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Fundamental solutions for steady-state heat transfer in an exponentially graded anisotropic material

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Abstract. Heat conduction in an anisotropic inhomogeneous medium is considered. The conductivities vary exponentially in one fixed but arbitrary direction. The Green's function corresponding to a point source is constructed. Two methods are used, one using Fourier transforms and one involving certain changes of variables in the governing partial differential equation. Solutions in both two and three dimensions are derived. They can be used as a basic ingredient in the formulation of boundary integral equations for graded anisotropic materials.

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1. Introduction

Thermal barrier coatings have been developed for a variety of high-temperature turbine engine applications. These ceramic coatings protect the turbine blades from high temperatures experienced during operation of the engine. However, problems of coating failure with early coatings precluded their widespread application. These failures were due to the large stresses which develop at the interface between the metallic substrate and the ceramic coating at high temperatures. In order to reduce these stresses, coatings have been developed which have graded properties through the thickness of the coating. Near the metallic substrate, the coating is designed to have properties similar to the substrate, in particular the coefficient of thermal expansion. Near the surface of the coating, the properties are similar to the pure ceramic which provides the greatest insulation against high temperatures. These coatings are examples of *functionally graded materials* (FGMs) [10, 18, 15]

One method of analysis for thermoelastic stresses in FGM systems is the boundary element method; its advantages for problems involving cracks are well known (see, for example, [8]). The immediate difficulty in applying boundary element methods to the analysis of an FGM-substrate interface problem is the lack of an appropriate fundamental solution for the FGM. Some work has been done on problems with spatially variable conductivity for both isotropic and, to a lesser extent, anisotropic media. For isotropic materials, different approaches for deriving boundary integral equations or fundamental solutions are presented in [5, 3, 16, 11, 12, 9]. Most of these rely on using a transformation of variables approach after writing the conductivity as a product of the spatial variable and, say, the temperature for the heat conduction problem. The review article [17] provides an overview of most of these methods. The approach in [11] uses a generalized forcing function to derive appropriate fundamental solutions. There has been some discussion in the literature on this method of analysis [2]. For anisotropic solids, the work in [6] and [7] provides approaches again based on transformation methods to derive fundamental solutions.

The work reported on here is the development of a fundamental solution for steady-state heat transfer in an FGM. We first present a detailed derivation based on Fourier transforms of the two-dimensional fundamental solution directly from the governing differential equation. Based on similarity with the fundamental solution of the Helmholtz equation, we then present an alternative formulation which extends the analysis to three dimensions. In this paper we will use the terms fundamental solution and Green's function interchangeably as we are only seeking the singular, or free-space, part of the Green's function.

2. Green's function formulation in two dimensions

Consider a two-dimensional solid occupying the (x_1, x_2) plane. The anisotropic thermal conductivities of the solid are given by

$$\mathbf{K}\,\mathrm{e}^{-2\mathrm{i}\alpha x_2} \quad \text{with} \quad \mathbf{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \tag{2.1}$$

where k_{11}, k_{12}, k_{21}, k_{22} and α are real constants. Thus, the material is exponentially graded in the x_2-direction.

We assume the matrix is symmetric and positive definite [4]. In particular, we have

$$k_{12} = k_{21} \tag{2.2}$$

and

$$\det \mathbf{K} = k_{11}k_{22} - k_{12}^2 > 0, \tag{2.3}$$

so that $\mathbf{K}$ is non-singular.

We follow [9] and initially restrict α to be purely real; the reason for this will become clear later. Our final result does not depend on this restriction. Indeed, the case of imaginary α has particular relevance in applications.

For steady-state heat conduction in the solid we have Fourier's law,

$$\nabla \cdot \mathbf{h} = \mathcal{Q}(\mathbf{x}) \tag{2.4}$$

where $\mathcal{Q}(\mathbf{x})$ is a heat source and the flux vector $\mathbf{h}$ is given by

$$h_j = -\left(k_{jm}\mathrm{e}^{-2\mathrm{i}\alpha x_2}\right)\frac{\partial u}{\partial x_m}. \tag{2.5}$$

Substituting (2.5) in (2.4), we have

$$k_{11}\frac{\partial^2 u}{\partial x_1^2} + 2k_{12}\frac{\partial^2 u}{\partial x_1 \partial x_2} + k_{22}\frac{\partial^2 u}{\partial x_2^2} - 2\mathrm{i}\alpha\left(k_{12}\frac{\partial u}{\partial x_1} + k_{22}\frac{\partial u}{\partial x_2}\right) + \mathcal{Q}(\mathbf{x})\,\mathrm{e}^{2\mathrm{i}\alpha x_2} = 0. \tag{2.6}$$

It is convenient to write this partial differential equation as

$$\left(L_0 + L_G\right)u(\mathbf{x}) + \mathcal{Q}(\mathbf{x})\,\mathrm{e}^{2\mathrm{i}\alpha x_2} = 0$$

where

$$L_0 = k_{11}\frac{\partial^2}{\partial x_1^2} + 2k_{12}\frac{\partial^2}{\partial x_1 \partial x_2} + k_{22}\frac{\partial^2}{\partial x_2^2}$$

and

$$L_G = -2\mathrm{i}\alpha\left(k_{12}\frac{\partial}{\partial x_1} + k_{22}\frac{\partial}{\partial x_2}\right).$$

To obtain the Green's function for the differential equation (2.6), we let $\mathcal{Q}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}')$ so

$$\left(L_0 + L_G\right)G(\mathbf{x};\mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}')\,\mathrm{e}^{2\mathrm{i}\alpha x_2'} \tag{2.7}$$

where $\mathbf{x}'$ is the source point, $\mathbf{x}$ is the field point, and $\delta(\mathbf{r})$ is Dirac's delta function. Note that the exponential term on the right side of (2.7) is evaluated at $x_2 = x_2'$ as discussed in [17].

In order to find G, we shall use Fourier transforms. We define the two-dimensional Fourier transform pair

$$\mathcal{F}\left(f(\mathbf{x})\right) = \hat{f}(\mathbf{q}) = \int_{-\infty}^{\infty} f(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathrm{d}\mathbf{x}$$
$$\mathcal{F}^{-1}\left(\hat{f}(\mathbf{q})\right) = f(\mathbf{x}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \hat{f}(\mathbf{q})\,\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathrm{d}\mathbf{q}. \tag{2.8}$$

Taking forward Fourier transforms of (2.7) we then have

$$\left(\hat{L}_0 + \hat{L}_G\right)\hat{G}(\mathbf{q}) = -\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}'}\mathrm{e}^{2\mathrm{i}\alpha x_2'}$$

where

$$\hat{L}_0 = -k_{11}q_1^2 - 2k_{12}q_1q_2 - k_{22}q_2^2$$

and

$$\hat{L}_G = 2\alpha\left(k_{12}q_1 + k_{22}q_2\right).$$

The Green's function is then given in Fourier space by

$$\hat{G}(\mathbf{q}) = \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}'}\mathrm{e}^{2\mathrm{i}\alpha x_2'}\left\{k_{11}q_1^2 + 2k_{12}q_1q_2 + k_{22}q_2^2 - 2\alpha(k_{12}q_1 + k_{22}q_2)\right\}^{-1}. \tag{2.9}$$

To obtain the final form of the Green's function we then have to invert (2.9) with (2.8). The inversion integral will be evaluated next.

2.1. Inversion integral

The two-dimensional inversion integral obtained by substituting (2.9) in (2.8) is

$$G(\mathbf{x};\mathbf{x}') = \frac{1}{4\pi^2} e^{2i\alpha x_2'} \int_{-\infty}^{\infty} \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}}{\Delta}\, d\mathbf{q}, \tag{2.10}$$

where

$$\Delta(q_1,q_2) = k_{11}q_1^2 + 2k_{12}q_1q_2 + k_{22}q_2^2 - 2\alpha(k_{12}q_1 + k_{22}q_2).$$

The fact that α is real ensures that Δ is also real.

In order to evaluate (2.10), we will use several changes of variables. First, consider the denominator Δ. The condition (2.3) implies that

$$\Delta(q_1,q_2) = 0$$

defines an ellipse in the (q_1,q_2)-plane. The changes of variable will map this ellipse into a circle. We will then integrate using plane polar coordinates (R,ψ), with $R = 0$ at the centre of the circle. We define the integral with respect to R as a Cauchy principal-value integral; any other interpretation would lead to additional regular solutions of the homogeneous form of (2.7).

Let

$$Q_1 = q_1\sqrt{k_{11}} \quad \text{and} \quad Q_2 = q_2\sqrt{k_{22}}.$$

This substitution changes Δ to

$$\Delta = Q_1^2 + aQ_1Q_2 + Q_2^2 - bQ_1 - cQ_2 \tag{2.11}$$

where

$$a = 2\frac{k_{12}}{\sqrt{k_{11}k_{22}}}, \quad b = \frac{2\alpha k_{12}}{\sqrt{k_{11}}} \quad \text{and} \quad c = 2\alpha\sqrt{k_{22}}.$$

The change from q_1 and q_2 to Q_1 and Q_2 is a change of scale that equalizes the coefficients of Q_1^2 and Q_2^2. The next change eliminates the cross-term Q_1Q_2: put

$$Q_1 = \lambda P + Q \quad \text{and} \quad Q_2 = \lambda P - Q$$

where

$$\lambda = \left\{(2-a)/(2+a)\right\}^{1/2}.$$

Then, (2.11) can be written as

$$\begin{aligned}\Delta &= (2-a)(P^2 + Q^2 + 2d_1P + 2d_2Q)\\ &= (2-a)\left\{(P+d_1)^2 + (Q+d_2)^2 - (d_1^2+d_2^2)\right\},\end{aligned} \tag{2.12}$$

where

$$d_1 = -\frac{\lambda(b+c)}{2(2-a)} \quad \text{and} \quad d_2 = -\frac{b-c}{2(2-a)}.$$

Equation (2.12) can be written conveniently using plane polar coordinates centred at $P = d_1$ and $Q = d_2$. Thus, we obtain

$$\Delta = (2-a)(R^2 - D^2) \tag{2.13}$$

where

$$P + d_1 = R\cos\psi, \quad Q + d_2 = R\sin\psi \quad \text{and} \quad D^2 = d_1^2 + d_2^2.$$

The expression (2.13) shows that $\Delta = 0$ when $R = D$.

Next, consider the exponential term in (2.10) under the variable transformations given above. Let $r_i = x_i - x_i'$. Then,

$$\begin{aligned} \mathbf{q}\cdot(\mathbf{x}-\mathbf{x}') &= q_1 r_1 + q_2 r_2 \\ &= \lambda\left(\frac{r_1}{\sqrt{k_{11}}} + \frac{r_2}{\sqrt{k_{22}}}\right) R\cos\psi + \left(\frac{r_1}{\sqrt{k_{11}}} - \frac{r_2}{\sqrt{k_{22}}}\right) R\sin\psi + C, \end{aligned}$$

where

$$C = -\lambda d_1\left(\frac{r_1}{\sqrt{k_{11}}} + \frac{r_2}{\sqrt{k_{22}}}\right) - d_2\left(\frac{r_1}{\sqrt{k_{11}}} - \frac{r_2}{\sqrt{k_{22}}}\right).$$

It turns out that the constant C simplifies significantly to $C = \alpha r_2$.

Now, let

$$\lambda\left(\frac{r_1}{\sqrt{k_{11}}} + \frac{r_2}{\sqrt{k_{22}}}\right) = S\cos\theta \quad \text{and} \quad \frac{r_1}{\sqrt{k_{11}}} - \frac{r_2}{\sqrt{k_{22}}} = S\sin\theta,$$

so that

$$\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}') = RS\cos(\psi-\theta) + C.$$

Finally, the Jacobian, $\mathcal{J}$, of the coordinate transformations given above is

$$\mathcal{J} = AR \quad \text{where} \quad A = -2\lambda/\sqrt{k_{11}k_{22}}.$$

The inversion integral is then

$$G(\mathbf{x};\mathbf{x}') = \frac{1}{4\pi^2}\mathrm{e}^{2\mathrm{i}\alpha x_2'}\mathrm{e}^{\mathrm{i}\alpha r_2}\frac{A}{2-a}\,⨍_0^\infty \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}RS\cos(\psi-\theta)}}{R^2-D^2} R\,\mathrm{d}\psi\,\mathrm{d}R.$$

The integral with respect to ψ is standard, and so

$$G(\mathbf{x};\mathbf{x}') = \frac{\mathrm{e}^{\mathrm{i}\alpha r_2}\mathrm{e}^{2\mathrm{i}\alpha x_2'}A}{2\pi(2-a)}\,⨍_0^\infty \frac{R\,J_0(RS)}{R^2-D^2}\mathrm{d}R$$

where J_0 is a Bessel function. The remaining integral can also be evaluated, by standard contour-integral methods. (One way is to consider the integral

$$I \equiv \int_{\mathcal{C}} \frac{z\,H_0^{(1)}(Sz)}{z^2-D^2}\,\mathrm{d}z$$

around a closed contour $\mathcal{C}$ in the complex z-plane, consisting of a piece of the real axis indented above the simple pole at $z = D$, a large quarter-circle $\mathcal{C}_0$, and a piece of the imaginary axis. Here, $H_0^{(1)}$ is a Hankel function, chosen so that the contribution from $\mathcal{C}_0$ vanishes as $\mathcal{C}_0$ recedes to infinity. There are no poles inside $\mathcal{C}$, whence $I = 0$. The result follows from $\mathrm{Re}(I) = 0$.) Hence,

$$G(\mathbf{x};\mathbf{x}') = -\frac{A\mathrm{e}^{\mathrm{i}\alpha r_2}\mathrm{e}^{2\mathrm{i}\alpha x_2'}}{4(2-a)}Y_0(DS) = \frac{\mathrm{e}^{\mathrm{i}\alpha(x_2+x_2')}}{4\sqrt{\det\mathbf{K}}}Y_0(\alpha\mathcal{R}) \tag{2.14}$$

where Y_0 is a Bessel function of the second kind and

$$\mathcal{R} = \sqrt{\frac{k_{22}}{\det \mathbf{K}}} \sqrt{k_{22} r_1^2 - 2k_{12} r_1 r_2 + k_{11} r_2^2}.$$

Equation (2.14) is a singular solution of (2.7), as needed for boundary element analysis. Any other regular solutions of the homogeneous form of (2.7) can be added to (2.14). For example, other particular singular solutions are

$$-\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\alpha(x_2+x_2')}}{4\sqrt{\det \mathbf{K}}} H_0^{(1)}(\alpha\mathcal{R}) \quad \text{and} \quad \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\alpha(x_2+x_2')}}{4\sqrt{\det \mathbf{K}}} H_0^{(2)}(\alpha\mathcal{R}), \tag{2.15}$$

where $H_0^{(2)}(\rho)$ is another Hankel function. The choice of fundamental solution will usually be dictated by the desired behaviour at infinity or how the solution simplifies when the grading parameter becomes complex.

As an example, suppose that $\alpha = \mathrm{i}\beta$, where β is real, so that the thermal conductivities are given by

$$\mathbf{K}\,\mathrm{e}^{2\beta x_2}.$$

Then, as $\mathrm{i}\pi H_0^{(1)}(\mathrm{i}\rho) = 2K_0(\rho)$, where K_0 is a modified Bessel function, we find that an appropriate fundamental solution is

$$G(\mathbf{x};\mathbf{x}') = -\frac{\mathrm{e}^{-\beta(x_2+x_2')}}{2\pi\sqrt{\det \mathbf{K}}} K_0(\beta\mathcal{R}) = G(\mathbf{x}';\mathbf{x}). \tag{2.16}$$

Note that $K_0(\rho)$ is exponentially small as $\rho \to \infty$, and has a logarithmic singularity as $\rho \to 0$. As might be expected, the solution (2.16) is real.

2.2. Reduction to isotropic form

For an isotropic, exponentially graded solid, $k_{ij} = k\delta_{ij}$ where k is a positive constant and δ_{ij} is the Kronecker delta. The various constants appearing in the analysis for the inversion integral are then $a = b = 0$, $c = 2\alpha\sqrt{k}$, $A = -2/k$, $D = \alpha\sqrt{k/2}$ and $\mathcal{R} = \sqrt{r_1^2 + r_2^2}$. Then the Green's function given by $(2.15)_1$ reduces to,

$$G(\mathbf{x};\mathbf{x}') = -\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\alpha(x_2+x_2')}}{4k} H_0^{(1)}(\alpha\mathcal{R}), \tag{2.17}$$

which agrees with the result in [9].

3. An alternative method: extension to three dimensions

The fundamental solutions found above involve Bessel and Hankel functions. They are similar to the well-known fundamental solutions of the Helmholtz equation and the modified Helmholtz equation. Therefore, we seek to transform the governing differential equation, (2.6), into the Helmholtz equation. In fact, we can proceed with grading in an arbitrary direction without additional effort. Thus, suppose that the thermal conductivities are given by

$$\mathbf{K}\,\exp(2\boldsymbol{\beta}\cdot\mathbf{x}),$$

where $\boldsymbol{\beta}$ is a constant vector giving the direction and magnitude of the grading. The symmetric matrix $\mathbf{K}$ can be 2×2 (as above) or 3×3.

In the absence of any heat source ($\mathcal{Q} \equiv 0$), the governing differential equation can be written in subscript form as

$$k_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + 2\beta_i k_{ij}\frac{\partial u}{\partial x_j} = 0. \tag{3.18}$$

Now consider a preliminary transformation so as to remove the first-order derivative term. Let

$$u = v\,\exp(\boldsymbol{\gamma}\cdot\mathbf{x}) \tag{3.19}$$

where the constant vector $\boldsymbol{\gamma}$ will be chosen later. We then have

$$\frac{\partial u}{\partial x_j} = \left(\frac{\partial v}{\partial x_j} + \gamma_j v\right)\exp(\boldsymbol{\gamma}\cdot\mathbf{x})$$

and

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \left(\frac{\partial^2 v}{\partial x_i \partial x_j} + \gamma_i\frac{\partial v}{\partial x_j} + \gamma_j\frac{\partial v}{\partial x_i} + \gamma_i\gamma_j v\right)\exp(\boldsymbol{\gamma}\cdot\mathbf{x}).$$

Substituting in (3.18) gives

$$k_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} + 2(\gamma_i + \beta_i)k_{ij}\frac{\partial v}{\partial x_j} + (\gamma_i\gamma_j + 2\beta_i\gamma_j)k_{ij}v = 0, \tag{3.20}$$

where we have used the symmetry of $\mathbf{K}$, $k_{ij} = k_{ji}$. Therefore, we choose $\boldsymbol{\gamma} = -\boldsymbol{\beta}$, whence $u = v\exp(-\boldsymbol{\beta}\cdot\mathbf{x})$ and v solves

$$k_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} - \beta_i\beta_j k_{ij}v = 0. \tag{3.21}$$

This equation is similar to the modified Helmholtz equation. In fact, as we have supposed that k_{ij} is positive definite, we can change the independent variable x_i so that the new equation *is* the modified Helmholtz equation, which has known fundamental solutions.

So, make a linear change of variables from x_i to y_i, using

$$y_i = q_{ij}x_j \quad \text{or} \quad \mathbf{y} = \mathbf{Q}\mathbf{x},$$

where the q_{ij} are to be chosen [14]. We have

$$\frac{\partial v}{\partial x_i} = \frac{\partial v}{\partial y_k}\frac{\partial y_k}{\partial x_i} = q_{ki}\frac{\partial v}{\partial y_k}$$

and

$$\frac{\partial^2 v}{\partial x_i\,\partial x_j} = q_{ki}q_{\ell j}\frac{\partial^2 v}{\partial y_k\,\partial y_\ell}.$$

Hence, (3.21) becomes

$$q_{ki}k_{ij}q_{\ell j}\frac{\partial^2 v}{\partial y_k \partial y_\ell} - \beta_i\beta_j k_{ij} v = 0. \tag{3.22}$$

Choose $\mathbf{Q}$ so that

$$\mathbf{QKQ}^T = \mathbf{I}, \tag{3.23}$$

and then (3.22) becomes

$$(\nabla_y^2 - \kappa^2)v = 0, \tag{3.24}$$

where $\kappa^2 = \beta_i k_{ij}\beta_j = \boldsymbol{\beta}^T\mathbf{K}\boldsymbol{\beta}$ and

$$\nabla_y^2 v \equiv \frac{\partial^2 v}{\partial y_i\,\partial y_i}$$

is the Laplacian in terms of $\mathbf{y}$.

Fundamental solutions for (3.24), in two or three dimensions, are simple functions of R, defined by

$$\begin{aligned} R^2 &= \mathbf{y}^T\mathbf{y} = y_i y_i \\ &= q_{ij}x_j q_{ik}x_k = \mathbf{x}^T\mathbf{Q}^T\mathbf{Q}\mathbf{x}. \end{aligned}$$

But (3.23) implies that $\mathbf{Q}^T\mathbf{Q} = \mathbf{K}^{-1}$, and so

$$R = \sqrt{\mathbf{x}^T\mathbf{K}^{-1}\mathbf{x}},$$

which means that we do not have to find $\mathbf{Q}$ explicitly in order to calculate R.

Hence, typical fundamental solutions of (3.21) are [1, 13]

$$G = \mathcal{A}K_0(\kappa R) \quad \text{in two dimensions} \tag{3.25}$$

and

$$G = \mathcal{A}\frac{\mathrm{e}^{-\kappa R}}{R} \quad \text{in three dimensions}, \tag{3.26}$$

where $\mathcal{A}$ can be chosen to provide the proper strength for the singularity at $R = 0$. A fundamental solution for the graded material can then be obtained from (3.25) or (3.26) with (3.19) simply by multiplying by $\exp(-\boldsymbol{\beta}\cdot\mathbf{x})$.

The fundamental solutions given by (3.25) and (3.26) correspond to a singularity at the origin, $\mathbf{x} = \mathbf{0}$. For a singularity at $\mathbf{x} = \mathbf{x}'$, simply replace $\mathbf{x}$ by $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ in the definition of R. Note that $\mathcal{A}$ can depend on $\mathbf{x}'$: compare with (2.16). Thus, in two dimensions, for example, we obtain

$$G(\mathbf{x};\mathbf{x}') = -\frac{K_0(\kappa R)}{2\pi\sqrt{\det\mathbf{K}}}\,\exp\{-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}')\} = G(\mathbf{x}';\mathbf{x}),$$

where

$$\kappa = \sqrt{\boldsymbol{\beta}^T\mathbf{K}\boldsymbol{\beta}} \quad \text{and} \quad R = \sqrt{(\mathbf{x}-\mathbf{x}')^T\mathbf{K}^{-1}(\mathbf{x}-\mathbf{x}')}.$$

4. Derivatives of G

Let us write

$$G(\mathbf{x};\mathbf{x}') = G(\mathbf{x}';\mathbf{x}) = G_0(R)\,\exp\{-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}')\},$$

where $G_0(R) = cK_0(\kappa R)$ in two dimensions, $G_0(R) = cR^{-1}\mathrm{e}^{-\kappa R}$ in three dimensions, and c is a constant. In applications, we usually need the conormal derivative of G; physically, this represents the normal heat flux, and is defined as the projection of the heat flux vector in the direction of the unit normal vector $\mathbf{n}$. Thus, with $\ell_j(\mathbf{x}) = n_i(\mathbf{x})\,k_{ij}\,\exp(2\boldsymbol{\beta}\cdot\mathbf{x})$, we require

$$\frac{\partial G}{\partial \nu} = -\ell_j(\mathbf{x})\,\frac{\partial G}{\partial x_j} \quad \text{and} \quad \frac{\partial G}{\partial \nu'} = -\ell_j(\mathbf{x}')\,\frac{\partial G}{\partial x_j'}.$$

We may also want the second derivative,

$$\frac{\partial^2 G}{\partial \nu\,\partial \nu'} = \ell_i(\mathbf{x})\,\ell_j(\mathbf{x}')\,\frac{\partial^2 G}{\partial x_i\,\partial x_j'}.$$

We have

$$\frac{\partial R}{\partial x_j} = -\frac{\partial R}{\partial x_j'} = \frac{1}{R}\left(\mathbf{K}^{-1}\mathbf{r}\right)_j$$

whence

$$\ell_j(\mathbf{x})\,\frac{\partial R}{\partial x_j} = \frac{1}{R}(\mathbf{n}\cdot\mathbf{r})\,\exp(2\boldsymbol{\beta}\cdot\mathbf{x})$$

and

$$\ell_j(\mathbf{x}')\,\frac{\partial R}{\partial x_j'} = -\frac{1}{R}(\mathbf{n}'\cdot\mathbf{r})\,\exp(2\boldsymbol{\beta}\cdot\mathbf{x}'),$$

where $\mathbf{n} \equiv \mathbf{n}(\mathbf{x})$ and $\mathbf{n}' \equiv \mathbf{n}(\mathbf{x}')$. Hence

$$\frac{\partial G}{\partial \nu} = \left\{\left(\mathbf{n}^T\mathbf{K}\boldsymbol{\beta}\right)\,G_0 - R^{-1}(\mathbf{n}\cdot\mathbf{r})\,G_0'\right\}\exp(\boldsymbol{\beta}\cdot\mathbf{r}), \tag{4.27}$$

$$\frac{\partial G}{\partial \nu'} = \left\{\left({\mathbf{n}'}^T\mathbf{K}\boldsymbol{\beta}\right)\,G_0 + R^{-1}(\mathbf{n}'\cdot\mathbf{r})\,G_0'\right\}\exp(-\boldsymbol{\beta}\cdot\mathbf{r}), \tag{4.28}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ and $G_0'(R) = dG_0/dR$. Similarly,

$$\frac{\partial^2 G}{\partial \nu\,\partial \nu'} = (\mathcal{A}_0 G_0 + \mathcal{A}_1 G_0' + \mathcal{A}_2 G_0'')\exp\{\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}')\}, \tag{4.29}$$

where

$$\mathcal{A}_0 = \left(\mathbf{n}^T\mathbf{K}\boldsymbol{\beta}\right)\left({\mathbf{n}'}^T\mathbf{K}\boldsymbol{\beta}\right), \quad \mathcal{A}_2 = -R^{-2}\,(\mathbf{n}\cdot\mathbf{r})\,(\mathbf{n}'\cdot\mathbf{r})$$

and

$$\mathcal{A}_1 = R^{-3}\,(\mathbf{n}\cdot\mathbf{r})\,(\mathbf{n}'\cdot\mathbf{r}) + R^{-1}\left\{\left(\mathbf{n}^T\mathbf{K}\boldsymbol{\beta}\right)(\mathbf{n}'\cdot\mathbf{r}) - \left({\mathbf{n}'}^T\mathbf{K}\boldsymbol{\beta}\right)(\mathbf{n}\cdot\mathbf{r}) - \mathbf{n}^T\mathbf{K}\mathbf{n}'\right\}.$$

These expressions for the conormal derivatives of the Green's functions are required in conventional collocation-based boundary element methods and also in the symmetric Galerkin method.

5. Discussion

We have described two methods for determining fundamental solutions for steady-state heat conduction in anisotropic, exponentially graded materials. It is of interest to compare the two methods. The first method is quite general, and leads to an expression for G as an inverse Fourier transform. Some ingenuity may be required in order to evaluate this multiple integral. The second method requires ingenuity in a different way: the aim is to transform the given differential equation into another differential equation with known fundamental solutions. For other physical problems, especially vector problems, a hybrid approach may be used profitably, where the original differential equation is first transformed into a new differential equation, which is then solved by Fourier transforms. These techniques may find further applications as inhomogeneous media become more common.

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