Journal of
Linear and Topological Algebra
Vol. 03*, No.* 03*,* 2014*,* 185- 196

Journal of Linear and Topological Algebra
J L T A

Module amenability and module biprojectivity of θ-Lau product of Banach algebras

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Received 2 October 2014; Revised 20 December 2014; Accepted 29 December 2014.

Abstract. In this paper we study the relation between module amenability of θ - Lau product $A \times_\theta B$ and that of Banach algebras A, B. We also discuss module biprojectivity of $A \times_\theta B$. As a consequent we will see that for an inverse semigroup S, $l^1(S) \times_\theta l^1(S)$ is module amenable if and only if S is amenable.

Keywords: Module amenability, module biprojectivity, θ-Lau product of Banach algebras, inverse semigroup.

1. Introduction

Lau product of Banach algebras, were introduced by A. T- M. Lau [6], for a special class of Banach algebras which are pre-duals of Von Numann algebras, such that the identity of the dual algebra is a multiplicative linear functional on the predual. The θ-Lau product was introduced by M. Sangani-Monfared in [7]. He defined θ-Lau product on $A \times B$ as

$$(a, b)(m, n) = (am + \theta(b)m + \theta(n)a, bn),$$

where $\theta \in \sigma(B)$, and A, B are Banach algebras, and then studied amenability and weak amenability of this Banach algebra. The norm on this space is as $\|(a, b)\| = \|a\| + \|b\|$. The Banach algebra generated by above multiplication on $A \times B$, is denoted by $A \times_\theta B$. In[4], it was studied some properties of $A \times_\theta B$, such as Character amenability, Gelfand space. M. Amini in [1] introduce the concept of module amenability. In this paper we

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

study the relation between module amenability and module biprojectivity, of θ - Lau product $A \times_\theta B$ and module amenability and module biprojectivity of Banach algebras A, B. By an example we will show that if $l^1(S)$ is unital then $l^1(S) \times_\theta l^1(S)$ is module amenable if and only if S is amenable. We also show that $l^1(\mathbb{N}_\vee) \times_\theta l^1(\mathbb{N}_\vee)$ is module biprojective.

Let $\mathfrak{U}$ be a Banach algebra and A be a Banach $\mathfrak{U}$-bimodule, with the compatible module actions

$$\alpha \cdot (am) = (\alpha \cdot a)m, (\alpha\beta) \cdot m = \alpha \cdot (\beta \cdot m), (\alpha, \beta \in \mathfrak{U}, a, m \in A).$$

A bounded map $D : A \longrightarrow X$ with $D(a + b) = D(a) + D(b)$, $D(ab) = D(a).b + a.D(b)$ and $D(\alpha \cdot a) = \alpha \cdot D(a)$, $D(a.\alpha) = D(a).\alpha$, $(\alpha \in \mathfrak{U}, a, b \in A)$ is called a module derivation. If there exists $x \in X$ such that $D(a) = a.x - x.a = \delta_x(a)$, $(a \in A)$ then D is called inner derivation. The set of all module derivations $D : A \longrightarrow X'$ is denoted by $Z_u(A, X')$ and the notation $N_u(A, X')$ for those which are inner. The quotient $\frac{Z_u(A,X')}{N_u(A,X')}$ is denoted by $H_u(A, X')$.

A Banach algebra A is module amenable if and only if $H_u(A, X') = \{0\}$, for each A-$\mathfrak{U}$-module X. Note that X is called an A-$\mathfrak{U}$-module if X is a Banach algebra which is at the same time a Banach A-bimodule and a Banach $\mathfrak{U}$-bimodule with compatibility of actions

$$(a.x).\alpha = a.(x.\alpha), \alpha.(a.x) = (\alpha.a).x, (\alpha \in \mathfrak{U}, a \in A, x \in X).$$

For such X, X' is also Banach module over A and $\mathfrak{U}$, with compatible actions under canonical actions of A, $\mathfrak{U}$, $\alpha.(a.f) = (\alpha.a).f$, $(a \in A, \alpha \in \mathfrak{U}, f \in X')$. In [2], it was defined module biprojectivity for a Banach algebra which is a Banach module over another Banach algebra.

Let X, Y be A-$\mathfrak{U}$-modules, module homomorphism from X to Y is a norm continous map $\varphi : X \longrightarrow Y$ with $\varphi(x \pm y) = \varphi(x) \pm \varphi(y)$, $\varphi(\alpha.x) = \alpha.\varphi(x)$, $\varphi(x.\alpha) = \varphi(x).\alpha$, $\varphi(a.x) = a.\varphi(x)$, $\varphi(x.a) = \varphi(x).a$, $(x, y \in X, \alpha \in \mathfrak{U}, a \in A)$. If A is a commutative $\mathfrak{U}$-module and acts on itself by multiplication from both sides, then it is also a Banach A-$\mathfrak{U}$-module. Consider the projective tensor product $A\widehat{\otimes}A$. It is well known that $A\widehat{\otimes}A$ is a Banach algebra with respect to the canonical multiplication defined by $(a \otimes b)(c \otimes d) = ac \otimes bd$ and extended by bi-linearity and continuity, [3]. Then $A\widehat{\otimes}A$ is a Banach A-$\mathfrak{U}$-module with canonical actions. Let I be the closed ideal of the projective tensor product $A\widehat{\otimes}A$ generated by elements of the form $\alpha.a \otimes b - a \otimes b.\alpha$ for $\alpha \in \mathfrak{U}$, $a, b \in A$. Consider the map $\pi_A : A\widehat{\otimes}A \longrightarrow A$ defined by $\pi_A(a \otimes m) = am$ and extended by linearity and continuity. Let J be the closed ideal of A generated by $\pi_A(I)$. Then the module projective tensor product $A\widehat{\otimes}_u A \cong A\widehat{\otimes}A/I$ and Banach algebra A/J are Banach $\mathfrak{U}$-module. The map $\widetilde{\pi_A} : A\widehat{\otimes}_u A \longrightarrow A/J$ defined by $\widetilde{\pi_A}(a \otimes m + I) = ab + J$, extended to an $\mathfrak{U}$-module morphism. If $A\widehat{\otimes}_u A$ and A/J are commutative $\mathfrak{U}$-module, then $A\widehat{\otimes}_u A$ and A/J are A/J - $\mathfrak{U}$-module and $\widetilde{\pi_A}$ is A/J-$\mathfrak{U}$-module homomorphism. A Banach algebra A is called module biprojective (as $\mathfrak{U}$-module) if $\widetilde{\pi_A}$ has a bounded right inverse which is an A/J-$\mathfrak{U}$-module morphism[2].

2. Module amenability

Throughout we assume that A, B are Banach $\mathfrak{U}$-bimodule with actions $A \times \mathfrak{U} \longrightarrow A, (a, \alpha) \longmapsto a.\alpha$, $\mathfrak{U} \times A \longrightarrow A, (\alpha, a) \longmapsto \alpha \circ a$, $B \times \mathfrak{U} \longrightarrow B, (b, \alpha) \longmapsto b \star \alpha$,

$\mathfrak{U} \times B \longrightarrow B, (\alpha, b) \longmapsto \alpha * b$. and $\theta \in \sigma(B)$ is such that $\theta(\alpha \star b)a = \theta(b)\alpha \circ a$, $\theta(b * \alpha) = \theta(b)\alpha \circ a$. Let $M = A \times_\theta B$.

Proposition 2.1 Let $\mathfrak{U}$ be a Banach algebra and A, B be Banach $\mathfrak{U}$-bimodule. for $\theta \in \sigma(B)$
1) $A \times_\theta B$ is a Banach A-bimodule,
2) $A \times_\theta B$ is a Banach B-bimodule,
3) $A \times_\theta B$ is a Banach $\mathfrak{U}$-bimodule,
4) $A \times_\theta B$ is a Banach B-$\mathfrak{U}$-module.

Proof. 1) We define the module actions as
$A \times (A \times_\theta B) \longrightarrow A \times_\theta B$ by $a.(m, n) = (a, 0)(m, n) = (am + \theta(n)a, 0)$,
and $(A \times_\theta B) \times A \longrightarrow A \times_\theta B$ by $(m, n) \centerdot a = (m, n)(a, 0) = (ma + \theta(n)a, 0)$. It is easy to see that, with above actions, $A \times_\theta B$ is an A-bimodule.
2)The module actions are defined as
$B \times (A \times_\theta B) \longrightarrow A \times_\theta B$ by $b.(m, n) = (0, b)(m, n) = (\theta(b)m, bn)$,
and $(A \times_\theta B) \times B \longrightarrow A \times_\theta B$ by $(m, n) \centerdot b = (m, n)(0, b) = (\theta(b)m, nb)$. It is easy to see that properties are satiesfied.
3) The module actions are defined as
$\mathfrak{U} \times (A \times_\theta B) \longrightarrow A \times_\theta B$ by $\alpha.(a, b) = (\alpha \circ a, \alpha * b)$,
$(A \times_\theta B) \times \mathfrak{U} \longrightarrow A \times_\theta B$ by $(a, b) \centerdot \alpha = (a.\alpha, b \star \alpha)$
4) By parts (2), (3), $A \times_\theta B$ is at the same time B-module and $\mathfrak{U}$-module, thus it is sufficient to check that actions are compatible.

$$\begin{aligned}
\alpha.\big(b.(a, n)\big) &= \alpha.\Big((0, b)(a, n)\Big) = \alpha.(\theta(b)a, bn) \\
&= \Big(\theta(b)\alpha \circ a, \alpha * (bn)\Big) = \Big(\theta(b \star \alpha)a, (\alpha * b)n\Big) = (\alpha * b).(a, n).
\end{aligned}$$

$$\begin{aligned}
b.(\alpha.(m, n)) &= b.(\alpha \circ m, \alpha * n) \\
&= \Big(\theta(b)(\alpha \circ m), b(\alpha * n)\Big) = \Big(\theta(b \star \alpha)m, (b * \alpha)n\Big) \\
&= (b * \alpha).(m, n).
\end{aligned}$$

Also

$$\begin{aligned}
(\alpha.(m, n)).b &= (\alpha \circ m, \alpha * n).b = (\alpha \circ m, \alpha * n)(0, b) \\
&= \Big(\theta(b)(\alpha \circ m), (\alpha * n)b\Big) = \Big(\alpha \circ (\theta(b)m), \alpha * (bn)\Big) \\
&= \alpha.((m, n)(0, b)) = \alpha.((m, n).b)
\end{aligned}$$

■

Proposition 2.2 Let $\mathfrak{U}$ be a Banach algebra, A and B be Banach $\mathfrak{U}$-bimodules , and $\theta \in \sigma(B)$. If X is a Banach A-$\mathfrak{U}$-module and Y is a Banach B-$\mathfrak{U}$-module then $X \times Y$ is a Banach $A \times_\theta B$-$\mathfrak{U}$-module.

Proof. Assume that module actions on X and Y , are as

$\mathfrak{U} \times X \longrightarrow X$ as $(\alpha, x) \longmapsto \alpha.x$, $X \times \mathfrak{U} \longrightarrow X$, as $(x, \alpha) \longmapsto x \circ \alpha$, $\mathfrak{U} \times Y \longrightarrow Y$ as $(\alpha, y) \longmapsto \alpha \vartriangle y$, $Y \times \mathfrak{U} \longrightarrow \mathfrak{U}$ as $(y, \alpha) \longmapsto y \nabla \alpha$. And $A \times X \longrightarrow X$ as $(a, x) \longmapsto a.x$, $X \times A \longrightarrow X$ as $(x, a) \longmapsto x \circ a$, $B \times Y \longrightarrow Y$, $(b, y) \longmapsto b \bullet y$, $y \times B \longrightarrow B$, as $(y, b) \longmapsto y \centerdot b$. We define
$\mathfrak{U} \times (X \times Y) \longrightarrow X \times Y$ by $\alpha.(x, y) = (\alpha.x, \alpha \vartriangle y)$ and $(X \times Y) \times \mathfrak{U} \longrightarrow X \times Y$ by $(x, y) \centerdot \alpha = (x \circ \alpha, y \nabla \alpha)$. Also $(X \times Y) \times (A \times_\theta B) \longrightarrow X \times Y$ by $(x, y) \centerdot (a, b) = (x \circ a + \theta(b)x, y \centerdot b)$ and $(A \times_\theta B) \times (X \times Y) \longrightarrow X \times Y$ By $(a, b).(x, y) = (a.x + \theta(b)x, b \bullet y)$. We can see that by above actions $X \times Y$ is at the same time a Banach $\mathfrak{U}$-bimodule and a Banach $A \times_\theta B$-bimodule. Only we prove that actions are compatible.

$$\begin{aligned}
\alpha.\Big((a, b).(x, y)\Big) &= \alpha.\Big(a.x + \theta(b)x, b \bullet y\Big) \\
&= \Big(\alpha.(a.x + \theta(b)\alpha.x), \alpha \vartriangle (b \bullet y)\Big) \\
&= \Big((\alpha \circ a).x + \theta(\alpha * b)x, (\alpha * b) \bullet y\Big) = \Big(\alpha.(a, b)\Big).(x, y).
\end{aligned}$$

$$\begin{aligned}
(a, b).(\alpha.(x, y)) &= (a, b).(\alpha.x, \alpha \vartriangle y) \\
&= \Big(a.(\alpha.x) + \theta(b)(\alpha.x), b \bullet (\alpha \vartriangle y)\Big) \\
&= \Big((a.\alpha).x + \theta(b \star \alpha)x, (b \star \alpha) \bullet y\Big) = ((a, b) \centerdot \alpha).(x, y).
\end{aligned}$$

Also

$$\begin{aligned}
(\alpha.(x, y)) \centerdot (a, b) &= (\alpha.x, \alpha \vartriangle y) \centerdot (a, b) \\
&= ((\alpha.x) \circ a + \theta(b)(\alpha.x), (\alpha \vartriangle y) \centerdot b) \\
&= (\alpha.(x \circ a) + \theta(b)\alpha.x, \alpha \vartriangle (y \centerdot b)) \\
&= \alpha.(x \circ a + \theta(b)x, y \bullet b) \\
&= \alpha.((x, y) \centerdot (a, b))
\end{aligned}$$

So $X \times Y$ is a $A \times_\theta B$-$\mathfrak{U}$-module. ■

Proposition 2.3 Let $\mathfrak{U}$ be a Banach algebra and A, B be Banach $\mathfrak{U}$-bimodules, and let X be a Banach A-$\mathfrak{U}$-module and Y be a Banach B-$\mathfrak{U}$-module then $D \in Z_u(A \times_\theta B, X' \times Y')$ if and only if $\exists D_1 \in Z_u(A, X')$, $D_2 \in Z_u(B, Y')$, $D_3 \in Z_u(B, X')$ and a bounded linear map $R : A \longrightarrow Y'$ with $R(\alpha \circ a) = \alpha.R(a)$,$(\alpha \in \mathfrak{U})$ such that
1) $D(a, b) = (D_1(a) + D_3(b), R(a) + D_2(b))$,
2) $D_1(\theta(b)m) = D_1(m) \odot \theta(b) + D_3(b).m$,
3) $D_1(\theta(n)c) = D_1(c) \odot \theta(n) + c.D_3(n)$,
4) $D_3(bn) = D_3(b) \odot \theta(n) + D_3(n) \odot \theta(b)$, where $(D_1(a) \odot \theta(b))(x) = D_1(a)(\theta(b)x)$.
5) $R(\theta(b)m) = b.R(m)$,
6) $b.R(m) = R(m).b$,
7)$R(am) = 0$.

Proof. Choose $D \in Z_u(A \times_\theta B, X' \times Y')$ so there are $d_1 : A \times_\theta B \longrightarrow X'$, $d_2 : A \times_\theta B \longrightarrow$

Y' such that D = (d_1, d_2), Set
$D_1 : A \longrightarrow X'$ as $D_1(a) = d_1(a, 0)$, $D_2 : B \longrightarrow Y'$ as $D_2(b) = d_2(0, b)$,
$D_3 : B \longrightarrow X'$ as $D_3(b) = d_1(0, b)$, $R : A \longrightarrow Y'$ as $R(a) = d_2(a, 0)$, Now

$$\begin{aligned} D(a,b) &= (d_1,d_2)((a,0)+(0,b)) \\ &= (d_1,d_2)(a,0)+(d_1,d_2)(0,b) \\ &= (d_1(a,0),d_2(a,0))+((d_1(0,b),d_2(0,b)) \\ &= \Big(d_1(a,0)+d_1(0,b)\Big)+\Big(d_2(a,0)+d_2(0,b)\Big) \\ &= (D_1(a)+D_3(b),R(a)+D_2(b)). \quad \textit{Since} \end{aligned}$$

$$\begin{aligned} D((a,b)(m,n)) &= D(am+\theta(n)a+\theta(b)m,bn) \\ &= (D_1(am)+D_1(\theta(n)a)+D_1(\theta(b)m) \\ &\quad + D_3(bn),R(am)+R(\theta(n)a)+R(\theta(b)m)+D_2(bn)). \end{aligned}$$

Also

$$\begin{aligned} &(a,b).D(m,n)+D(a,b).(m,n) = (a,b).(D_1(m)+D_3(n),R(m)+D_2(n)) \\ &+ (D_1(a)+D_3(b),R(a)+D_2(b)).(m,n) \\ &= \Big(a.D_1(m)+a.D_3(n)+D_1(m)\odot\theta(b)+D_3(n)\odot\theta(b),b.R(m)+b.D_2(n)\Big) \\ &+ \Big(D_1(a).m+D_3(b).m+D_1(a)\odot\theta(n)+D_3(b)\odot\theta(n),R(a).n+D_2(b).n\Big) \\ &= \Big(a.D_1(m)+D_1(a).m+a.D_3(n)+D_3(b).m+D_1(a)\odot\theta(n)+D_1(m)\odot\theta(b) \\ &+ D_3(n)\odot\theta(b)+D_3(b)\odot\theta(n),R(a).n+b.R(m)+b.D_2(n)+D_2(b).n\Big). \end{aligned}$$

Since D is a derivation by taking $a = n = 0$ we get $D_1(\theta(b)m) = D_3(b).m + D_1(m) \odot \theta(b)$ and $R(\theta(b)m) = b.R(m)$. Take $b = m = 0$ then $D_1(\theta(n)a) = a.D_3(n) + D_1(a) \odot \theta(n)$ and $R(\theta(b)a) = R(a).n$. Take $a = m = 0$ then $(D_3(bn) = D_3(n) \odot \theta(b) + D_3(b) \odot \theta(n)$ and $D_2(bn) = b.D_2(n) + D_2(b).n$ so $D_2 \in Z(B, Y')$, $D_3 \in Z(B, X')$. Take $b = n = 0$ to get $D_1(am) = D_1(a).m + a.D_1(m)$ and $R(am) = 0$. Also

$$D_1(\alpha \circ a) = d_1(\alpha \circ a, 0) = d_1(\alpha.(a, 0)) = \alpha.d_1(a, 0) = \alpha.D_1(a).$$

In the same way $D_2(\alpha * b) = \alpha.D_2(b)$, $D_3(\alpha * b) = \alpha.D_3(b)$.
Note that since $D \in Z_u(A \times_\theta B, X' \times Y')$ so
$D((a, b) + (m, n)) = D(a, b) + D(m, n)$ and $D(\alpha.(a, b)) = \alpha.D(a, b)$ thus

$$\begin{aligned} D(\alpha.(a,b)) &= D(\alpha\circ a,\alpha * b) = (d_1,d_2)(\alpha\circ a,\alpha * b) \\ &= (d_1(\alpha\circ a,\alpha * b),d_2(\alpha\circ a,\alpha * b)). \end{aligned}$$

On the other hand

$$\begin{aligned}\alpha.D(a,b) &= \alpha.((d_1,d_2)(a,b)) = \alpha.(d_1(a,b),d_2(a,b))\\ &= (\alpha.d_1(a,b),\alpha.d_2(a,b)).\end{aligned}$$

So $d_1(\alpha\circ a,\alpha * b)=\alpha.d_1(a,b)$, and $d_2(\alpha\circ a,\alpha * b)=\alpha.d_2(a,b)$.
Also we can prove that $d_1((a,b)+(m,n)) = d_1(a,b)+d_1(m,n)$ and $d_2((a,b)+(m,n)) = d_2(a,b)+d_2(m,n)$. So

$$\begin{aligned}D_1(a+m) &= d_1(a+m,0) = d_1((a,0)+(m,0))\\ &= d_1(a,0)+d_1(m,0) = D_1(a)+D_1(m).\end{aligned}$$

Similarly $D_2(b+n)=D_2(b)+D_2(n)$ and $D_3(b+d)=D_3(b)+D_3(d)$.
Consequently $D_1\in Z_u(A,X'), D_2\in Z_u(B,Y'), D_3\in Z_u(B,X')$ and properties of proposition are satisfied.
Let $D_1\in Z_u(A,X')$, $D_2\in Z_u(B,Y')$, $D_3\in Z_u(B,X')$ and bounded linear mapping R be such that the statement of proposition are satisfy, then

$$\begin{aligned}D((a,b)(m,n)) &= D(am+\theta(n)a+\theta(b)m,bn)\\ &= (D_1(am)+D_1(\theta(n)a)+D_1(\theta(b)m)\\ &+ D_3(bn), R(am)+R(\theta(n)a)+R(\theta(b)m)+D_2(bn))\\ &= \Big(a.D_1(m)+D_1(a).m+D_1(a)\odot\theta(n)\\ &+ a.D_3(n)+D_3(b).m+D_1(m)\odot\theta(b)\\ &+ D_3(n)\odot\theta(b)+D_3(b)\odot\theta(n), n.R(a)+R(m).b\\ &+ b.D_2(n)+D_2(b).n\Big).\end{aligned}$$

On the other hand

$$\begin{aligned}&(a,b).D(m,n)+D(a,b).(m,n)\\ &= (a,b).(D_1(m)+D_3(n),R(m)+D_2(n))\\ &+ (D_1(a)+D_3(b),R(a)+D_2(b)).(m,n)\\ &= \Big(a.D_1(m)+a.D_3(n)+\theta(b)\odot D_1(m)+\theta(b)\odot D_3(n), b.R(m)+b.D_2(n)\Big)\\ &+ \Big(D_1(a).m+D_3(b).m+D_1(a)\odot\theta(n)+D_3(b)\odot\theta(n), R(a).n+D_2(b).n\Big)\\ &= \Big(a.D_1(m)+D_1(a).m+\theta(n)\odot D_1(a)+a.D_3(n)+\theta(n)\odot D_3(b)\\ &+ D_3(b).m+\theta(b)\odot D_1(m)+\theta(b)\odot D_3(n), b.R(m)+R(a).n+b.D_2(n)+D_2(b).n\Big).\end{aligned}$$

So $D((a,b)(m,n)) = D(a,b).(m,n)+(a,b).D(m,n)$, for all $(a,b),(m,n) \in A \times_\theta B$ thus $D \in Z(A \times_\theta B, X' \times Y')$.

$$\begin{aligned}D(\alpha.(a,b)) &= D(\alpha \circ a, \alpha * b) = (D_1(\alpha \circ a) + D_3(\alpha * b), R(\alpha \circ a) + D_2(\alpha * b))\\ &= (\alpha.D_1(a) + \alpha.D_3(b), \alpha.R(a) + \alpha.D_2(b))\\ &= \alpha.(D_1(a) + D_3(b), R(a) + D_2(b)) = \alpha.D(a,b).\end{aligned}$$

And

$$\begin{aligned}D((a,b)+(m,n)) &= D(a+m, b+n)\\ &= (D_1(a+m) + D_3(b+n), R(a+m) + D_2(b+n))\\ &= (D_1(a) + D_1(m) + D_3(b) + D_3(n), R(a) + R(m) + D_2(b) + D_2(n))\\ &= (D_1(a) + D_3(b), R(a) + D_2(b)) + (D_1(m) + D_3(n), R(m) + D_2(n))\\ &= D(a,b) + D(m,n).\end{aligned}$$

Hence $D \in Z_u(A \times_\theta B, X' \times Y')$ ■

Corollary 2.4 By assumption of previous proposition, $D = \delta_{(\varphi,\psi)}(\varphi \in X', \psi \in Y')$ if and only if $\big(D_1 = \delta_\varphi, D_2 = \delta_\psi, D_3 = 0, R(a) = 0$, for all $a \in A.\big)$

Proof. Let $D = \delta_{(\varphi,\psi)}$ so for each $a \in A$ and $b \in B$ we have

$$\begin{aligned}D(a,b) &= \delta_{(\varphi,\psi)}(a,b)\\ &= (\varphi,\psi).(a,b) - (a,b).(\varphi,\psi)\\ &= (\varphi.a + \varphi \odot \theta(b), \psi.b) - (a.\varphi + \theta(b) \odot \varphi, b.\psi)\\ &= (\varphi.a + \varphi \odot \theta(b) - a.\varphi - \theta(b) \odot \varphi, \psi.b - b.\psi).\end{aligned}$$

Take $a = 0$, so $D_3(b) = \varphi \odot \theta(b) - \theta(b) \odot \varphi = 0$, $D_2(b) = \psi.b - b.\psi = \delta_\psi(b)$. Take $b = 0$, so $D_1(a) = \varphi.a - a.\varphi = \delta_\varphi(a)$ and $R(a) = 0$.
For the converse let $D_1 = \delta_\varphi$, $D_2 = \delta_\psi$, $D_3 = 0$, $R = 0$, so

$$\begin{aligned}D(a,b) &= (D_1(a) + D_3(b), R(a) + D_2(b))\\ &= (a.\varphi - \varphi.a + \varphi \odot \theta(b) - a.\varphi - \theta(b) \odot \varphi, \psi.b - b.\psi)\\ &= \delta_{(\varphi,\psi)}(a,b).\end{aligned}$$

■

Theorem 2.5 Let $\mathfrak{U}$ be a Banach algebra and A, B be Banach $\mathfrak{U}$-bimodules. If A is unital Banach algebra then $A \times_\theta B$ is module amenable if and only if both A, B are module amenable.

Proof. Assume that X is a Banach A-$\mathfrak{U}$-module and Y is a Banach B-$\mathfrak{U}$-module, by proposition 2.2, $X \times Y$ is a Banach $A \times_\theta B$-$\mathfrak{U}$-module, and let $D_1 \in Z_u(A, X')$, $D_2 \in$

$Z_u(B, Y')$. Define $D_3 : B \longrightarrow X'$ by $D_3(b) = D_1(\theta(b)1)$ then

$$\begin{aligned} D_3(bn) &= D_1(\theta(bn)1) = D_1(\theta(b)\theta(n)1) \\ &= D_3(b).(\theta(n)1) + D_1(\theta(n)1) \odot \theta(b) \\ &= D_3(b) \odot \theta(n) + D_3(n) \odot \theta(b). \end{aligned}$$

Also

$$\begin{aligned} D_1(\theta(n)a) &= D_1((\theta(n)1)a) = D_1(\theta(n)1).a + (\theta(n)1).D_1(a) \\ &= D_3(n).a + D_1(a) \odot \theta(n). \end{aligned}$$

$$\begin{aligned} D_1(\theta(b)m) &= D_1(\theta(b)1)m) = D_1(\theta(b)1).m + (\theta(b)1).D_1(m) \\ &= D_3(b).m + D_1(m) \odot \theta(b). \end{aligned}$$

Since $D_1 \in Z_u(A, X')$ so $D_3 \in Z_u(B, X')$. Now define D: $A \times_\theta B \longrightarrow X' \times Y'$ as $D(a, b) = (D_1(a) + D_3(b), D_2(b))$. So by above proposition $D \in Z_u(A \times_\theta B, X' \times Y')$. Since $A \times_\theta B$ is module amenable, D is inner so $\exists(\varphi, \psi) \in X' \times Y'$ such that $D = \delta_{(\varphi,\psi)}$ thus by corollary $D_1 = \delta_\varphi$, $D_2 = \delta_\psi$, this means that A, B are module amenable. For the converse let A, B are module amenable. Let $X \times Y$ be a Banach $A \times_\theta B$-$\mathfrak{U}$-module and $D \in Z_u(A \times_\theta B, X' \times Y')$.
Define $q_X : X \times Y \longrightarrow X$, by $q_X(x, y) = x$, and $q_Y : X \times Y \longrightarrow Y$, by $q_Y(x, y) = y$. It is easy to check that X is a Banach A-$\mathfrak{U}$-module and Y is B-$\mathfrak{U}$-module with module multiplications
$X \times A \longrightarrow X$ defined by $x.a = q_X($ (x , 0).(a , 0)),
$A \times X \longrightarrow X$ defined by $a.x = q_X($ (a , 0).(x , 0)),
$Y \times B \longrightarrow Y$ defined by $y.b = q_Y($ (0 , y).(0 , b)),
$B \times Y \longrightarrow Y$ defined by $b.y = q_Y($ (0 , b).(0 , y)), and
$X \times \mathfrak{U} \longrightarrow X$ with $x \circ \alpha = q_X((x, 0) \centerdot \alpha)$, $\mathfrak{U} \times X \longrightarrow X$ with $\alpha.x = q_X(\alpha.(x, 0))$,
$Y \times \mathfrak{U} \longrightarrow Y$ with $y \nabla \alpha = q_Y((0, y) \centerdot \alpha)$, $\mathfrak{U} \times Y \longrightarrow Y$ with $\alpha \vartriangle y = q_Y(\alpha.(0, y))$ with compatible actions.
Now Since $D \in Z_u(A \times_\theta B, X' \times Y')$, by proposition 2.3, there are $D_1 \in Z_u(A, X')$, $D_3 \in Z_u(B, X')$, $D_2 \in Z_u(B, Y')$ and $R : A \longrightarrow Y'$, such that $D(a, b) = (D_1(a) + D_3(b), R(a) + D_2(b))$.
Since $D_1 \in Z_u(A, X')$ and A is module amenable so $\exists\varphi \in X'$ such that $D_1 = \delta_\varphi$, also since $D_2 \in Z_u(B, Y')$ and B is module amenable so $\exists\psi \in Y'$ such that $D_2 = \delta_\psi$. Since $D_1(\theta(b)1) = D_1(1) \odot \theta(b) + D_3(b).1$, then $D_3(b).1 = D_1(\theta(b)1)$ for all $b \in B$, so

$$D_3(b).1 = \delta_\varphi(\theta(b)1) = \varphi.(\theta(b)1) - (\theta(b)1).\varphi = (\varphi \odot \theta(b)).1 - (\theta(b) \odot \varphi).1 = 0.$$

Thus $D_3 = 0$. Also since $R(am) = 0$, let $m = 1$, so $R(a) = 0$, for all $a \in A$. Hence by corollarly $A \times_\theta B$ is module amenable. ■

***Example* 2.6** Let S be an amenable inverse semigroup with idempotent E, such that $l^1(S)$ be unital. Since $l^1(S)$ is $l^1(E)$-bimodule with the multiplication, right action and trivial left action by [1, theorem 3.1], $l^1(S)$ is module amenable if and only if S is amenable so $l^1(S) \times_\theta l^1(S)$ is module amenable if and only if $l^1(S)$ is module amenable if and only if S is amenable, $(\theta \in \sigma(l^1(S))$.

3. Module biprojectivity

From now on we use the following maps:
$P_A : A \longrightarrow M$, with $P_A(a) = (a, 0)$, $q_A : M \longrightarrow A$, as $q_A(a, b) = a$, $r_A : M \longrightarrow A$, $r_A(a, b) = a + \theta(b)1$, $S_B : B \longrightarrow M$, $S_B(b) = (-\theta(b)1, b)$ where 1 is the unit of A. and $P_B : B \longrightarrow M$, $b \longmapsto (0, b)$, $q_B : M \longrightarrow B$, $(a, b) \longmapsto b$, $S_B \otimes S_B(b \otimes n + I_2) = (-\theta(b)1, b) \otimes (-\theta(n)1, n) + I$, $P_A \otimes P_A(a \otimes m + I_1) = (a, 0) \otimes (m, 0) + I$, $q_B \otimes q_B((a, b) \otimes (m, n) + I) = b \otimes n + I_2$, where I, I_1 and I_2 are introduce bellow.

Let A and B are commutative Banach $\mathfrak{U}$-bimodules. Let I be the closed ideal of the projective tensor product $M\widehat{\otimes}M$ generated by elements of the form $\alpha.(a, b) \otimes (m, n) - (a, b) \otimes (m, n).\alpha$, $(\alpha \in \mathfrak{U}, (a, b), (m, n) \in M)$. And J be the closed ideal of M generated by $\pi_M(I)$. Let I_1 be the closed ideal of the projective tensor product $A\widehat{\otimes}A$ generated by elements of the form $\alpha \circ a \otimes m - a \otimes m.\alpha$ for $\alpha \in \mathfrak{U}$, $a, m \in A$, and I_2 be the closed ideal of the projective tensor product $B\widehat{\otimes}B$ generated by elements of the form $\alpha * b \otimes n - b \otimes n \star \alpha$ for $\alpha \in \mathfrak{U}$, $b, n \in B$ and j_1, j_2 be the closed ideals of A, B(respectively) generated by $\pi_A(I_1)$ and $\pi_B(I_2)$. Let A and B be commutative $\mathfrak{U}$-bimodules then

$$\begin{aligned}
&\pi_M\big(\alpha.(a, b) \otimes (m, n) - (a, b) \otimes (m, n).\alpha\big)\\
&= \pi_M\big((\alpha \circ a, \alpha * b) \otimes (m, n) - (a, b) \otimes (m.\alpha, n \star \alpha)\big)\\
&= \big((\alpha \circ a)m + \theta(b)\alpha \circ m + \theta(n)\alpha \circ a, (\alpha * b)n\big)\\
&- \big(a(m.\alpha) + \theta(b)(m.\alpha) + \theta(n)\alpha \circ a, b(n \star \alpha)\big) = (0, 0).
\end{aligned}$$

So $\pi_M(I) = \{(0, 0)\}$ thus $J = \{(0, 0)\}$. In a same way $j_1 = \{0\}$ and $j_2 = \{0\}$.

Proposition 3.1 For Banach $\mathfrak{U}$-bimodule $M = A \times_\theta B$, $M\widehat{\otimes}_u M$ is a commutative $\mathfrak{U}$-module.

Proof. Since $\alpha.(a, b) \otimes (m, n) - (a, b) \otimes (m, n).\alpha \in I$, so for $x = \sum_{i=1}^{n}(a_i, b_i) \otimes (m_i, n_i)$ we have $\alpha.x - x.\alpha \in I$ so $\alpha.x + I = x.\alpha + I$, that is $M\widehat{\otimes}_u M$ is always a commutative $\mathfrak{U}$-module. ■

Lemma 3.2 With the above notations, the following statements hold.
1) $(q_B \otimes q_B)(\alpha.(a, b) \otimes (m, n)) = \alpha.(q_B \otimes q_B)((a, b) \otimes (m, n))$
2) $\widetilde{\pi_M} \circ (P_A \otimes P_A) = P_A \circ \widetilde{\pi_A}$,
3) $\widetilde{\pi_B} \circ (q_B \otimes q_B) = q_B \circ \widetilde{\pi_M}$,
4) $P_A \otimes P_A(\alpha.a \otimes m + I_1) = \alpha.P_A \otimes P_A(a \otimes m + I_1)$,
5) $q_B \otimes q_B((a, b).(m, n) \otimes (c, d)) = b.q_B \otimes q_B((m, n) \otimes (c, d))$,
when A is unital with unit 1 then
6) $\widetilde{\pi_M} \circ (S_B \otimes S_B) = S_B \circ \widetilde{\pi_B}$,
7) $(a, b).P_A \otimes P_A(m \otimes c + I_1) = P_A \otimes P_A((a + \theta(b)1)m \otimes c + I_1)$,
8) $P_A \otimes P_A(m \otimes c + I_1).(a, b) = P_A \otimes P_A(m \otimes c(a + \theta(b)1) + I_1)$,
9) $(a, b).(S_B \otimes S_B)(d \otimes n + I_2) = (S_B \otimes S_B)(bd \otimes n + I_2)$,
10) $(S_B \otimes S_B)(d \otimes n + I_2).(a, b) = (S_B \otimes S_B)(d \otimes nb + I_2)$,
11) $S_B \otimes S_B(\alpha.b \otimes n + I_2) = \alpha.S_B \otimes S_B(b \otimes n + I_2)$.

Proof. We only prove 2, 9. The others are in a similar way.

$$\begin{aligned} 2)\widetilde{\pi_M} \circ (P_A \otimes P_A)(a \otimes m + I_1) &= \widetilde{\pi_M}((a, 0) \otimes (m, 0) + I) \\ &= (a, 0)(m, 0) = (am, 0) \\ &= P_A(am) = P_A(\widetilde{\pi_A}(a \otimes m + I_1)) \\ &= P_A \circ \widetilde{\pi_A}(a \otimes m + I_1) \end{aligned}$$

$$\begin{aligned} 6)(a, b).(S_B \otimes S_B)(d \otimes n + I_2) &= (a, b).((-\theta(d)1, d) \otimes (-\theta(n)1, n) + I) \\ &= (-\theta(d)a - \theta(b)\theta(d)1 + \theta(d)a, bd) \otimes (-\theta(n)1, n) + I \\ &= (-\theta(bd)1, bd) \otimes (-\theta(n)1, n) + I \\ &= (S_B \otimes S_B)(bd \otimes n + I_2) \end{aligned}$$

■

Theorem 3.3 Let A and B be commutative $\mathfrak{U}$-bimodules. Then module biprojectivity of $M = A \times_\theta B$ implies module biprojectivity of B.

Proof. Suppose that M be module biprojective and I, I_2,J and j_2, be as above. Since M is module biprojective, so for $\widetilde{\pi_M}$ there exist $\omega_M : M \longrightarrow M\widehat{\otimes}M$ such that $\widetilde{\pi_M} \circ \omega_M = Id_M$.

$$Consider\ \ B \overset{P_B}{\longrightarrow} M \overset{\omega_M}{\longrightarrow} M\widehat{\otimes}M \overset{q_B \otimes q_B}{\longrightarrow} B\widehat{\otimes}B.$$

Now set $\omega_B = (q_B \otimes q_B) \circ \omega_M \circ P_B$, then

$$\begin{aligned} \widetilde{\pi_B} \circ \omega_B(b) &= \widetilde{\pi_B} \circ (q_B \otimes q_B) \circ \omega_M \circ P_B(b) \\ &= q_B \circ \widetilde{\pi_M} \circ \omega_M(0, b) \\ &= q_B(0, b) = b. \end{aligned}$$

Also

$$\begin{aligned} \omega_B(\alpha * b) &= (q_B \otimes q_B) \circ \omega_M \circ P_B(\alpha * b) \\ &= (q_B \otimes q_B) \circ \omega_M(0, \alpha * b) \\ &= (q_B \otimes q_B) \circ \omega_M(\alpha.(0, b)) \\ &= (q_B \otimes q_B)(\alpha.\omega_M(0, b) \\ &= \alpha.(q_B \otimes q_B)(\omega_M(0, b) \qquad by\ \ (lemma, part1)) \\ &= \alpha.(q_B \otimes q_B) \circ \omega_M \circ P_B(b) \\ &= \alpha.\omega_B(b) \end{aligned}$$

And

$$\begin{aligned}\omega_B(bd) &= (q_B \otimes q_B)\circ \omega_M \circ P_B(bd)\\ &= (q_B \otimes q_B)\circ \omega_M(0,bd)\\ &= (q_B \otimes q_B)\circ \omega_M((0,b)(0,d))\\ &= (q_B \otimes q_B)((0,b)\omega_M(0,d))\\ &= b.\omega_B(d). \qquad by\ \ (lemma, part5)\end{aligned}$$

Similar for right side. So ω_B is a right inverse for $\widetilde{\pi_B} : B\widehat{\otimes}B \longrightarrow B/j_2 = B$ which is an B/j_2-$\mathfrak{U}$-module. Hence B is module biprojective. ∎

Theorem 3.4 Let A and B be commutative $\mathfrak{U}$-bimodules. If both A and B are module biprojective and A is unital, then $M = A \times_\theta B$ is module biprojective.

Proof. Since A, B are module biprojective so there exists ω_A and ω_B such that $\widetilde{\pi_A}\circ\omega_A = Id_A$ and $\widetilde{\pi_B}\circ\omega_B = Id_B$. Define

$$\omega_M = P_A \otimes P_A \circ \omega_A \circ r_A + S_B \otimes S_B \circ \omega_B \circ q_B \ \ since$$

.

$$\begin{aligned}\widetilde{\pi_M}\circ\omega_M(a,b) &= P_A \otimes P_A \circ \omega_A \circ r_A(a,b) + S_B \otimes S_B \circ \omega_B \circ q_B(a,b)\\ &= \widetilde{\pi_M}\circ P_A \otimes P_A \circ \omega_A(a+\theta(b)1) + \widetilde{\pi_M}\circ S_B \otimes S_B \circ \omega_B(b)\\ &= P_A \circ \widetilde{\pi_A}\circ\omega_A(a+\theta(b)1) + S_B \circ \widetilde{\pi_B}\circ \omega_B(b)\\ &= (a+\theta(b)1, 0) + (-\theta(b)1, b) = (a,b).\end{aligned}$$

Also

$$\begin{aligned}\omega_M(\alpha.(a,b)) &= \omega_M(\alpha\circ a, \alpha * b)\\ &= P_A \otimes P_A \circ \omega_A \circ r_A(\alpha\circ a, \alpha * b) + S_B \otimes S_B \circ \omega_B \circ q_B(\alpha\circ a, \alpha * b)\\ &= P_A \otimes P_A \circ \omega_A(\alpha\circ a + \theta(\alpha * b)1) + S_B \otimes S_B \circ \omega_B(\alpha * b)\\ &= P_A \otimes P_A \circ \omega_A(\alpha\circ a + \theta(b)\alpha\circ 1) + S_B \otimes S_B \circ \omega_B(\alpha * b)\\ &= \alpha.P_A \otimes P_A \circ \omega_A(a+\theta(b)1) + \alpha.S_B \otimes S_B \circ \omega_B(b)\\ &= \alpha.\omega_M(a,b).\end{aligned}$$

And

$$\begin{aligned}\omega_M((a,b)(m,n)) &= P_A \otimes P_A \circ \omega_A \circ r_A((a,b)(m,n)) + S_B \otimes S_B \circ \omega_B \circ q_B((a,b)(m,n))\\ &= P_A \otimes P_A \circ \omega_A((a+\theta(b)1)(m+\theta(n)1)) + S_B \otimes S_B \circ \omega_B(bn)\\ &= (a,b).P_A \otimes P_A \circ \omega_A(m+\theta(n)1) + (a,b).S_B \otimes S_B \circ \omega_B(n)\\ &= (a,b).(P_A \otimes P_A \circ \omega_A \circ r_A(m,n) + S_B \otimes S_B \circ \omega_B \circ q_B(m,n))\\ &= (a,b).\omega_M(m,n).\end{aligned}$$

Similarly for right actions. So $\omega_M : M/J = M \longrightarrow M\widehat{\otimes}M$ is a right inverse for $\widetilde{\pi_M} : M\widehat{\otimes}M \longrightarrow M/J = M$, thus M is module biprojective. ■

***Example* 3.5** Let $S = \mathbb{N}$ with product $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ as $(m, n) \longmapsto m \vee n = \max\{m, n\}$, then S is a countable, abelian inverse semigroup with the identity 1. Clearly $E_S = S$. This semigroup is denoted by $\mathbb{N}_\vee$. $l^1(\mathbb{N}_\vee)$ is unital with unit δ_1 and is module biprojective(as an $l^1(\mathbb{N}_\vee)$) thus $l^1(\mathbb{N}_\vee) \times_\theta l^1(\mathbb{N}_\vee)$, $(\theta \in \sigma(l^1(\mathbb{N}_\vee)))$is module biprojective.

Acknowledgement

This research was supported by Islamic Azad University Central Tehran Branch and the authors acknowledge it with thanks.

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