MPC for tracking with optimal closed-loop performance

A. Ferramosca, D. Limon, I. Alvarado, T. Alamo and E.F. Camacho

Abstract— **This paper deals with the tracking problem for constrained linear systems using a model predictive control (MPC) law. As it is well known, MPC provides a control law suitable for regulating a constrained linear system to a given target steady state. Asymptotic stability and constraint fulfilment for any finite prediction horizon is typically ensured by means of a suitable choice of the terminal cost and constraint. However, when the target operating point changes, the feasibility of the controller may be lost and the controller fails to track the reference. Recently, a novel MPC formulation has been proposed to solve this problem, ensuring feasibility and asymptotic convergence to any admissible steady state. On the other hand, this control law can not ensure the local optimality of the proposed controller, which is a desirable property of predictive controllers.**

In this paper, this controller is extended considering a generalized offset cost function. Sufficient conditions on this function are given to ensure the local optimality property. Besides, this novel formulation allows to consider as target operation points, states which may be not equilibrium points of the linear systems. In this case, it is proved in this paper that the proposed control law steers the system to an admissible steady state (different to the target) which is optimal with relation to the offset cost function. Thanks to the proposed generalization, the offset cost function could be chosen according to some steady performance criterium.

Therefore, the proposed controller for tracking achieves an optimal closed-loop performance during the transient as well as an optimal steady state in case of not admissible target. These properties are illustrated in an example.

I. INTRODUCTION

Model predictive control (MPC) is one of the most successful techniques of advanced control in the process industry. This is due to its control problem formulation, the natural usage of the model to predict the expected evolution of the plant, the optimal character of the solution and the explicit consideration of hard constraints in the optimization problem. Thanks to the recent developments of the underlying theoretical framework, MPC has become a mature control technique capable to provide controllers ensuring stability, robustness, constraint satisfaction and tractable computation for linear and for non linear systems [1].

The control law is calculated by predicting the evolution of the system and computing the admissible sequence of control inputs which makes the system evolve satisfying the constraints. This problem can be posed as an optimization problem. To obtain a feedback policy, the obtained sequence of control inputs is applied in a receding horizon manner, solving the optimization problem at each sample time. Considering a suitable penalization of the terminal state and an additional terminal constraint, asymptotic stability and constraints satisfaction of the closed loop system can be proved [2]. Moreover, if the terminal cost is the infinite-horizon optimal cost of the unconstrained system, then the MPC control law results to be optimal in a neighborhood of the steady state. This property is the so-called local optimality property and allows to design finite horizon MPC controllers for constrained system with an optimal closed-loop performance (in the latter neighborhood).

Most of the results on MPC consider the regulation problem, that is steering the system to a fixed steady state (typically the origin), but when the target operating point changes, the feasibility of the controller may be lost and the controller fails to track the reference. In [3] an MPC for tracking is proposed, which is able to lead the system to any admissible set point in an admissible way. The main characteristics of this controller are: an artificial steady state is considered as a decision variable, a cost that penalizes the error with the artificial steady state is minimized, an additional term that penalizes the deviation between the artificial steady state and the target steady state is added to the cost function (the so-called *offset cost function*) and an invariant set for tracking is considered as extended terminal constraint. This controller ensures that under any change of the steady state target, the closed loop system maintains the feasibility of the controller and ensures the convergence to the target if admissible. The main drawback of this controller is the potential loss of the optimality property due to the addition of the artificial steady state together with the proposed cost function.

In this paper, this controller is extended considering a general offset cost function. Sufficient conditions on this function are given to ensure the local optimality property. Besides, this novel formulation allows to consider as target operation points, states which may be not equilibrium points of the linear systems. This is particularly interesting for instance when the target is provided by an upper-level target optimizer based on more complex models or when the problem deals with under or over-actuated systems and the target operating point is not consistent with the model. In this paper it is proved that in this case the proposed control law steers the system to an admissible steady state (different to the target) which minimizes the offset cost function. Thanks to the proposed generalization of the offset cost function, this could be chosen according to some steady performance

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criterium.

This paper is organized as follows: in the following section the constrained tracking problem is stated. In section III the new MPC for tracking is presented and in section IV the property of local optimality is introduced and proved. Finally an illustrative example is shown and some conclusions are drawn.

II. Problem Description

Let a discrete-time linear system be described by:

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$ is the current state of the system, $u \in \mathbb{R}^m$ is the current input, $y \in \mathbb{R}^p$ is the current output and x^+ is the successor state. Note that no assumption is considered on the dimension of the states, inputs and outputs. Hence, under-actuated (namely $p > m$) or over-actuated systems (namely $p < m$) might be considered.

The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$ respectively. The system is subject to hard constraints on state and control:

$$(x(k), u(k)) \in \mathcal{Z}$$

for all $k \geq 0$. $\mathcal{Z} \subset \mathbb{R}^{n+m}$ is a compact convex polyhedra containing the origin in its interior.

Assumption 1: The pair (A,B) is stabilizable.

Under this assumption, the set of steady states and inputs of the system (1) is a n_θ-dimensional linear subspace of $\mathbb{R}^{n+m}$ [4] given by

$$(x_s, u_s) = M_\theta \theta$$

Every pair of steady state and input $(x_s, u_s) \in \mathbb{R}^{n+m}$ is characterized by a given parameter $\theta \in \mathbb{R}^{n_\theta}$.

The problem we consider is the design of an MPC controller $\kappa_N^O(x, z_t)$ to track a (possible time-varying) target operation point $z_t = (x_t, u_t)$. If the target operating point z_t is an admissible steady state, the closed loop system evolves to this target state without offset. If z_t is not consistent with the linear model considered for predictions, namely, it is not an admissible steady state of system (1), the closed loop system evolves to an admissible steady state which minimizes a given performance index.

III. Enhanced Formulation of the MPC for tracking

In this section we present a novel formulation of the MPC for tracking which generalizes and improves the one presented by the authors in [3]. The way this controller handle the tracking problem is characterized by (i) considering an artificial steady state and input as decision variables, (ii) penalizing the deviation of the predicted trajectory with the artificial steady conditions, (iii) adding a quadratic offset-cost function to penalize the deviation between the artificial and the target equilibrium point and (iv) considering an extended terminal constraint. In this paper, this controller is extended to the case of considering a general offset cost function. As it will be demonstrated later on, under mild assumptions, this function provides significant properties to the controlled system.

The proposed cost function of the MPC is given by:

$$\begin{aligned} V_N^O(x, z_t; \mathbf{u}, \bar{\theta}) &= \sum_{i=0}^{N-1} \|x(i) - \bar{x}_s\|_Q^2 + \|u(i) - \bar{u}_s\|_R^2 \\ &\quad + \|x(N) - \bar{x}_s\|_P^2 + V_O(\bar{z}_s - z_t) \end{aligned}$$

where $x(i)$ denotes the prediction of the state i-samples ahead, the pair $(\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}$ is the artificial steady state and input parametrized by $\bar{\theta}$ and z_t is the target operating point. The controller is derived from the solution of the optimization problem $P_N^O(x, z_t)$ given by

$$\begin{aligned} V_N^{O*}(x, z_t) &= \min_{\mathbf{u}, \bar{\theta}} V_N^O(x, z_t, \mathbf{u}, \bar{\theta}) \\ s.t.\ & x(0) = x, \\ & x(j+1) = Ax(j) + Bu(j), \\ & (x(j), u(j)) \in \mathcal{Z}, \quad j = 0, \cdots, N-1 \\ & (\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}, \\ & (x(N), \bar{\theta}) \in \Omega_{t,K}^w \end{aligned}$$

Considering the receding horizon policy, the control law is given by

$$\kappa_N^O(x, z_t) = u^*(0; x, z_t)$$

Since the set of constraints of $P_N^0(x, z_t)$ does not depend on z_t, its feasibility region does not depend on the target operating point z_t. Then there exists a polyhedral region $\mathcal{X}_N \subseteq X$ such that for all $x \in \mathcal{X}_N$, $P_N^O(x, z_t)$ is feasible. This is the set of initial states that can be admissibly steered to the projection of $\mathcal{O}_{t,K}^w$ onto x in N steps or less.

Consider the following assumption on the controller parameters:

Assumption 2:

1) Let $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ be positive definite matrices.
2) The offset cost function is a convex function such that
$$\alpha_1 \|(\bar{x}_s - x_t)\|_1 \leq V_O(\bar{z}_s - z_t) \leq \alpha_2 \|(\bar{x}_s - x_t)\|_1$$
where α_1, α_2 are positive real constant.
3) Let $K \in \mathbb{R}^{m \times n}$ be a stabilizing control gain such that $(A + BK)$ is Hurwitz.
4) Let $P \in \mathbb{R}^{n \times n}$ be a positive definite matrix such that:
$$(A+BK)^T P (A+BK) - P = -(Q + K^T R K)$$
5) Let $\mathcal{O}_{t,K}^w \subseteq \mathbb{R}^{n+n_\theta}$ be an admissible polyhedral invariant set for tracking for system (1) subject to (2), for a given gain K [3].

The set of admissible steady states and inputs contained in the invariant set for tracking $\mathcal{O}_{t,K}^w$ is given by:

$$\mathcal{Z}_s = \{(x_t, u_t) = M_\theta \theta : (x_t, \theta) \in \mathcal{O}_{t,K}^w\}$$

This set is potentially the set of all admissible steady states and inputs [3].

Taking into account the proposed conditions on the controller parameters, in the following theorem it is proved asymptotic stability and constraints satisfaction of the controlled system.

Theorem 1 (Stability): Consider that assumptions 1 and 2 hold and consider a given target operation point $z_t = (x_t, u_t)$. Then for any feasible initial state $x_0 \in \mathcal{X}_N$, the system controlled by the proposed MPC controller $\kappa_N^O(x, z_t)$ fulfils the constraints along the time and, besides

(i) If $z_t = (x_t, u_t) \in \mathcal{Z}_s$ then z_t is an asymptotically stable steady state for the closed loop system.

(ii) In other case, the steady state and input $z_s^* = (x_s^*, u_s^*)$ given by

$$z_s^* = \arg \min_{z_s \in \mathcal{Z}_s} V_O(z_s - z_t)$$

is an asymptotically stable steady state for the closed loop system.

Proof: The first part of the proof is devoted to prove the feasibility of the controlled system, that is, $x(k+1) \in \mathcal{X}_N$, for all $x(k) \in \mathcal{X}_N$, and z_t. Consider the optimal solution of $P_N(x(k), z_t)$, then the successor state is $x(k+1) = Ax(k) + B\kappa_N^O(x(k), z_t)$. Define the following sequences:

$$\begin{aligned} \mathbf{u}(x(k+1), z_t) &\triangleq [u^*(1; x(k), z_t), \cdots, u^*(N-1; x(k), z_t), \\ &\quad K(x^*(N-1; x(k), z_t) - \bar{x}_s^*(x(k), z_t)) \\ &\quad + \bar{u}_s^*(x(k), z_t)] \\ \bar{\theta}(x(k+1), z_t) &\triangleq \bar{\theta}^*(x(k), z_t) \end{aligned}$$

Then, $(\mathbf{u}, \bar{\theta})$ is a feasible solution for the optimization problem $P_N(x(k+1), z_t)$ due to:

- Since $x(x(k+1), z_t) = x^*(1; x(k), z_t)$, then $x(i; x(k+1), z_t) = x^*(i+1; x(k), z_t)$ $i = 0, 1, \cdots, N-1$; then the first $N-1$ terms of the trajectory are admissible. Admissibility of $x(N; x(k+1), z_t)$ is derived from the fact that $(x(N-1; x(k+1), z_t), \bar{\theta}(x(k+1), z_t)) \in \Omega_{t,K}^w$ and hence the control action $u(N-1; x(k+1), z_t)$ ensures that $(x(N; x(k+1), z_t), \bar{\theta}(x(k+1), z_t)) \in \Omega_{t,K}^w$.
- Feasibility of $\mathbf{u}^*(x(k), z_t)$ and admissibility of set $\Omega_{t,K}^w$ ensures the feasibility of $\mathbf{u}(x(k+1), z_t)$.
- The terminal constraint satisfaction stems from the same arguments.

Convergence is derived proving that the optimal cost is a Lyapunov function. Consider the proposed feasible solution. Taking into account the properties of the feasible nominal trajectories for $x(k+1)$, the condition *(iv)* of Assumption 2 and using standard procedures in MPC [2] it is possible to obtain:

$$\begin{aligned} \Delta V_N^O(x, z_t) &= V_N^O(x(k+1), z_t; \mathbf{u}, \bar{\theta}) - V_N^{O*}(x(k), z_t) \\ &\leq -\|x^*(x(k), z_t) - \bar{x}_s^*(x(k), z_t)\|_Q^2 \\ &\quad -\|u^*(0; x(k), z_t) - \bar{u}_s^*(x(k), z_t)\|_R^2 \\ &\leq -\|x^*(x(k), z_t) - \bar{x}_s^*(x(k), z_t)\|_Q^2 \end{aligned}$$

By optimality, we have that $V_N^{O*}(x(k+1), z_t) \leq V_N^O(x(k+1), z_t; \mathbf{u}, \bar{\theta})$ and then:

$$\begin{aligned} \Delta V_N^{O*}(x, z_t) &= V_N^{O*}(x(k+1), z_t) - V_N^{O*}(x(k), z_t) \\ &\leq -\|x^*(x(k), z_t) - \bar{x}_s^*(x(k), z_t)\|_Q^2 \end{aligned}$$

Taking into account that $Q > 0$, then

$$\lim_{k \to \infty} \|x^*(x(k), z_t) - \bar{x}_s^*(x(k), z_t)\|_Q^2 = 0$$

and hence the system evolves to an admissible steady state $\bar{x}_s^* \in \mathcal{Z}_s$. In virtue of lemma 2 we can deduce that $(\bar{x}_s^*, \bar{u}_s^*)$ is the minimizer of the offset cost function $V_O(z_s - z_t)$, proving the second assertion of the theorem. The first one is a direct consequence of the latter. ■

In the following section it is demonstrated that the proposed controller could provide a locally optimal control law.

IV. Local Optimality

Consider that system (1) is controlled by the control law $u = \kappa(x, z_t)$ to steer the system to the target $z_t = (x_t, u_t) \in \mathcal{Z}_s$. Consider also a quadratic cost function of the closed-loop system evolution when the initial state is x, given by

$$V_\infty(x, z_t, \kappa(\cdot, z_t)) \sum_{i=0}^{\infty} \|x(i) - x_t\|_Q^2 + \|\kappa(x(i), z_t) - u_t\|_R^2$$

where $x(i) = \phi(i; x, \kappa(\cdot, z_t))$ is calculated from the recursion $x(j+1) = Ax(j) + B\kappa(x(j), z_t)$ for $j = 0, \cdots, i-1$ with $x(0) = x$. A control law $\kappa_\infty(x, z_t)$ is said to be optimal if it is admissible (namely, the constraints are fulfiled along the closed loop evolution) and it is the one which minimizes the cost $V_\infty(x, z_t, \kappa(\cdot, z_t))$ for all admissible x. It is clear that the optimal control law (the so-called Linear Quadratic Regulator) is the best control law to be designed according to the given performance index. The optimal cost function is denoted as $V_\infty^*(x, z_t) = V_\infty(x, z_t, \kappa_\infty(\cdot, z_t))$. The calculation of the optimal control law $\kappa_\infty(x, z_t)$ may be computationally unaffordable for constrained systems, while for unconstrained, it can be obtained from the solution of a Riccati's equation.

Model predictive controllers can be considered as suboptimal controllers since the cost function is only minimized for a finite prediction horizon. The standard MPC control law to regulate the system to the target z_t, $\kappa_N^r(x, z_t)$, is derived from the following optimization problem $P_N^r(x, z_t)$

$$\begin{aligned} V_N^{r*}(x, z_t) &= \min_{\mathbf{u}, \bar{\theta}} \sum_{i=0}^{N-1} \|x(i) - \bar{x}_s\|_Q^2 + \|u(i) - \bar{u}_s\|_R^2 \\ &\quad + \|x(N) - \bar{x}_s\|_P^2 \\ s.t. \quad & x(0) = x, \\ & x(j+1) = Ax(j) + Bu(j), \\ & (x(j), u(j)) \in \mathcal{Z}, \quad j = 0, \cdots, N-1 \\ & (\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}, \\ & (x(N), \bar{\theta}) \in \Omega_{t,K}^w \\ & \|\bar{x}_s - x_t\|_1 = 0 \end{aligned}$$

This optimization problem is feasible for any x in a polyhedral region denoted as $\mathcal{X}_N^r(z_t)$. Under certain assumptions [2], for any feasible initial state $x \in \mathcal{X}_N^r(z_t)$, the control law $\kappa_N^r(x, z_t)$ steers the system to the target fulfilling the constraints. However, this control law is suboptimal in the sense that it does not minimizes $V_\infty(x, z_t, \kappa_N^r(\cdot, z_t))$. Fortunately, as stated in the following lemma, if the terminal cost function is the optimal cost of the unconstrained LQR, then the resulting finite horizon MPC is equal to the constrained LQR in a neighborhood of the terminal region [5], [6].

Lemma 1: Consider that assumptions 1 and 2 hold. Consider that the terminal control gain K is the one of the unconstrained linear quadratic regulator and $z_t = M_\theta \theta_t$. Define the set $\Upsilon_N(z_t) \subset \mathbb{R}^n$ as

$$\Upsilon_N(z_t) = \{\bar{x} \in \mathbb{R}^n : (\phi(N; \bar{x}, \kappa_\infty(\cdot, z_t), \theta_t) \in \mathcal{O}_{t,K}^w\}$$

Then for all $x \in \Upsilon_N(z_t)$, $V_N^{r,*}(x, z_t) = V_\infty^*(x, z_t)$ and $\kappa_N^r(x, z_t) = \kappa_\infty(x, z_t)$.

This lemma directly stems from [5, Thm. 2].

The proposed MPC for tracking might not ensure this local optimality property under assumptions of lemma 1 due to the artificial steady state and input and the functional cost to minimize. However, as it is demonstrated in the following property, under some conditions on the offset cost function $V_O(\cdot)$, this property holds.

Property 1 (Local optimality):
Consider that assumptions 1 and 2 hold. Then there exists a $\alpha^* > 0$ such that for all $\alpha_1 \geq \alpha^*$:

- The proposed MPC for tracking is equal to the MPC for regulation, that is $\kappa_N^O(x, z_t) = \kappa_N^r(x, z_t)$ and $V_N^{O*}(x, z_t) = V_N^{r*}(x, z_t)$ for all $x \in \mathcal{X}_N^r(z_t)$.
- If the terminal control gain K is the one of the unconstrained linear quadratic regulator, then the MPC for tracking control law $\kappa_N^O(x, z_t)$ is equal to the optimal control law $\kappa_\infty(x, z_t)$ for all $x \in \Upsilon(z_t)$.

Proof: First, define the following optimization problem $P_{N,\alpha}^m(x, z_t; \alpha)$ as:

$$\begin{aligned} V_{N,\alpha}^{m*}(x, z_t, \alpha) \;=\; & \min_{\mathbf{u},\bar{\theta}} \sum_{i=0}^{N-1} \|x(i) - \bar{x}_s\|_Q^2 + \|u(i) - \bar{u}_s\|_R^2 \\ & \quad + \|x(N) - \bar{x}_s\|_P^2 + \alpha \|\bar{x}_s - x_t\|_1 \\ s.t. \;\; & x(0) = x, \\ & x(j+1) = Ax(j) + Bu(j), \\ & (x(j), u(j)) \in \mathcal{Z}, \quad j = 0, \cdots, N\text{–}1 \\ & (\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}, \\ & (x(N), \bar{\theta}) \in \Omega_{t,K}^w \end{aligned}$$

This optimization problem $P_{N,\alpha}^m(x, z_t; \alpha)$ results from the optimization problem $P_N^r(x, z_t)$ with the last constraint posed as an exact penalty function [7]. Therefore, there exists a finite constant $\alpha^* > 0$ such that for all $\alpha \geq \alpha^*$, $V_{N,\alpha}^{m*}(x, z_t) = V_N^{r*}(x, z_t)$ for all $x \in \mathcal{X}_N^r(z_t)$ [7], [8].

Consider the problem $P_N^O(x, z_t)$. From the assumption on $V_O(\cdot)$, we derive that

$$V_{N,\alpha_1}^{m*}(x, z_t) \leq V_N^{O*}(x, z_t) \leq V_{N,\alpha_2}^{m*}(x, z_t)$$

Since $\alpha_2 \geq \alpha_1 \geq \alpha^*$, we have that for all $x \in \mathcal{X}_N^r(z_t)$

$$V_N^{r*}(x, z_t) \leq V_N^{O*}(x, z_t) \leq V_N^{r*}(x, z_t)$$

and hence $V_N^{O*}(x, z_t) = V_N^{r*}(x, z_t)$.

The second claim is derived from lemma 1 observing that $\Upsilon_N(z_t) \subseteq \mathcal{X}_N^r(z_t)$. ■

A. Determination of the lower bound of $V_O(\cdot)$ for optimality

The aim of this section is to present a method to compute the value of α^* presented in property 1 such that for all $\alpha_1 \geq \alpha^*$, then $V_N^{O*}(x, z_t) = V_N^{r*}(x, z_t)$.

In virtue of the well-known result on the exact penalty functions [7], the constant α^* can be chosen as the Lagrange multiplier of the equality constraint $\|\bar{x}_s - x_t\|_1 = 0$ of the optimization problem $P_N^r(x, z_t)$. Since the optimization problem depends on the parameters (x, z_t), the value of this Lagrange multiplier also depends on (x, z_t). In order to ensure the local optimality property, the constant α^* should be chosen as the maximum of the Lagrange multiplier in the set of the parameters $(x, z_t) \in \mathcal{X}_N \times \mathcal{Z}_s$.

Firstly, notice that using standard techniques of convex optimization [8], the optimization problem $P_N^r(x, z_t)$ can be casted as a multiparametric quadratic programming (mp-QP) problem [6], which can be defined as:

$$\begin{aligned} \min_z \quad & \frac{1}{2} z' H z \\ s.t. \quad & Gz \leq W + Sx \\ & Fz = Y + Tx \end{aligned} \tag{2}$$

The Karush-Kuhn-Tucker (KKT) optimality conditions [8] for this problem are given by:

$$Hz + G'\lambda + F'\nu = 0 \tag{3a}$$
$$\lambda(Gz - W - Sx) = 0 \tag{3b}$$
$$\lambda \geq 0 \tag{3c}$$
$$Gz - W - Sx \leq 0 \tag{3d}$$
$$Fz - Y - Tx = 0 \tag{3e}$$

Solving (3a) for z and substituting in the other equations, we obtain a new set of constraints for the Lagrange dual problem associated with the problem (2) which depends on (λ, ν, x). Then the value of α^* should be chosen as the maximum ν in the feasible set of (λ, ν, x). This can be calculated by solving the following optimization problem

$$\begin{aligned} \max_{\lambda,\nu,x} \quad & \nu \\ s.t. \quad & \lambda'(GH^{-1}G'\lambda + GH^{-1}F'\nu + W + Sx) = 0 \\ & \lambda \geq 0 \\ & -(GH^{-1}G'\lambda + GH^{-1}F'\nu + W + Sx) \leq 0 \\ & FH^{-1}G'\lambda + FH^{-1}F'\nu + Y + Tx = 0 \end{aligned}$$

Then α^* is taken as the optimal value of this optimization problem.

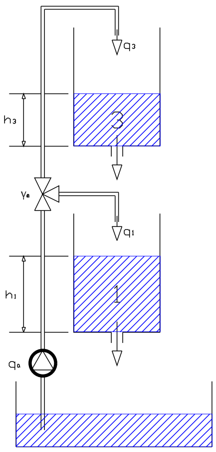

Fig. 1. The two tanks system.

V. Example

The objective of this example is double. The first aim is to illustrate that the proposed controller can deal with non-square systems and to show the role of the offset cost function in the prioritization of the outputs in the case of set point with offset. The second aim is to show the role of the offset cost function in recovering the property of local optimality.

We considered a two cascaded tanks system. A scheme of the system is presented in figure 1. The nonlinear model of the system is:

$$\begin{aligned} \frac{dh_1}{dt} &= -\frac{a_1}{A}\cdot\sqrt{2gh_1} + \frac{a_3}{A}\cdot\sqrt{2gh_3} + \frac{\gamma}{A}\cdot q \\ \frac{dh_3}{dt} &= -\frac{a_3}{A}\cdot\sqrt{2gh_3} + \frac{1-\gamma}{A}\cdot q \end{aligned}$$

where h_1 and h_3 are the levels of water in each tank and q is the inlet flow. The cross-section of the tanks is $A = 0.06m^2$, the cross-sections of the outlets are $a_1 = 6.7371e^{-4}m^2$ and $a_2 = 4.0423e^{-4}m^2$, and $\gamma = 0.4$.

Linearizing the model in an operating point given by $h_1^0 = 0.68$ m, $h_2^0 = 0.65$ m and $q^0 = 2$ m^3/h, and defining the variables $x_i = h_i - h_i^o$ and $u = q - q^o$ where $i = 1, 3$ we have that:

$$\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} \frac{-1}{\tau_1} & \frac{1}{\tau_3} \\ 0 & \frac{-1}{\tau_3} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma}{A} \\ \frac{1-\gamma}{A} \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \end{aligned}$$

where $\tau_i = \frac{A}{a_i}\sqrt{\frac{2h_i^0}{g}} \geq 0$, $i = 1, 3$, are the time constants of each tank.

The system is constrained to $0.30 \leq x_1 \leq 1.36$, $0.30 \leq x_2 \leq 1.30$ and $0 \leq u \leq q_{max}$, where $q_{max} = 4$.

The aim of the first test is to show the property of offset minimization of the controller. The claim is that, in case of a not consistent target operation point, the system evolves to the state such that the water volume in the tanks is the most similar to the one relative to the desired point. Therefore, the difference of the volumes of water in the tanks has been chosen as offset cost function, that is $V_O = A((x_{1s}+x_{2s})-(x_{1t}+x_{2t}))$. The optimal point is the one that minimizes this difference. In the test, two references have been considered. The first reference, $Ref_1 = (1.2, 1.17)$, is an admissible set point. The second one, $Ref_2 = (1, 0.6)$, is a not consistent operation point. The initial state is $x_0 = (0.32, 1.26)$. An MPC with $N = 3$ has been considered. The weighting matrices have been chosen as $Q = I_2$ and $R = 100 \times I_1$.

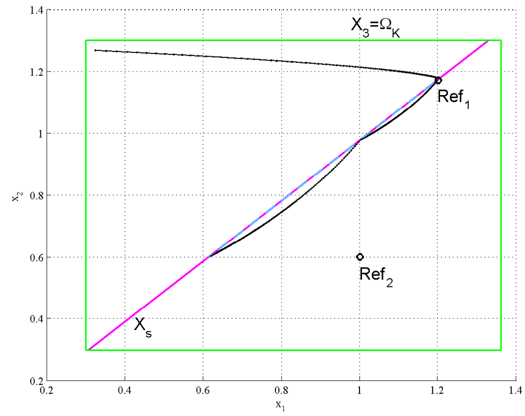

Fig. 2. Evolution of the levels.

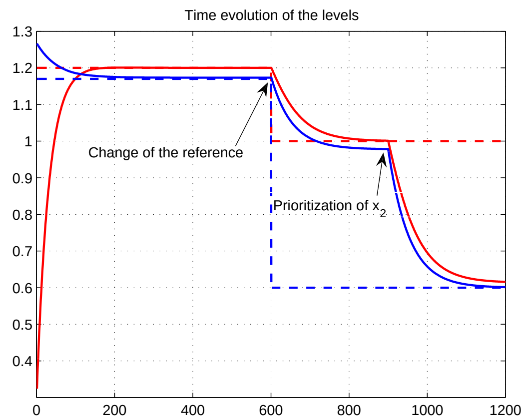

Fig. 3. Time evolution of the plant.

The maximal invariant set for tracking $\Omega_{t,K}$, the region of attraction $\mathcal{X}_3$, the set of equilibrium levels $\mathcal{X}_s = Proj_x(\mathcal{Z}_s)$ and the evolution of the levels for a given reference are shown in figure 2. As it can be seen, since $Ref_1 \in \mathcal{X}_s$, the system reaches the first reference without any offset. At the sample time 600 the reference changes, becoming a not consistent point. Note how the controller leads the system to the closest equilibrium point, in the sense that the offset cost function is minimized, providing an optimal closed-loop performance. In order to show the role of the offset cost function in the prioritization of the outputs, at the sample

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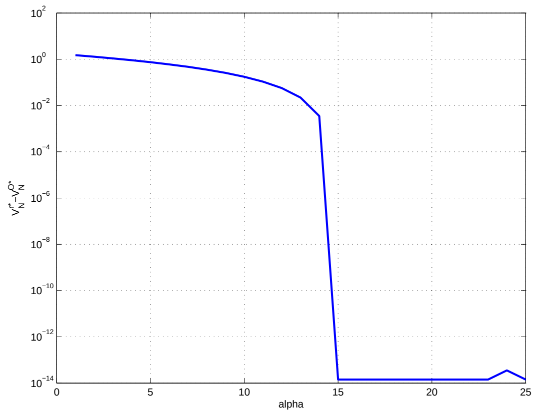

Fig. 4. Difference between the regulation cost and the tracking cost versus α.

time 900 the x_2 state has been prioritized on-line by means of a suitable weight in the offset cost function. As it can be seen in figure 3, the system moves to the closest equilibrium point that minimizes the distance in the direction determined by this prioritization (in this case the x_2 state).

TABLE I

$V_N^{r*} - V_N^{O*}$ FOR DIFFERENT VALUES OF α

α	$V_N^{r*} - V_N^{O*}$
14	0.0035
14.5	$2.0802e - 004$
14.6	$2.9923e - 005$
14.65	$9.8764e - 007$
14.66	$9.7060e - 009$
14.6611	$1.4211e - 014$
15	$1.4211e - 014$
16	$1.4211e - 014$

To illustrate the property of the local optimality, the difference between the optimal cost of the MPC for tracking proposed in this paper, V_N^{O*}, with the optimal cost of the MPC for regulation, V_N^{r*}, has been considered. To this aim, the MPC for tracking optimal cost has been calculated considering as the offset cost function a 1-norm cost, that is $V_O = \alpha\|\bar{x}_s - x_t\|_1$ where α is a parameter. In figure 4 the value of $V_N^{r*} - V_N^{O*}$ for a given state versus α is plotted. As it can be seen, $V_N^{r*} - V_N^{O*}$ drops to zero dramatically for a certain value of α. This proves the benefit of the new formulation of the MPC for tracking. Note how the value of $V_N^{r*} - V_N^{O*}$ drops to practically zero when $\alpha = 15$. As we said in section IV, this happens because the value of α becomes greater than the value of the Lagrange multiplier of the equality constraint of the regulation problem V_N^{r*}. To point out this fact, consider that, for this example, the value of the Lagrange multiplier of the equality constraint of the regulation problem V_N^{r*}, is $\alpha^* = 14.6611$. In table I the value of $V_N^{r*} - V_N^{O*}$ in case of different values of the parameter α is presented. Note how the value seriously decrease when α becomes equals to α^*. So, using the procedure described in section IV, we can determine the value of α^* such that $V_N^{O*}(x, z_t) = V_N^{r*}(x, z_t)$.

VI. CONCLUSIONS

In this paper an enhanced formulation of the MPC for tracking is presented. This formulation generalizes the original one by considering a general convex function as offset cost. This offset cost function allows to consider as target operating points states and inputs not consistent with the prediction model. This case is particularly interesting for non-square plants or for instance, when the target calculated by means of a non-linear model.
Under some assumptions, it is proved that the proposed controller steers the system to the target if this is admissible. If not, the controller converges to an admissible steady state optimum according to the offset cost function. Besides, the closed-loop evolution is also optimal in the sense that provides the best possible performance index.
The properties of the controller are illustrated by an example where the proposed controller is applied to an under-actuated plant.

APPENDIX

Lemma 2: Let the assumptions of theorem 1 hold. Consider a desired steady state $z_t = (\mathrm{x}_t, \mathrm{u}_t)$ and assume that for a given state x, the optimal solution of $P_N^O(x, z_t)$ is such that $\|x - \bar{\mathrm{x}}_s^*(x, z_t)\|_Q = 0$ (i.e. $x = \bar{\mathrm{x}}_s^*(x, z_t)$), then $V_N^{O*}(\bar{x}_s^*, z_t) = V_O(\bar{z}_s^* - z_t)$.

Proof: The proof is obtained by contradiction. Consider that $\bar{z}_s^*$ is not the minimizer of the offset function. Then there exists a $\tilde{z}_s \neq \bar{z}_s^*$, such that $V_O(\tilde{z}_s - z_t) < V_O(\bar{z}_s^* - z_t)$. Consider the following statments:

1) There exists an $\alpha \in [0, 1)$ such that $V_O(\tilde{z}_s - z_t) = \alpha V_O(\bar{z}_s^* - z_t)$.
2) There exists a $\hat{\lambda} \in [0, 1)$ such that for every $\lambda \in [\hat{\lambda}, 1)$, $\hat{z}_s = \lambda\bar{z}_s^* + (1 - \lambda)\tilde{z}_s$ is admissible, then:
   - $\bar{z}_s^* - \hat{z}_s = (1 - \lambda)(\bar{z}_s^* - \tilde{z}_s)$.
   - $\hat{z}_s - \tilde{z}_s = \lambda(\bar{z}_s^* - \tilde{z}_s)$.
   - $\hat{z}_s - z_t = \lambda(\bar{z}_s^* - z_t) + (1 - \lambda)(\tilde{z}_s - z_t)$.

Defining as $\mathbf{u}$ the sequence of control actions derived from the control law $u = K(x - \bar{x}_s) + \bar{u}_s$ [3], it is inferred that $(\mathbf{u}, \bar{\mathrm{x}}_s^*, \bar{\theta})$ is a feasible solution for $P_N^O(\bar{\mathrm{x}}_s^*, z_t)$. Then, from assumption 2,

$$\begin{aligned}
V_N^{O*}(\bar{x}_s^*, z_t) &\leq V_N^O(\bar{x}_s^*, z_t; \mathbf{u}, \hat{z}_s) \\
&= \sum_{i=0}^{N-1} \overbrace{\|x(i) - \hat{x}_s\|_Q^2 + \|K(x(i) - \hat{x}_s)\|_R^2}^{\|x(i)-\hat{x}_s\|^2_{(Q+K^TRK)}} \\
&\quad + \|x(N) - \hat{x}_s\|_P^2 + V_O(\hat{z}_s - z_t) \\
&= \|\bar{\mathrm{x}}_s^* - \hat{x}_s\|_P^2 + V_O(\hat{z}_s - z_t)
\end{aligned}$$

Since $\mathcal{Z}_s$ is compact, there exists a $\beta_1 > 0$ such that $\|\bar{x}_s^* - \tilde{x}_s\|_P^2 \leq \beta_1\|\bar{x}_s^* - \tilde{x}_s\|_q$. Then, considering the previous statements:

$$\begin{aligned}\|\bar{x}_s^* - \hat{x}_s\|_P^2 &= (1-\lambda)^2\|\bar{x}_s^* - \tilde{x}_s\|_P^2 \\ &\leq \beta_1(1-\lambda)^2\|\bar{x}_s^* - \tilde{x}_s\|_1 \\ &= \beta_1(1-\lambda)^2\|\bar{x}_s^* - x_t + x_t - \tilde{x}_s\|_1 \\ &\leq \beta_1(1-\lambda)^2\left[\|\bar{x}_s^* - x_t\|_1 + \|\tilde{x}_s - x_t\|_1\right] \\ &\leq \beta_1(1-\lambda)^2\left[\|\bar{z}_s^* - z_t\|_1 + \|\tilde{z}_s - z_t\|_1\right] \\ &\leq \frac{\beta_1}{\alpha_1}(1-\lambda)^2\left[V_O(\bar{z}_s^* - z_t) + V_O(\tilde{z}_s - z_t)\right] \\ &\leq \frac{\beta_1}{\alpha_1}(1+\alpha)(1-\lambda)^2 V_O(\bar{z}_s^* - z_t)\end{aligned}$$

Hence, taking $\beta = \dfrac{\beta_1}{\alpha_1}(1+\alpha)$:

$$\begin{aligned}\|\bar{x}_s^* - \hat{x}_s\|_P^2 + V_O(\hat{z}_s - z_t) &\leq \beta(1-\lambda)^2 V_O(\bar{z}_s^* - z_t) \\ &\quad + V_O(\lambda(\bar{z}_s^* - z_t) + (1-\lambda)(\tilde{z}_s - z_t)) \\ &\leq \beta(1-\lambda)^2 V_O(\bar{z}_s^* - z_t) \\ &\quad + \lambda V_O(\bar{z}_s^* - z_t) + (1-\lambda)V_O(\tilde{z}_s - z_t) \\ &\leq \left[\beta(1-\lambda)^2 + \alpha + (1-\alpha)\lambda\right] V_O(\bar{z}_s^* - z_t)\end{aligned}$$

Then it can be proved that exists a $\tilde{\lambda} \in [\hat{\lambda}, 1)$ such that for all $\lambda \in [\tilde{\lambda}, 1)$, $\beta(1-\lambda)^2 + \alpha + (1-\alpha)\lambda - 1 < 0$. So, taking into account that:

$$\begin{aligned}\beta(1-\lambda)^2 + \alpha + (1-\alpha)\lambda - 1 &= \beta(1-\lambda)^2 - (1-\alpha)(1-\lambda) \\ &= (1-\lambda)\left[\beta(1-\lambda) - (1-\alpha)\right]\end{aligned}$$

we want to find a λ such that

$$(1-\lambda)\left[\beta(1-\lambda) - (1-\alpha)\right] < 0$$

that is:

$$\lambda > \frac{\beta - (1-\alpha)}{\beta}$$

Therefore, defining $\tilde{\lambda} = max\left(\dfrac{\beta-(1-\alpha)}{\beta}, \hat{\lambda}\right)$, then for all $\lambda \in [\tilde{\lambda}, 1)$, we have that $V_N^{O*}(\mathrm{x}_s^*, z_t) < V_O(\bar{z}_s^* - z_t)$, which contradict the fact that $\bar{z}_s^*$ is not the minimizer of the offset cost function. ■

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