COMPARING EXPECTED WAIT TIMES OF A $M/M/1$ QUEUE

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1. Abstract

In this report we calculate transient probabilities $p_i(t|\, n \text{ at time } 0)$ for all i, of an $M/M/1$ queue, based on task of time t, n customers at time 0, the arrival rate λ, and the service rate μ. Using these probabilities we can arrive at the expected system time $E\left(T(a)\right)$ of the task and the queue if the task is completed after the customer receives service, and the expected wait time $E\left(T(b)\right)$ of the task and the queue if the customer finishes his task and then returns to the queue. We discover when completing the additional task first becomes more efficient based on various n, t, μ, and λ.

2. Problem

We have an $M/M/1$ queueing system. A customer arrives and sees n people in the system. This customer must receive service from the server and has an additional task of fixed length t. The customer has two choices:

(a) Join the queue first, then do the task
(b) Do the task first, then join the queue

We will determine at what point the second option becomes more desirable based on arrival rate λ, service rate μ, number of customers seen initially n, and the length of time of the additional task t.

3. Notation

The following notation will be used throughout the report

- t is the length of time it takes to complete the additional task.
- n is the number of people in line at time 0.
- i is the number of people in line after you return from your additional task.
- $E\left(T(a)\right)$ is the expected time to complete your tasks if you do your additional task after.
- $E\left(T(b)\right)$ is the expected time to complete your tasks if you do your additional task first.
- ρ is the ratio $\frac{\lambda}{\mu}$.
- L-Expected number of customers in an M/M/1 queue $L = \frac{\rho}{1-\rho}$.
- W-Expected time spent in an M/M/1 queue $W = \frac{1}{\mu-\lambda}$.

4. Introduction

We assume that we have an $M/M/1$ queueing system and interarrival and service time are exponentially distributed (no memory property).

Definition 1. *An $M/M/1$ queue is a single server queueing system with the following properties:*

1. *Arrivals are a Poisson process (interarrival times are exponential)*
2. *Service times are exponential*
3. *There is one server*
4. *The capacity of the system is infinite*

Definition 2. *Steady state is defined as the long run average state of the system also known as the equilibrium.*

5. Base Case

We can assume t is large enough so that when returning to the queue the system is in steady state (equilibrium). Assume you had n customers when you fist observed the queue. We will compare $E\left(T(a)\right)$ and $E\left(T(b)\right)$ to decide when doing the additional task first becomes the more efficient option.

1. $E\left(T(a)\right) = n(\frac{1}{\mu}) + \frac{1}{\mu} + t$
2. $E\left(T(b)\right) = t + E(W) = t + \frac{1}{\mu - \lambda}$

We compare when $E\left(T(a)\right) > E\left(T(b)\right)$ (at this point we will choose option b).

$$
\begin{aligned}
& E\left(T(a)\right) > E\left(T(b)\right) \\
\iff & n(\tfrac{1}{\mu}) + \tfrac{1}{\mu} + t > t + \tfrac{1}{\mu - \lambda} \\
\iff & n + 1 > \tfrac{1}{1 - \frac{\lambda}{\mu}} \\
\iff & n - \rho n - \rho > 0 \\
\iff & n(1 - \rho) > \rho \\
\iff & n > L.
\end{aligned}
$$

We can see that if the number of people in line first observed n, is less than the expected number of people in line L, we will wait to do our additional task. At the point when the number of people in line is exactly the expected number we are indifferent to choosing option a or option b. Finally, when we observe more people than the expected number we will do our additional task and then return to the queue. This observation is true only when our queue is in equilibrium.

6. General Case

This case does not assume t is large, when you return to the queue it is not necessarily in equilibrium. Assume n is the number of customers initially observed and i is the number of customers observed after returning to the queue.

(a) $E\left(T(a)\right) = n(\frac{1}{\mu}) + \frac{1}{\mu} + t$,

(b) $E\left(T(b)\right) = t + E(W^{*}) = t + \sum_{i=0}^{\infty} p_i(t|\text{ n at time } 0)\frac{i+1}{\mu}$.

There are many different ways to evaluate $p_i(t|\text{ n at time } 0)$, and many different researcher have proposed methods they claim to be faster computationally than others. We look at two different methods in particular and discover one that best suits our needs. The two we focus on were developed by Sharma [1] and Conolly and Langaris [2], but there have been methods created such as A.M.K. Tarabia (2002) [3], L. Kleinrock (1975) [4]. As well as many others.

The formula by Sharma [1] can be used to evaluate $p_i(t|\text{ n at t=0})$:

$$p_i(t|\text{ n at time } 0) = (1-\rho)\rho^i + e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty}\left(\frac{(\lambda t)^k}{k!}\sum_{m=0}^{i+k+n}(k-m)\frac{(\mu t)^{m-1}}{m!}\right)$$
$$+ e^{-(\lambda+\mu)t}\sum_{k=0}^{\infty}(\lambda t)^{i+k-n}(\mu t)^k\left(\frac{1}{k!(i+k-n)!} - \frac{1}{(i+k)!(k-n)!}\right).$$

Notice there are some issues with negative factorials in this formula (for the purpose of our calculations). We will use an equivalent formula that removes any problems involving these factorials. Developed by Conolly and Langaris [2] this is a generalized version of Sharma and Shobha (where our initially state n can be any number, not just 0).

This formula (we will call formula 1.1) is used to do all calculations in this report:

$$p_i(t) = (1-\rho)\rho^i + e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty}\left(\frac{(\lambda t)^k}{k!}\sum_{m=0}^{k+i+n+1}(k-m)\frac{(\mu t)^{m-1}}{m!}\right)$$
$$+ e^{-(\lambda+\mu)t}\rho^i\sum_{k=0}^{\infty}\frac{(\lambda t)^{k+1}(\mu t)^{k+max(i,n)}}{k!}\left(\frac{(\lambda t)^{-min(i,n)-1}}{(k+|n-i|)!} - \frac{(\mu t)^{min(i,n)+1}}{(k+n+i+2)!}\right).$$

To evaluate these large summations we use the programming language R. We cap off these summations at values between 50 and 80 and claim this does not affect our final answer to any significance.

Since R cannot do factorials over 170 we rearrange formula 1.1 so that R has enough memory to evaluate these large factorials. The memory limit can be avoided by rearranging the formula in a way that will avoid direct computation of the factorials. Factorials will be paired up with a variable whose exponent is equal to the factorial. i.e. the calculation of expressions in the form: $\frac{x^{m+2}}{(m+2)!}$.

Focusing on what is inside the final summation of formula 1.1 we have:

$$\frac{(\lambda t)^{k+1}(\mu t)^{k+max(i,n)}}{k!}\left(\frac{(\lambda t)^{-min(i,n)-1}}{(k+|n-i|)!}-\frac{(\mu t)^{min(i,n)+1}}{(k+n+i+2)!}\right),$$

we can rearrange this to give us

$$\frac{(\mu t)^k}{k!}\frac{(\lambda t)^k(\mu t)^{max(i,n)}(\lambda t)^{-min(i,n)}}{(k+|n-i|)!}-\frac{(\lambda t)^k}{k!}\frac{(\lambda t)(\mu t)(\mu t)^k(\mu t)^{max(i,n)}(\mu t)^{min(i,n)}}{(k+n+i+2)!}.$$

To remove the max and min functions we can separate into cases $i<n$ and $i\geq n$. Which gives us the following:

CASE i<n

$$\begin{aligned}&\frac{(\mu t)^k}{k!}\frac{(\lambda t)^k(\mu t)^n(\lambda t)^{-i}}{(k+n-i)!}-\frac{(\lambda t)^k}{k!}\frac{(\lambda t)(\mu t)(\mu t)^k(\mu t)^n(\mu t)^i}{(k+n+i+2)!}\\&=\frac{(\mu t)^k}{k!}\frac{(\lambda t)^{k+n-i}}{(k+n-i)!}\left(\frac{\mu}{\lambda}\right)^n-\frac{(\lambda t)^k}{k!}\frac{(\mu t)^{k+n+i+2}}{(k+n+i+2)!}\left(\frac{\lambda}{\mu}\right).\end{aligned}$$

CASE i≥n

$$\begin{aligned}&\frac{(\mu t)^k}{k!}\frac{(\lambda t)^k(\mu t)^i(\lambda t)^{-n}}{(k+i-n)!}-\frac{(\lambda t)^k}{k!}\frac{(\lambda t)(\mu t)(\mu t)^k(\mu t)^i(\mu t)^n}{(k+n+i+2)!}\\&=\frac{(\mu t)^k}{k!}\frac{(\lambda t)^{k+i-n}}{(k+i-n)!}\left(\frac{\mu}{\lambda}\right)^i-\frac{(\lambda t)^k}{k!}\frac{(\mu t)^{k+i+n+2}}{(k+i+n+2)!}\left(\frac{\lambda}{\mu}\right).\end{aligned}$$

This allows us to create a function in R that does division before exponents and factorials. By doing division first R can handle the value computationally since these values do not grow as quickly. This new function can calculate factorials (that large exponentials are being divided by) of almost any size, without exhausting R's memory power.

Using this new (computationally acceptable by R) formula we can evaluate these probabilities and expected wait times for different values of n, i, μ, λ, and t.

We can put all of this together to give us the entire formula that we will be using for our calculations:

$$
\begin{aligned}
E\left(T(b)\right) &= t + \sum_{i=0}^{\infty} p_i(t|\text{ n at time 0})\tfrac{i+1}{\mu} \\
&= t + \sum_{i=0}^{\infty}\left((1-\rho)\rho^i + e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty}\left(\frac{(\lambda t)^k}{k!}\sum_{m=0}^{k+i+n+1}(k-m)\frac{(\mu t)^{m-1}}{m!}\right)\right. \\
&+ e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty}\frac{(\lambda t)^{k+1}(\mu t)^{k+max(i,n)}}{k!}\left(\frac{(\lambda t)^{-min(i,n)-1}}{(k+|n-i|)!} - \frac{(\mu t)^{min(i,n)+1}}{(k+n+i+2)!}\right)\left.\vphantom{\sum}\right)\tfrac{i+1}{\mu}.
\end{aligned}
$$

Breaking it into cases and rearranging to be computationally acceptable to R we get:

<u>CASE i<n</u>

$$
\begin{aligned}
E\left(T(b)\right) &= t + \sum_{i=0}^{\infty}\left((1-\rho)\rho^i + e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty}\left(\frac{(\lambda t)^k}{k!}\sum_{m=0}^{k+i+n+1}\frac{k-m}{\mu t}\frac{(\mu t)^{m}}{m!}\right)\right. \\
&+ e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty}\left(\frac{(\mu t)^k}{k!}\frac{(\lambda t)^{k+n-i}}{(k+n-i)!}\left(\frac{\mu}{\lambda}\right)^n - \frac{(\lambda t)^k}{k!}\frac{(\mu t)^{k+n+i+2}}{(k+n+i+2)!}\left(\frac{\lambda}{\mu}\right)\right)\left.\vphantom{\sum}\right)\tfrac{i+1}{\mu}.
\end{aligned}
$$

<u>CASE i≥n</u>

$$
\begin{aligned}
E\left(T(b)\right) &= t + \sum_{i=0}^{\infty}\left((1-\rho)\rho^i + e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty}\left(\frac{(\lambda t)^k}{k!}\sum_{m=0}^{k+i+n+1}\frac{k-m}{\mu t}\frac{(\mu t)^{m}}{m!}\right)\right. \\
&+ e^{-(\lambda+\mu)t}\rho^i \sum_{k=0}^{\infty}\left(\frac{(\mu t)^k}{k!}\frac{(\lambda t)^{k+i-n}}{(k+i-n)!}\left(\frac{\mu}{\lambda}\right)^i - \frac{(\lambda t)^k}{k!}\frac{(\mu t)^{k+i+n+2}}{(k+i+n+2)!}\left(\frac{\lambda}{\mu}\right)\right)\left.\vphantom{\sum}\right)\tfrac{i+1}{\mu}.
\end{aligned}
$$

These formulas were coded into R to compute the expected wait times for different cases of option b.

7. Calculations

The following tables are expected wait times comparing both option a (do additional task after) $E\left(T(a)\right)$ and option b (do additional task first) $E\left(T(b)\right)$. See other report for inner calculations of $p_i(t|\, n$ at time 0). Those calculations are for specific values of i (the number of people observed when we return to the queue). We need to sum over all i, in order to achieve the expected wait times.

In the tables below the values of t are organized by page, the tables represent different values of μ, the rows different values of n, the columns λ, and each column shows both the expected wait time of option a, and that of option b. The shaded boxes represent the point at which option b becomes more desirable (leaving to do our additional task and returning to the queue). Notice that when ρ is held constant, change μ and λ gives different expected wait times.

There is an interesting pattern in the calculations to notice. Comparing expected values of $\lambda = \frac{19}{20}, \mu = 1, t = 2$ and $\lambda = \frac{19}{10}, \mu = 2, t = 1$ we can see that the second cases is twice as fast as the first (doubled service time, doubled interarrival time and halved time of additional task). As suspected the expected time of the queue in the first case is double that of the second. This can be seen by comparing the tables when $t = 1$ and $t = 2$ (with the exception of rounding errors).

Table for $\mu = 2, t = 0.01$

n	$\lambda = \frac{1}{10}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = 1$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{2}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{10}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = 2$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	0.51	0.510495	0.51	0.514951	0.51	0.517426	0.51	0.519406	0.51	0.519901
$n=1$	1.01	1.000599	1.01	1.005099	1.01	1.007598	1.01	1.009598	1.01	1.010098
$n=2$	1.51	1.500501	1.51	1.505001	1.51	1.507501	1.51	1.509501	1.51	1.510001
$n=3$	2.01	2.0005	2.01	2.005	2.01	2.0075	2.01	2.0095	2.01	2.01
$n=4$	2.51	2.5005	2.51	2.505	2.51	2.5075	2.51	2.5095	2.51	2.51
$n=5$	3.01	3.0005	3.01	3.005	3.01	3.0075	3.01	3.0095	3.01	3.01
$n=6$	3.51	3.5005	3.51	3.505	3.51	3.5075	3.51	3.5095	3.51	3.51
$n=7$	4.01	4.0005	4.01	4.005	4.01	4.0075	4.01	4.0095	4.01	4.01
$n=8$	4.51	4.5005	4.51	4.505	4.51	4.5075	4.51	4.5095	4.51	4.51
$n=9$	5.01	5.0005	5.01	5.005	5.01	5.0075	5.01	5.0095	5.01	5.01
$n=10$	5.51	5.5005	5.51	5.505	5.51	5.5075	5.51	5.5095	5.51	5.51
$n=30$	15.51	15.5005	15.51	15.505	15.51	15.5075	15.51	15.5095	15.51	15.51

Table for $\mu = 1, t = 0.01$

n	$\lambda = \frac{1}{20}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = \frac{1}{2}$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{4}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{20}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = 1$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	1.01	1.010498	1.01	1.014975	1.01	1.017463	1.01	1.019453	1.01	1.01995
$n=1$	2.01	2.00055	2.01	2.00505	2.01	2.00755	2.01	2.00955	2.01	2.01005
$n=2$	3.01	3.0005	3.01	3.005	3.01	3.0075	3.01	3.0095	3.01	3.01
$n=3$	4.01	4.0005	4.01	4.005	4.01	4.0075	4.01	4.0095	4.01	4.01
$n=4$	5.01	5.0005	5.01	5.005	5.01	5.0075	5.01	5.0095	5.01	5.01
$n=5$	6.01	6.0005	6.01	6.005	6.01	6.0075	6.01	6.0095	6.01	6.01
$n=6$	7.01	7.0005	7.01	7.005	7.01	7.0075	7.01	7.0095	7.01	7.01
$n=7$	8.01	8.0005	8.01	8.005	8.01	8.0075	8.01	8.0095	8.01	8.01
$n=8$	9.01	9.0005	9.01	9.005	9.01	9.0075	9.01	9.0095	9.01	9.01
$n=9$	10.01	10.005	10.01	10.005	10.01	10.0075	10.01	10.0095	10.01	10.01
$n=10$	11.01	11.0005	11.01	11.005	11.01	11.0075	11.01	11.0095	11.01	11.01
$n=30$	31.01	31.0005	31.01	31.005	31.01	31.0075	31.01	31.0095	31.01	31.01

Table for $\mu = \frac{1}{2}, t = 0.01$

n	$\lambda = \frac{1}{40}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = \frac{1}{4}$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{8}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{40}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = \frac{1}{2}$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	2.01	2.010499	2.01	2.014988	2.01	2.017481	2.01	2.019476	2.01	2.019975
$n=1$	4.01	4.000525	4.01	4.005025	4.01	4.007525	4.01	4.009525	4.01	4.010025
$n=2$	6.01	6.0005	6.01	6.005	6.01	6.0075	6.01	6.0095	6.01	6.01
$n=3$	8.01	8.0005	8.01	8.005	8.01	8.0075	8.01	8.0095	8.01	8.01
$n=4$	10.01	10.0005	10.01	10.005	10.01	10.0075	10.01	10.0095	10.01	10.01
$n=5$	12.01	12.0005	12.01	12.005	12.01	12.0075	12.01	12.0095	12.01	12.01
$n=6$	14.01	14.0005	14.01	14.005	14.01	14.0075	14.01	14.0095	14.01	14.01
$n=7$	16.01	16.0005	16.01	16.005	16.01	16.0075	16.01	16.0095	16.01	16.01
$n=8$	18.01	18.0005	18.01	18.005	18.01	18.0075	18.01	18.0095	18.01	18.01
$n=9$	20.01	20.0005	20.01	20.005	20.01	20.0075	20.01	20.0095	20.01	20.01
$n=10$	22.01	22.0005	22.01	22.005	22.01	22.0075	22.01	22.0095	22.01	22.01
$n=30$	62.01	62.0005	62.01	62.005	62.01	62.0075	62.01	62.0095	62.01	62.01

Table for $\boldsymbol{\mu = 2, t = 1}$

n	$\lambda=\frac{1}{10}$ $L=\frac{1}{19}$ $\rho=\frac{1}{20}$		$\lambda=1$ $L=1$ $\rho=\frac{1}{2}$		$\lambda=\frac{3}{2}$ $L=3$ $\rho=\frac{3}{4}$		$\lambda=\frac{19}{10}$ $L=19$ $\rho=\frac{19}{20}$		$\lambda=2$ $\rho=1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	1.5	1.522018	1.5	1.754062	1.5	1.906524	1.5	2.038862	1.5	2.073256
$n=1$	2	1.596451	2	1.887858	2	2.070984	2	2.226246	2	2.266132
$n=2$	2.5	1.808363	2.5	2.170012	2.5	2.384894	2.5	2.562223	2.5	2.607197
$n=3$	3	2.153335	3	2.563924	3	2.799015	3	2.989674	3	3.037638
$n=4$	3.5	2.585416	3.5	3.020842	3.5	3.265545	3.5	3.462307	3.5	3.511611
$n=5$	4	3.060552	4	3.505963	4	3.754349	4	3.953381	4	4.003175
$n=6$	4.5	3.55277	4.5	4.001514	4.5	4.251084	4.5	4.45083	4.5	4.50077
$n=7$	5	4.050648	5	4.500345	5	4.750243	5	4.950184	5	5.000171
$n=8$	5.5	4.550137	5.5	5.000071	5.5	5.250049	5.5	5.450037	5.5	5.500034
$n=9$	6	5.050026	6	5.500013	6	5.750009	6	5.950007	6	6.00006
$n=10$	6.5	5.550005	6.5	6.000002	6.5	6.250002	6.5	6.450001	6.5	6.500001
$n=30$	16.5	15.55	16.5	16	16.5	16.25	16.5	16.45	16.5	16.5

Table for $\boldsymbol{\mu = 1, t = 1}$

n	$\lambda=\frac{1}{20}$ $L=\frac{1}{19}$ $\rho=\frac{1}{20}$		$\lambda=\frac{1}{2}$ $L=1$ $\rho=\frac{1}{2}$		$\lambda=\frac{3}{4}$ $L=3$ $\rho=\frac{3}{4}$		$\lambda=\frac{19}{20}$ $L=19$ $\rho=\frac{19}{20}$		$\lambda=1$ $\rho=1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	2	2.031805	2	2.334745	2	2.514676	2	2.663844	2	2.701809
$n=1$	3	2.408805	3	2.787255	3	3.004367	3	3.181041	3	3.225587
$n=2$	4	3.15062	4	3.577266	4	3.816829	4	4.009556	4	4.057873
$n=3$	5	4.072583	5	4.516834	5	4.764317	5	4.962587	5	5.012189
$n=4$	6	5.054198	6	5.503061	6	5.752572	6	5.952238	6	6.002162
$n=5$	7	6.050664	7	6.500476	7	6.750396	7	6.950342	7	7.00033
$n=6$	8	7.050091	8	7.500065	8	7.250053	8	7.950046	8	8.000044
$n=7$	9	8.050011	9	8.500008	9	8.750006	9	8.950005	9	9.000005
$n=8$	10	9.050001	10	9.5	10	9.75	10	9.95	10	10
$n=9$	11	10.05	11	10.5	11	10.75	11	10.95	11	11
$n=10$	12	11.05	12	11.5	12	11.75	12	11.95	12	12
$n=30$	32	31.05	32	31.5	32	31.75	32	31.95	32	32

Table for $\boldsymbol{\mu = \frac{1}{2}, t = 1}$

n	$\lambda=\frac{1}{40}$ $L=\frac{1}{19}$ $\rho=\frac{1}{20}$		$\lambda=\frac{1}{4}$ $L=1$ $\rho=\frac{1}{2}$		$\lambda=\frac{3}{8}$ $L=3$ $\rho=\frac{3}{4}$		$\lambda=\frac{19}{40}$ $L=19$ $\rho=\frac{19}{20}$		$\lambda=\frac{1}{2}$ $\rho=1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	3	3.039419	3	3.400324	3	3.605267	3	3.771312	3	3.8131
$n=1$	5	4.260014	5	4.68467	5	4.922077	5	5.112695	5	5.16044
$n=2$	7	6.082114	7	6.527664	7	6.77548	7	6.973865	7	7.023478
$n=3$	9	8.053808	9	8.503236	9	8.752958	9	8.952754	9	9.002705
$n=4$	11	10.05037	11	10.50031	11	10.75028	11	10.95026	11	11.00026
$n=5$	13	12.05003	13	12.50002	13	12.75002	13	12.95002	13	13.00002
$n=6$	15	14.05	15	14.5	15	14.75	15	14.95	15	15
$n=7$	17	16.05	17	16.5	17	16.75	17	16.95	17	17
$n=8$	19	18.05	19	18.5	19	18.75	19	18.95	19	19
$n=9$	21	20.05	21	20.5	21	20.75	21	20.95	21	21
$n=10$	23	22.05	23	22.5	23	22.75	23	22.95	23	23
$n=30$	63	62.05	63	62.5	63	62.75	63	62.95	63	63

Table for $\mu = 2, t = 2$										
n	$\lambda = \frac{1}{10}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = 1$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{2}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{10}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = 2$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	2.5	2.525505	2.5	2.84723	2.5	3.098214	2.5	3.333099	2.5	3.396155
$n=1$	3	2.538566	3	2.912387	3	3.199772	3	3.464503	3	3.534942
$n=2$	3.5	2.594355	3.5	3.068756	3.5	3.41114	3.5	3.715956	3.5	3.795749
$n=3$	4	2.728046	4	3.324677	4	3.722051	4	4.063234	4	4.1511
$n=4$	4.5	2.960145	4.5	3.66941	4.5	4.110151	4.5	4.477798	4.5	4.5713
$n=5$	5	3.287219	5	4.080843	5	4.550232	5	4.934233	5	5.031096
$n=6$	5.5	3.688387	5.5	4.535362	5.5	5.021102	5.5	5.413932	5.5	5.512556
$n=7$	6	4.138035	6	5.014223	6	5.508189	6	5.905256	6	6.004704
$n=8$	6.5	4.614986	6.5	5.50528	6.5	6.002945	6.5	6.401844	6.5	6.50164
$n=9$	7	5.105432	7	6.001816	7	6.500985	7	6.900603	7	7.000533
$n=10$	7.5	5.60182	7.5	6.500581	7.5	7.000307	7.5	7.400184	7.5	7.500162
$n=30$	17.5	15.6	17.5	16.5	17.5	17	17.5	17.4	17.5	17.5

Table for $\mu = 1, t = 2$										
n	$\lambda = \frac{1}{20}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = \frac{1}{2}$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{4}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{20}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = 1$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	3	3.044037	3	3.508124	3	3.813048	3	4.077723	3	4.146512
$n=1$	4	3.192902	4	3.775715	4	4.141968	4	4.452491	4	4.532265
$n=2$	5	3.616727	5	4.340024	5	4.769789	5	5.124445	5	5.214395
$n=3$	6	4.306669	6	5.127847	6	5.59803	6	5.979348	6	6.075276
$n=4$	7	5.170833	7	6.041684	7	6.53109	7	6.924614	7	7.023221
$n=5$	8	6.121103	8	7.011926	8	7.508697	8	7.906762	8	8.00635
$n=6$	9	7.105539	9	8.003028	9	8.502168	9	8.90166	9	9.001554
$n=7$	10	8.101296	10	9.00069	10	9.500486	10	9.900368	10	10.00034
$n=8$	11	9.100273	11	10.00014	11	10.5001	11	10.90007	11	11.00007
$n=9$	12	10.10005	12	11.00003	12	11.50002	12	11.90001	12	12.00001
$n=10$	13	11.10001	13	12	13	12.5	13	12.9	13	13
$n=30$	33	31.1	33	32	33	32.5	33	32.9	33	33

Table for $\mu = \frac{1}{2}, t = 2$										
n	$\lambda = \frac{1}{40}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = \frac{1}{4}$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{8}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{40}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = \frac{1}{2}$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	4	4.063611	4	4.669489	4	5.029351	4	5.327689	4	5.403619
$n=1$	6	4.817611	6	5.57451	6	6.008734	6	6.362082	6	6.451174
$n=2$	8	6.301239	8	7.154531	8	7.633657	8	8.019113	8	8.115746
$n=3$	10	8.145167	10	9.033668	10	9.528635	10	9.925174	10	10.02438
$n=4$	12	10.1084	12	11.00612	12	11.50514	12	11.90448	12	12.00432
$n=5$	14	12.10133	14	13.00095	14	13.50079	14	13.90068	14	14.00066
$n=6$	16	14.10018	16	15.00013	16	15.50011	16	15.90009	16	16.00009
$n=7$	18	16.10002	18	17.00002	18	17.50001	18	17.90001	18	18.00001
$n=8$	20	18.1	20	19	20	19.5	20	19.9	20	20
$n=9$	22	20.1	22	21	22	21.5	22	21.9	22	22
$n=10$	24	22.1	24	23	24	23.5	24	23.9	24	24
$n=30$	64	62.1	64	63	64	63.5	64	63.9	64	64

Table for $\mu = 2, t = 3$

n	$\lambda = \frac{1}{10}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = 1$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{2}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{10}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = 2$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	3.5	3.526147	3.5	3.897223	3.5	4.227346	3.5	4.555946	3.5	4.64645
$n=1$	4	3.528732	4	3.935418	4	4.300867	4	4.661644	4	4.760395
$n=2$	4.5	3.542371	4.5	4.031598	4.5	4.45869	4.5	4.868489	4.5	4.978997
$n=3$	5	3.583976	5	4.200585	5	4.701673	5	5.163653	5	5.285947
$n=4$	5.5	3.676007	5.5	4.447919	5.5	5.021464	5.5	5.529403	5.5	5.661534
$n=5$	6	3.838582	6	4.768977	6	5.403673	6	5.946943	6	6.086282
$n=6$	6.5	4.08167	6.5	5.151545	6.5	5.831905	6.5	6.399479	6.5	6.543567
$n=7$	7	4.40229	7	5.580104	7	6.291112	7	6.87386	7	7.020802
$n=8$	7.5	4.78744	7.5	6.039752	7.5	6.769449	7.5	7.360878	7.5	7.509398
$n=9$	8	5.21986	8	6.518543	8	7.258682	8	7.854693	8	8.004021
$n=10$	8.5	5.68319	8.5	7.008144	8.5	7.753662	8.5	8.351919	8.5	8.501631
$n=30$	18.5	15.65	18.5	17	18.5	17.75	18.5	18.35	18.5	18.5

Table for $\mu = 1, t = 3$

n	$\lambda = \frac{1}{20}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = \frac{1}{2}$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{4}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{20}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = 1$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	4	4.048954	4	4.617824	4	5.027553	4	5.397695	4	5.495582
$n=1$	5	4.110213	5	4.798115	5	5.278147	5	5.703084	5	5.814291
$n=2$	6	4.331753	6	5.211536	6	5.784361	6	6.274707	6	6.401025
$n=3$	7	4.779767	7	5.847858	7	6.499237	7	7.040793	7	7.178468
$n=4$	8	5.446664	8	6.65197	8	7.354535	8	7.927473	8	8.071885
$n=5$	9	6.274132	9	7.559598	9	8.289584	9	8.878533	9	9.026292
$n=6$	10	7.196458	10	8.521102	10	9.2636	10	9.859573	10	10.00877
$n=7$	11	8.16567	11	9.506788	11	10.25426	11	10.85294	11	11.00268
$n=8$	12	9.154798	12	10.502	12	11.25123	12	11.85083	12	12.00075
$n=9$	13	10.15134	13	11.50054	13	12.25032	13	12.85022	13	12.00075
$n=10$	14	11.15035	14	12.50013	14	13.25008	14	13.85005	14	14.00005
$n=30$	34	31.15	34	32.5	34	33.25	34	33.85	34	34

Table for $\mu = \frac{1}{2}, t = 3$

n	$\lambda = \frac{1}{40}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = \frac{1}{4}$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{8}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{40}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = \frac{1}{2}$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	5	5.078636	5	5.865477	5	6.357166	5	6.77456	5	6.881963
$n=1$	7	5.54984	7	6.546013	7	7.141667	7	7.635876	7	7.761617
$n=2$	9	6.690242	9	7.879998	9	8.563142	9	9.118558	9	9.25849
$n=3$	11	8.321794	11	9.615367	11	10.34264	11	10.9278	11	11.07449
$n=4$	13	10.19604	13	11.52983	13	12.27347	13	12.8694	13	13.0185
$n=5$	15	12.16061	15	13.50668	15	14.25518	15	14.85422	15	15.00401
$n=6$	17	14.15214	17	15.50132	17	16.25101	17	16.85081	17	17.00077
$n=7$	19	16.15038	19	17.50023	19	18.25017	19	18.85014	19	19.00013
$n=8$	21	18.15006	21	19.50004	21	20.25003	21	20.85002	21	21.00002
$n=9$	23	20.15001	23	21.50001	23	22.25	23	22.85	23	23
$n=10$	25	22.15	25	23.5	25	24.25	25	24.85	25	25
$n=30$	65	62.15	65	63.5	65	64.25	65	64.85	65	65

Table for $\mu = 2, t = 7$

n	$\lambda = \frac{1}{10}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = 1$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{2}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{10}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = 2$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	7.5	7.52632	7.5	7.97147	7.5	8.51813	7.5	9.17516	7.5	9.37045
$n=1$	8	7.52632	8	7.97966	8	8.55084	8	9.24070	8	9.44550
$n=2$	8.5	7.52638	8.5	8.00155	8.5	8.62375	8.5	9.37236	8.5	9.59296
$n=3$	9	7.52661	9	8.04387	9	8.74286	9	9.56802	9	9.80778
$n=4$	9.5	7.52746	9.5	8.11398	9.5	8.91205	9.5	9.82323	9.5	10.08307
$n=5$	10	7.52996	10	8.21917	10	9.13269	10	10.13172	10	10.41075
$n=6$	10.5	7.53621	10.5	8.36571	10.5	9.40360	10.5	10.48609	10.5	10.78223
$n=7$	11	7.54997	11	8.55805	11	9.72135	11	10.87851	11	11.18904
$n=8$	11.5	7.57693	11.5	8.79824	11.5	10.08082	11.5	11.30133	11.5	11.62338
$n=9$	12	7.62474	12	9.08576	12	10.47584	12	11.74758	12	12.07844
$n=10$	12.5	7.70228	12.5	9.41766	12.5	10.89988	12.5	12.21126	12.5	12.54857
$n=30$	22.5	15.85003	22.5	19	22.5	20.74998	22.5	22.14916	22.5	22.5

Table for $\mu = 1, t = 7$

n	$\lambda = \frac{1}{20}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = \frac{1}{2}$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{4}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{20}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = 1$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	8	8.05247	8	8.82849	8	9.55786	8	10.30427	8	10.51219
$n=1$	9	8.05485	9	8.88910	9	9.68668	9	10.49852	9	10.72350
$n=2$	10	8.06830	10	9.04392	10	9.96575	10	10.88113	10	11.13131
$n=3$	11	8.11304	11	9.32205	11	10.40145	11	11.43250	11	11.70911
$n=4$	12	8.22178	12	9.74059	12	10.98490	12	12.12391	12	12.42381
$n=5$	13	8.43286	13	10.30082	13	11.69541	13	12.92321	13	13.24129
$n=6$	14	8.77818	14	10.98925	14	12.50614	14	13.79985	14	14.13080
$n=7$	15	9.27259	15	11.78265	15	13.38972	15	14.72818	15	15.06750
$n=8$	16	9.91044	16	12.65440	16	14.32231	16	15.68881	16	16.03316
$n=9$	17	10.66997	17	13.57978	17	15.28552	17	16.66834	17	17.01552
$n=10$	18	11.52177	18	14.53903	18	16.26658	18	17.65826	18	18.00693
$n=30$	38	31.35	38	34.5	38	36.25	38	37.65	38	38

Table for $\mu = \frac{1}{2}, t = 7$

n	$\lambda = \frac{1}{40}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = \frac{1}{4}$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{8}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{40}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = \frac{1}{2}$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n=0$	9	9.10040	9	10.31849	9	11.23283	9	12.07395	9	12.29812
$n=1$	11	9.18007	11	10.62302	11	11.68149	11	12.63776	11	12.89017
$n=2$	13	9.49544	13	11.33953	13	12.60328	13	13.70648	13	13.99297
$n=3$	15	10.19326	15	12.48047	15	13.93495	15	15.16308	15	15.47728
$n=4$	17	11.31823	17	13.97269	17	15.56535	17	16.87776	17	17.20994
$n=5$	19	12.80649	19	15.70659	19	17.38235	19	18.74258	19	19.08466
$n=6$	21	14.54398	21	17.58216	21	19.30080	21	20.68456	21	21.03138
$n=7$	23	16.42471	23	19.52987	23	21.26791	23	22.66189	23	23.01073
$n=8$	25	18.37623	25	21.50997	25	23.25582	25	24.65378	25	25.00339
$n=9$	27	20.35845	27	23.50307	27	25.25175	27	26.65112	27	27.00100
$n=10$	29	22.35251	29	25.50088	29	27.25049	29	28.65031	29	29.00027
$n=30$	69	62.35	69	65.5	69	67.25	69	68.65	69	69

Table for $\mu = 2, t = 10$

n	$\lambda = \frac{1}{10}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = 1$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{2}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{10}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = 2$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n = 0$	10.5	10.52632	10.5	10.98742	10.5	11.63752	10.5	12.50992	10.5	12.78102
$n = 1$	11	10.52632	11	10.99071	11	11.65906	11	12.56295	11	12.8439
$n = 2$	11.5	10.52632	11.5	10.99964	11.5	11.70751	11.5	12.67012	11.5	12.9681
$n = 3$	12	10.52632	12	11.01732	12	11.78786	12	12.83094	12	13.15061
$n = 4$	12.5	10.52634	12.5	11.04761	12.5	11.90434	12.5	13.04344	12.5	13.3872
$n = 5$	13	10.52641	13	11.09499	13	12.06007	13	13.30434	13	13.67269
$n = 6$	13.5	10.52662	13.5	11.16434	13.5	12.25686	13.5	13.60934	13.5	14.00132
$n = 7$	14	10.52718	14	11.26054	14	12.49509	14	13.95347	14	14.36707
$n = 8$	14.5	10.52856	14.5	11.38815	14.5	12.77372	14.5	14.33138	14.5	14.76397
$n = 9$	15	10.53163	15	11.55095	15	13.09049	15	14.73771	15	15.18639
$n = 10$	15.5	10.53791	15.5	11.75159	15.5	13.44209	15.5	15.16734	15.5	15.62918
$n = 30$	25.5	16.01065	25.5	20.5002	25.5	22.9998	25.5	24.99175	25.5	25.5

Table for $\mu = 1, t = 10$

n	$\lambda = \frac{1}{20}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = \frac{1}{2}$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{4}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{20}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = 1$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n = 0$	11	11.05261	11	11.89594	11	12.80307	11	13.8034	11	14.09062
$n = 1$	12	11.05287	12	11.92885	12	12.89569	12	13.96251	12	14.26791
$n = 2$	13	11.05451	13	12.01518	13	13.09974	13	14.27958	13	14.61373
$n = 3$	14	11.06104	14	12.177	14	13.42689	14	14.74492	14	15.11188
$n = 4$	15	11.08059	15	12.43435	15	13.88012	15	15.34198	15	15.74097
$n = 5$	16	11.12787	16	12.80163	16	14.4538	16	16.05002	16	16.47715
$n = 6$	17	11.22445	17	13.28506	17	15.13529	17	16.84675	17	17.29669
$n = 7$	18	11.39604	18	13.88189	18	15.90745	18	17.71086	18	18.17804
$n = 8$	19	11.66736	19	14.58157	19	16.7515	19	18.62364	19	19.10307
$n = 9$	20	12.05631	20	15.36823	20	17.64934	20	19.56989	20	20.05756
$n = 10$	21	12.56981	21	16.22375	21	18.58531	21	20.53808	21	21.03101
$n = 30$	41	31.5	41	36	41	38.5	41	40.4999	41	41

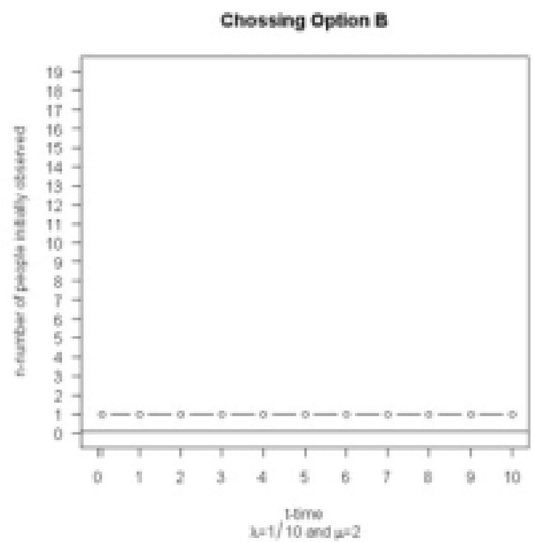
Table for $\mu = \frac{1}{2}, t = 10$

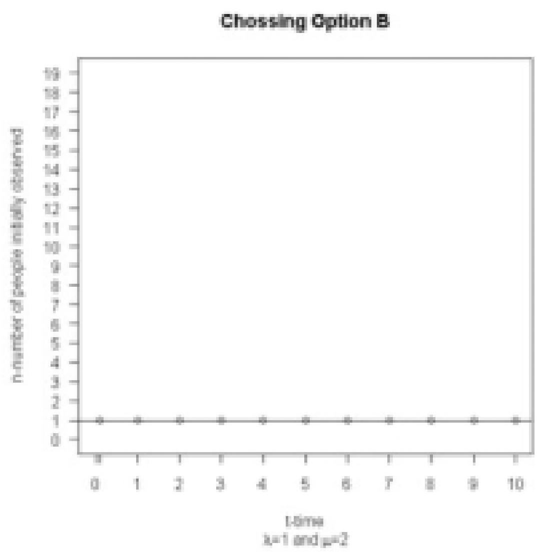
n	$\lambda = \frac{1}{40}$ $L = \frac{1}{19}$ $\rho = \frac{1}{20}$		$\lambda = \frac{1}{4}$ $L = 1$ $\rho = \frac{1}{2}$		$\lambda = \frac{3}{8}$ $L = 3$ $\rho = \frac{3}{4}$		$\lambda = \frac{19}{40}$ $L = 19$ $\rho = \frac{19}{20}$		$\lambda = \frac{1}{2}$ $\rho = 1$	
	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$	$E(T(a))$	$E(T(b))$
$n = 0$	12	12.1038	12	13.50217	12	14.67188	12	15.80222	12	16.10975
$n = 1$	14	12.12675	14	13.69883	14	15.01338	14	16.26888	14	16.60795
$n = 2$	16	12.23739	16	14.18445	16	15.73733	16	17.17395	16	17.55582
$n = 3$	18	12.54056	18	15.01281	18	16.83109	18	18.44828	18	18.87053
$n = 4$	20	13.14072	20	16.18262	20	18.23851	20	20.00269	20	20.45625
$n = 5$	22	14.09347	22	17.645	22	19.88188	22	21.74929	22	22.22391
$n = 6$	24	15.38611	24	19.32659	24	21.68413	24	23.61569	24	24.10294
$n = 7$	26	16.95431	26	21.1537	26	23.58288	26	25.5503	26	26.04438
$n = 8$	28	18.71517	28	23.06737	28	25.53489	28	27.52052	28	28.01796
$n = 9$	30	20.5944	30	25.02756	30	27.51376	30	29.50787	30	30.00684
$n = 10$	32	22.53849	32	27.01055	32	29.5051	32	31.50284	32	32.00245
$n = 30$	72	62.5	72	67	72	69.5	72	71.5	72	72

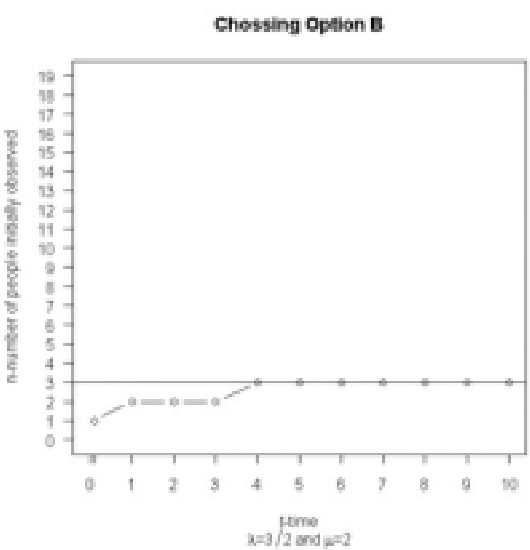
8. Graphical Analysis

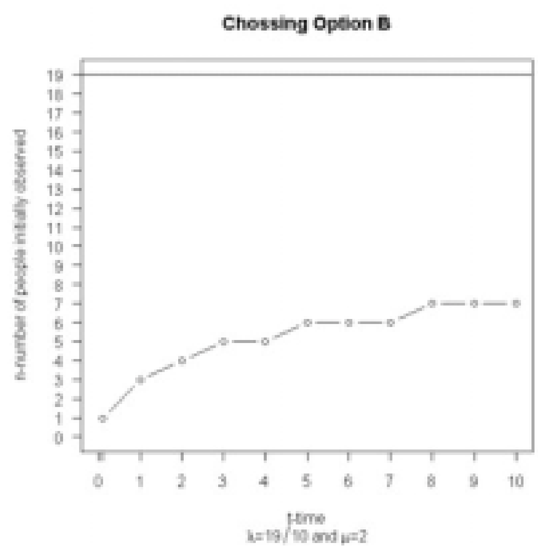
The following set of graphs represent at what point option b is more desirable. The horizonal lines drawn on each graph represent L; the average number of customers waiting in line. Notice, as in the calculations, the value of ρ remains constant but the graphs change.

The following set of graphs represent when $\mu = 2$.

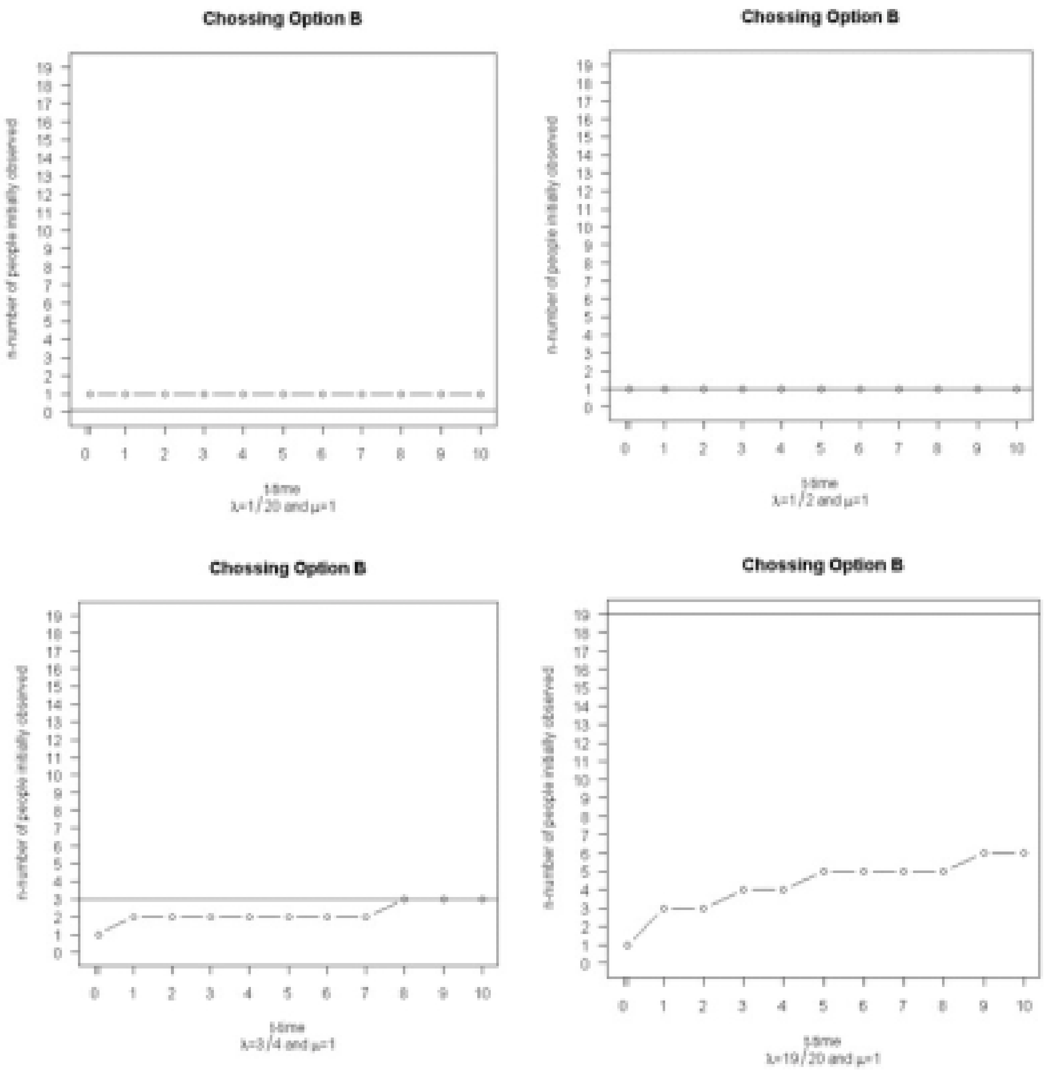




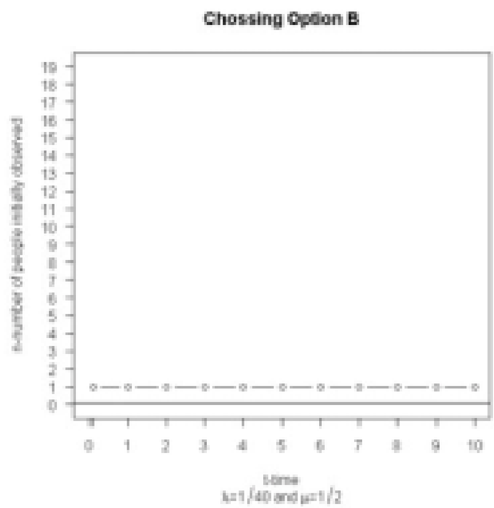


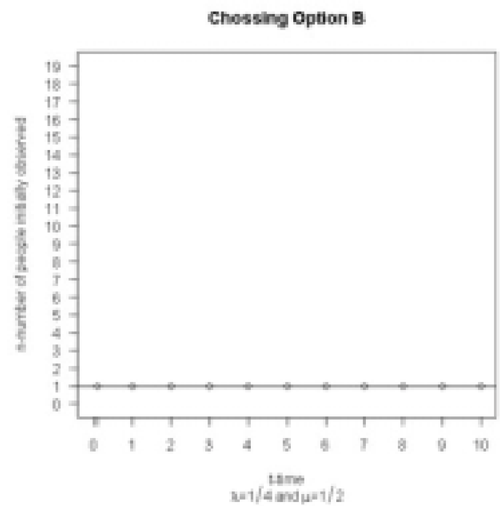


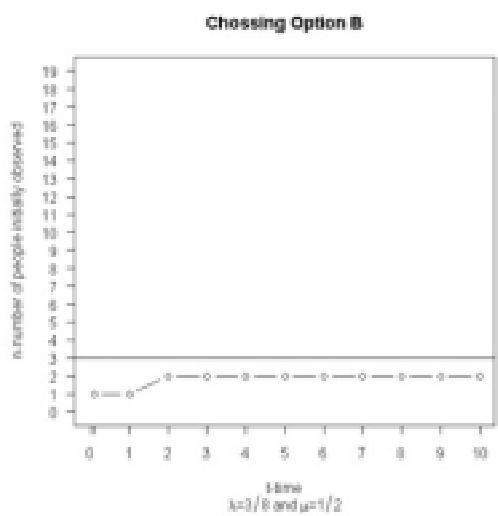
This set of graphs represent when $\mu = 1$. Notice there are does not need to be as many people in line before we choose option b. Since the service rate is slower it becomes more efficient to leave the queue to do our additional task first.

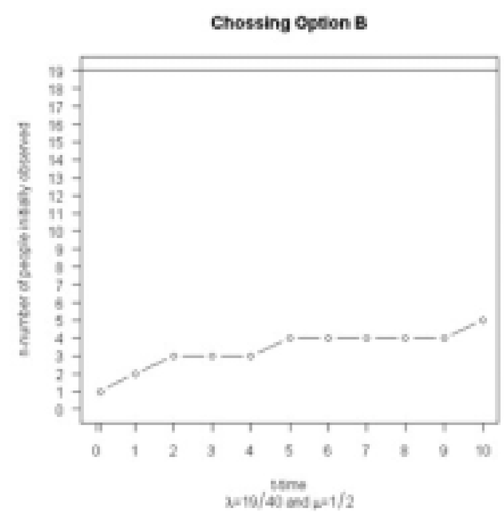


The last set of graphs represent when $\mu = \frac{1}{2}$. We can again notice that there needs to be less people in line before we leave the queue.









If we were to round L up to the nearest person, say $L*$, in these cases we always choose to do our additional task before the number of people in line reaches $L*$. This observation is true for all t. If we extended the graphs to show points for large t, we would see that our points would never surpass $L*$. This argument is represented by the base cases shown in an above section. When t gets large enough the queue is in steady state and we proved that option b becomes more favorable when $n > L$. This explains why the points of the graph will never be above $L*$. Another thing to notice is that these points will always reach $L*$ when t is large enough.

The pattern that as μ increases (ceteris paribus) it takes more people in line before we leave to do our additional task can be seen by looking at the different sets of graphs. We can see that the customer is less likely to leave as the service time of the queue is increases.

It is also noticeable that as λ increases approaching μ (ceteris paribus) it takes more people in line before we would leave to do our additional task. This is represented by the groups of graphs. We are less likely to leave if the arrival rate is high since there is a higher probability that more people join then exit the queue before we return.

The other variable to consider is t. As the length of time of our additional task increases the number of people in line n also needs to be larger before we leave the queue. The longer our additional task takes the less likely we are to leave.

In the tables a relationship was observed between the expected wait times when the arrival rate, service rate, and time of additional task are all doubled. Since the expected times of option a and option b both double, the point at which you decided to do option b remains constant. This is represented in the graphs by many different cases: e.g. $\lambda = \frac{19}{10}, \mu = 2, t = 3$ and $\lambda = \frac{19}{20}, \mu = 1, t = 6$

9. Conclusion and Acknowledgement

Steady State:
Option b is always better as soon as $n > L$ (for any μ and λ).

General Case:
Option b will always be better before $n > L$ (for any μ and λ) because when t is large enough we reach steady state. As μ, λ or t increase, it takes more people in line before option b is more efficient. If we double the speed of the system the expected times of option a and option b double. Since both expected system times double the point at which option b is more efficient remains the same.

Notice:
One would expect that we would always leave the line if the number of customers observed is less then the expected number of customers. Clearly this is not the case. When $\lambda = \frac{19}{10}, \mu = 2, t = 0.01$ our $L = 19$ customers. We leave the line as soon as we observe 1 customer. This is significantly different then leaving only when we observe 19 customers. This observation is only true when our queue is in steady state. Although the point at which we leave the queue is always less than the

expected number of customer it does not necessarily have to be equal.

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10. References

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