the solution is expressed by

$$f = \sqrt{1 - \frac{2\kappa M}{r} + \frac{\lambda r^2}{3} + \frac{\kappa q^2}{2r^2} + \alpha\frac{\kappa \widetilde{N}^2}{2r^2}} \tag{8.8}$$

and

$$u = \frac{\widetilde{N}}{fr}\,, \qquad v = \frac{3\beta}{2}\,\frac{\widetilde{N}}{fr}\,, \qquad \tau = -\frac{\beta+6}{4}\,\frac{\widetilde{N}}{fr}\,, \tag{8.9}$$

where $\widetilde{N}$ is an integration constant. The dimensionless coupling constants are subject to the *constraint*

$$\gamma = -\frac{8}{3}\,\frac{\beta}{\beta+6}\,, \tag{8.10}$$

i.e. only two of the post-Riemannian coupling constants (α, β, γ) in (7.1) remain independent, while the third one, γ, is determined by (8.10). These results have been found with the help of the computer algebra system REDUCE [36] making use also of its Excalc package [37], see [38].

Let us summarize the properties of the MAG–Maxwell solution that is presented here. The 0-form f, which fixes the orthonormal coframe (8.1), has four contributions, see (8.8). The terms containing the mass parameter M, the cosmological constant λ, and the electric charge q correspond exactly to the (general relativistic) Reissner–Nordström solution with cosmological constant. The additional term with the dilation charge $\widetilde{N}$ has a similar structure as the previous term with the electric charge q. The non-metricity has the explicit form

$$Q^{\alpha\beta} = \frac{\widetilde{N}}{fr}\,\big[o^{\alpha\beta} + \tfrac{2}{3}\,\beta\big(\vartheta^{(\alpha}e^{\beta)} \rfloor - \tfrac{1}{4}\,o^{\alpha\beta}\big)\big]\vartheta^{\hat{0}} \tag{8.11}$$

carrying, besides the dilation piece, a shear part—the second term in (8.11) with the factor β. The torsion 2-form evaluates to

$$T^{\alpha} = -\frac{\beta+6}{12}\,\frac{\widetilde{N}}{fr}\,\vartheta^{\alpha}\wedge\vartheta^{\hat{0}}\,, \tag{8.12}$$

and the Faraday 2-form

$$F = \frac{q}{r^2}\,\vartheta^{\hat{1}}\wedge\vartheta^{\hat{0}} \tag{8.13}$$

has the same innocent appearance as that of a point charge in *flat* Minkowski space. It is clear, however, that all relevant geometric objects, coframe, connection, torsion, curvature, etc, 'feel'—via the 0-form f—the presence of the electric charge. However, as one can recognize from (8.11)–(8.13), the Maxwell field is otherwise disconnected from nonmetricity and torsion. This exemplifies and is in full accordance with our general statement concerning the coupling of the Maxwell equations to post-Riemannian structures.

9. Discussion

There is so much experimental evidence in favour of the conservation laws of electric charge and magnetic flux that one can hardly doubt the correctness of axioms 1 and 3 from a physical point of view. The form of the Maxwell equations is then fixed, and we have no trouble in predicting how they change in spacetimes with Riemannian and post-Riemannian geometrical structure: *they do not change at all*. They are stable against such 'deformations'. Thereby the equivalence principle turns out to be rather trivial in this context. The only 'freedom' one has is to modify the constitutive law. Incidentally, if the limits of classical physics are reached, then, on the level of quantum mechanics, a fresh look at the equivalence principle is needed, see [39].

Coming back to the article of Vandyck [16], we recognize that the different options for generalizing the Maxwell equations are artificial ones in the sense that they violate the well established axioms 1 and 3, namely the conservation of electric charge and magnetic flux. These options can only emerge if one forgets the underlying physical structure of Maxwell's theory. Clearly, whether one uses the calculus of tensor analysis (see the appendix) or that of exterior differential forms, does not make any difference, if one starts off with our axioms.

In the framework of the Poincaré gauge theory of gravitation, the spacetime of which carries, besides the metric, a propagating torsion, we also found exact electrically charged solutions, see [40]. In this latter context, as well as in the case of the new charged solution of MAG that was presented in section 8, we used Maxwell's theory as described in sections 2–5, and everything is well behaved and consistent with our analysis of how to couple the Maxwell equations to post-Riemannian structures. There is almost no freedom for an alternative coupling of Maxwell's equations to gravity within Riemannian or post-Riemannian spacetimes. Equations (2.3), (4.2) and (5.2) solve the problem completely.

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Appendix. The tensor analysis version of metric-free electrodynamics

We decompose excitation, field strength and current into (holonomic) coordinate components:

$$G = \frac{1}{2!}\, G_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j\,, \qquad F = \frac{1}{2!}\, F_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j\,, \qquad J = \frac{1}{3!}\, J_{ijk}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k\,. \tag{A.1}$$

If we use the Levi-Civita antisymmetric unit tensor *density* $\epsilon^{ijkl} = \pm 1, 0$, which is metric-free,

$$\mathcal{G}^{ij} := \frac{1}{2!}\, \epsilon^{ijkl}\, G_{kl}\,, \qquad \mathcal{J}^i := \frac{1}{3!}\, \epsilon^{ijkl}\, J_{jkl}\,, \tag{A.2}$$

then Maxwell's equations read

$$\partial_k \mathcal{G}^{ik} = \mathcal{J}^i\,, \qquad \partial_{[i} F_{jk]} = 0\,. \tag{A.3}$$

The constitutive law for the vacuum can be put in the linear form

$$\mathcal{G}^{ij} = \tfrac{1}{2}\chi^{ijkl} F_{kl}\,, \qquad \chi^{(ij)kl} = \chi^{ij(kl)} = \chi^{[ijkl]} = 0\,, \qquad \chi^{ijkl} = \chi^{klij}\,, \tag{A.4}$$

with the specific metric dependent 'modulus'

$$\chi^{ijkl} := 2\sqrt{|\det g_{mn}|}\; g^{k[i} g^{j]l}\,. \tag{A.5}$$

Equations (A.3)–(A.5) remain valid in post-Riemannian spacetimes. Note that the representation of the electromagnetic excitation $\mathcal{G}^{ij}$ as a density, see Schrödinger [41], is vital for these considerations and distinguishes our approach from that of Vandyck.

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