A Representation for Whole Numbers and Their Factors

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Abstract

In this paper I will describe models for whole numbers and their divisors using an adaptation of Hasse diagrams. The model gives us a unique visual representation for each whole number, because they are constructed based on the prime factorization of the numbers. Since every number has a unique prime factorization, the model for each number will be unique. The models are easy to make in the classroom and yield beautiful constructions. But most importantly, since they are a visual representation of important relations between a number and its factors, many theorems can be "rediscovered" by students when they investigate their constructions.

Rationale

Mathematical concepts can be represented in a variety of forms, including words, algebraic notation, diagrams, figures, graphs, or manipulative models. Studies in the area of language and semiosis, which have recently been applied to mathematics education, argue that learning of a concept does not take place unless the learner is able to coordinate different representations of that concept – that is, use different representations and translate a concept from one representation to another [1]. It is so important that teachers understand the role that the representations of a concept plays in students' learning that one of the National Council of Teachers of Mathematics' standards for school mathematics [2] deals specifically with the use of representations. As stated in that document: "Representations can help students organize their thinking. Students' use of representations can help make mathematical ideas more concrete and available for reflection" (p. 67).

Each representation highlights specific features of a concept. To fully grasp a concept, if that is ever attainable, one should explore as many different representations of that concept as possible. By examining and using different representations we can always gain more knowledge about a mathematical object or idea.

We use many representations of whole numbers in schools. Some of them include number words, numerals, base-ten blocks, Cuisinaire® rods, as well as discrete models, such as counters. Each of these models emphasizes or conveys a particular set of properties or characteristics of whole numbers, as well as relations between them: One representation may make the place-value concept used in our number system particularly evident; another may emphasize additive relationships between different numbers (for example, ten-frames make visible pairs of numbers that add up to ten); yet another may develop the notion of whole number as the cardinality of a set, or the use of whole numbers for counting (for example, when representing the number four by a set of four counters). The prime-factored form of a number is another representation for whole numbers – one that emphasizes the multiplicative relationships within them [3].

According to the NCTM's Principles and Standards for School Mathematics, students in grades 3-5 should recognize equivalent representations for the same number and generate them by decomposing and composing numbers. In the middle school grades, students should use factors, multiples, and prime factorization to solve problems [2].

However, for many students the prime-factored representation of a number is not meaningful. They often feel more comfortable by changing the number into base-ten form to be able to solve problems. In an interesting remark, Brown reports that a student justified this strategy by saying: "I need to see the actual numbers" [3], p. 134. While this is a valid approach, by doing this students lose valuable information that is carried by the prime-factored representation, as Brown herself pointed out.

While a variety of manipulative models are available which can help students build understanding of additive properties of numbers, we have been apparently lacking a good physical model that would highlight one of the most important properties of whole numbers: the fact that each of them can be written as a unique product of prime factors. In this paper I will describe such model – by no means original, but maybe unknown or under-used outside my home country, Brazil. This model can fill the gap we find when we look for multiplicative models among the available resources.

The lattice model

The model explored here as a representation for whole numbers and its factors is basically an application of the concept of Hasse diagrams. It was popularized in Brazil by Grossi as a classroom concrete material [4], but the concept itself can be found in any introduction to Hasse diagrams.

To build the models you will need Styrofoam balls and wooden sticks painted in different colors. Chenille-covered wire pieces are a good substitute for the wooden sticks, since they are quite firm and come in different colors.

In a nutshell, these are the principles used to construct a lattice model:

a) For each prime, we use sticks of different colors.
b) Each time we use a different prime (thus, a different color), we change the direction in which we place the sticks.
c) Each stick will tell us an operation to perform. The result will be written in foam balls. For example, if the prime 2 is represented by the color red, each time we go through a red stick we have to perform the operation "times 2" (or "divided by 2" if we go backwards).
d) We start with number 1 in a corner. The number represented by that lattice will end up in the opposite corner.

For example, the representation of number eight (2 × 2 × 2) requires three sticks to represent the operation "times 2". And it will show the four factors of number eight (Figure 1).

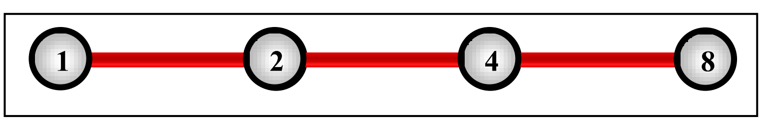

Figure 1: *The lattice model for the number 8.*

Once we introduce another prime in the picture, for example, if our task is to represent the number 24, which is 2 × 2 × 2 × 3, we will need to place sticks of a different color, and in a different direction. (Figure 2)

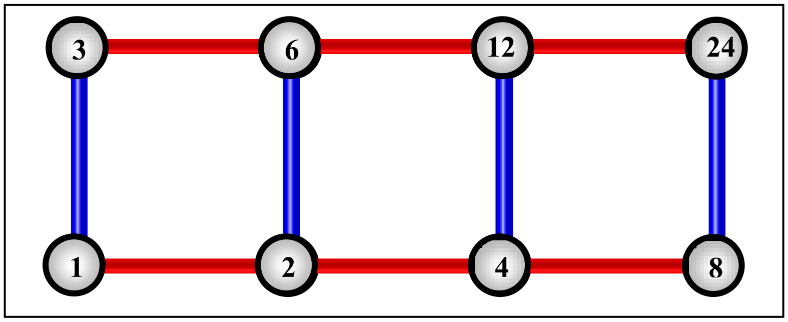

Figure 2: *The lattice model for the number 24.*

If the number we want to represent has 3 different primes in its composition, we will need a third color of stick to represent that third prime, and we will place it in a different direction. For example, the representation for number 60 = 2 × 2 × 3 × 5, will require the use of thee dimensions (Figure 3).

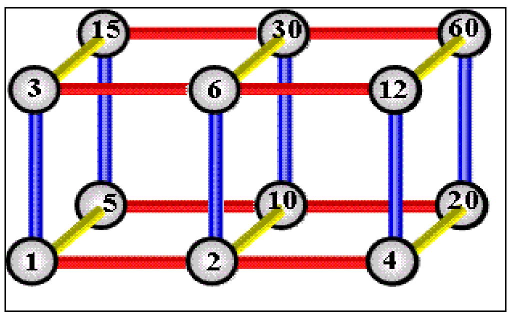

Figure 3: *The lattice model for the number 60.*

One interesting aspect that the construction of these models bring about is the multiplicative nature of the structure. For example, in the model for the number 60, one might initially think that we would need only one stick for the prime 3, since this number appears only once in the prime factorization of 60. But this is when the peculiarity of the multiplicative structure of whole numbers comes into play. Unlike additive structures, multiplicative structures repeatedly add self-similar smaller structures. This feature needs to be represented in any model of multiplicative structures and I believe it is evident in this model. The construction is also a good way to explore the fundamental theorem of arithmetic, which states that every positive integer different than 1 can be written as a unique product of prime numbers. Since every whole number has a unique prime factorization, the model for each number will be unique.

Even though each model will be unique, we can also notice that different numbers may have models that have the same basic structure. For example, any number with prime factorization $p_1^2 \times p_2 \times p_3$, where p_1, p_2, and p_3, are distinct primes, will have a structure similar to the one in figure 3. The only thing that will make it unique is the colors of the sticks (different primes will have different colors) and the value of the factors labeled in the spheres. Similarly, any number of the form p^3, where p is a prime, will have the same configuration as that shown in figure 1. These generalizations help students formulate conjectures, such as “any number with an odd number of factors will be a perfect square”.

Finally, I want to point out that we can have models of numbers that have more than three prime in their sturcutre. The diagram may look too complicated to be helpful (see figure 4), but the physical models are not complicated and are fun to build.

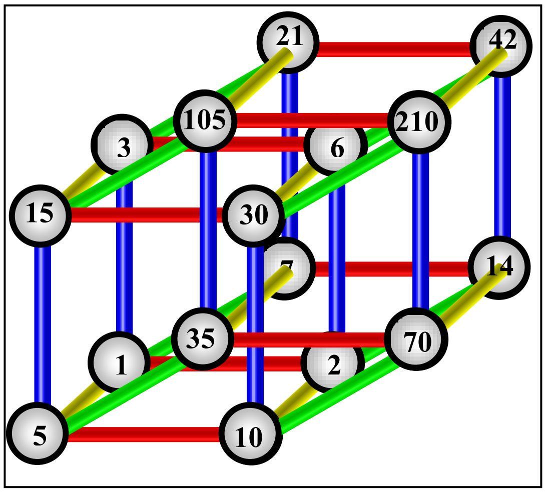

Figure 4: *Model of a number with four distinct primes in its structure.*

We typically do a great deal of work with elementary school students to help them develop number sense based on additive relations between numbers. For example, we encourage students to think of the number 10 as 6+4 or 7+3, or yet as one less than 11 or one more than 9. If we broaden our perspective and look at the multiplicative structure of this number, we will be able to think of 10 as 2×5 and a multiple of 1, 2, 5, and 10.

In the lattice models we can see the primes as the "building blocks", or rather, the "building sticks" for the number in the modeled. This is an important metaphor to complement the way students think of whole numbers. They have an inherent and unique structure, where primes are the building blocks. An intuition for that structure, or, in other words, a "number sense" of the multiplicative structure of a number requires construction by the students and is often neglected. The lattice model is a great way to help students build this kind of number sense.

References

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