On optimal bidding in sequential procurement auctions

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ABSTRACT

We investigate the problem of optimal bidding for a firm that in each period procures items to meet a random demand by participating in a finite sequence of auctions. We develop a new model for a firm where its item valuation derives from the sale of the acquired items via their demand distribution, sale price, acquisition cost, salvage value and lost sales. We establish monotonicity properties for the value function and the optimal dynamic bid strategy and we present computations.

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1. Introduction

In this paper we study procurement auctions for firms that use sequential auctions for incremental procurement of items or services in order to meet a random demand at the end of the procurement period. Specifically, we develop a new model for a firm that in each period procures items by participating in auctions and then it sells the acquired items by the end of the period. There is a random demand with known distribution and any unsold items are salvaged. The objective of the firm is to have a bidding policy that maximizes the expected value of its profit over a fixed number of N auctions that take place within the period. In this new study the firm's valuations derive from the sale of the acquired items via their demand distribution, sale price, acquisition cost, salvage value and lost sales penalties. One can think of this model as the "Newsvendor Model" where items are acquired incrementally in procurement auctions. To our knowledge this is the first study of a newsvendor model where procurement is done by bidding in auctions.

Incremental procurement of items to satisfy random demand is one of the ways companies deal with very high demand scenarios, examples are found in holiday sales and in the retail sector. Such auctions have become a crucial part of the procurement process for many firms and they are viewed as an efficient mechanism of setting the price for a good or a service. Other examples where

such a framework is used for procurement is in auctions for agricultural commodities such as cattle, wheat, rice, coffee, etc. (cf., [3,17,18,9]). In some instances items such as units of electricity, (cf., [5,6]), and flowers (cf., [16,7]) are also acquired in a similar fashion.

An introduction to the study of multiple auctions can be found in the book by Milgrom [12]. Klemperer [10] has an introduction to the theory of auctions which includes references to multiple object auctions. The first paper in sequential auctions was by Ortega-Reichert [15]. He considered a two-person two-auction scenario. He proved that the optimal bids in the first auction for the two auction scenario would be less than a corresponding single auction scenario, for the asymmetric Nash equilibrium sequence of pure bidding strategies. Rothkopf and Oren [14] characterized a sequential auction as a multi-stage control process where the state represented the competitor's strategy and state transitions represented the competitors' reaction to a strategy used by the bidder. The control was the bidders strategy. They showed that a bidder should bid less aggressively in initial auctions if he believes this will lead his competitors to bid less aggressively in future auctions. Milgrom and Weber [13] study a model where k identical objects are sold to n bidders where every bidder can only acquire one item. Prior to the auctions each bidder receives a private signal and after each auction the seller announces the winning bid. They proved that in this case a bidder does not deceive and change his bid. Further, Weber [19] considers models of sequential auctions for which the winning bid price is a martingale that on average does not increase or decrease with each auction. Engelbrecht-Wiggans [4] showed that in the case of auctions of "stochastically equivalent" objects where bidders have independent valuations,

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the winning bid price decreases with each auction if the individual bidder valuations are bounded distributions. In [2] the authors give an overview of current state research in procurement auctions and also study a model of procurement auctions under general conditions. In [11] the authors study a problem where a seller has multiple items (k) to sell and offers to buy individual items arrive sequentially. The seller either "marks" the item or rejects. Marking a buyer's offer means that the seller will sell one of the items to that buyer. Once k items have been marked the seller sells the items to the buyers at the same cost which is the minimum of the k marked offers. There is also a cost c associated with rejecting an offer.

Most papers on sequential auctions are from the auctioneer's point of view. There are very few papers which deal with the "bidder's" perspective. Also, most of the papers in the field of multiple auctions restrict the demand of each bidder to one. Herein, we consider the bidder's point of view when the bidder has a multi-unit demand.

To benchmark our model we compare the expected total reward for the newsvendor that follows this auction procedure (the "auction-newsvendor") with that of a traditional newsvendor (the "classic-newsvendor") that can acquire all items at a fixed cost. It is shown that as the item acquisition cost c increases the optimal expected total reward of the classic-newsvendor model decreases and at some value of c the expected total reward of the classic-newsvendor will become smaller than that of the auction-newsvendor and it will stay smaller. Knowing this "cutoff acquisition cost" is important in applications.

The paper is organized as follows. In Section 2 we define the problem as a Markovian decision process. In Section 3 under sensible assumptions, it is shown that the optimal bid is a decreasing function of the number of remaining auctions, an increasing function of the number of other auction participants ("opponents") and a decreasing function of the number of items at hand. We also establish monotonicity properties for the value function and present computations. In addition to their theory value, these monotonicity properties can be used to obtain efficient algorithms for calculating optimal biding policies.

In Section 4, we compare our auction model with the newsvendor problem with salvage. We present computational results to show that the expected total reward of our model becomes higher than the traditional newsvendor model as the cost price in the newsvendor model increases. In the final Section 5, we present concluding remarks.

2. Problem formulation

In each time period, the buyer procures items through a sequence of N auctions which he then sells. The buyer's demand D is a random variable with a known discrete distribution. Let $f_D(d) = P(D = d)$, $F_D(d) = P(D \le d)$, and $\bar{F}_D(d) = 1 - F_D(d)$. The sales price r is assumed to be known (where when the sales price is a random variable R with a known distribution we take $r = E(R) < \infty$). As in a standard newsvendor model, excess demand is lost with a penalty and unsold items at the end of the period have the same salvage value. Let $\delta(x)$ denote the penalty associated with x units of excess demand and let s be the unit salvage value. We assume that $s < r$.

In each auction the number of opposing bidders (*opponents*) m is known and each of the $1 + m$ bidders submits a sealed bid. At the end of each auction the winning bid is announced and one of the highest bidders wins the auction. The objective of the buyer is to maximize his expected profit.

It is assumed that the set of all bids available (to the buyer and all opponents) is a finite set $\{a_0, a_1, \ldots, a_p\}$ where $a_0 < a_1 < \cdots < a_p$. For simplicity we will use the same symbol a to represent both the bid price and the action of the buyer bidding amount a. We assume that $a_0 = 0$ represents the action of not bidding.

Let $p_m(a)$ denote the known probability that the buyer wins an auction when his bid is a and there are m opponents present, where $p_m(a_0) = 0$. For convenience let $\bar{p}_m(a) = 1 - p_m(a)$.

Let Z_n be the number of opponents participating in the nth auction. It is assumed that Z_n for $n = 1, 2, \ldots, N$ is a discrete time Markov chain with transition probabilities:

$$q_{mm'}(n) = P(Z_{n+1} = m' | Z_n = m),$$

and an initial distribution which is denoted for simplicity by:

$$q_m(1) = P(Z_1 = m).$$

It is assumed that whenever there is a tie in an auction involving the buyer he loses. This assumption is made to simplify the exposition. Other tie breaking procedures like deciding the winner randomly will not change the analysis but would complicate the exposition.

We model this problem as a Markov decision process below.

1. The set $\mathcal{X} = \{(n, m, x), n = 0, \ldots, N, m = 1, \ldots, x = 0, 1, \ldots\}$, is the state space, where n represents the number of remaining auctions, m represents the number of bidders participating in the current auction, $x \ge 0$ represents the inventory level at the beginning of the current, $(N - n)$th, auction. Note that:
   (i) If $n = 0$ then $m = 0$.
   (ii) State $(0, 0, x)$ represents the state of the system when all auctions are over.
   (iii) Possible states prior to the start of the N auctions are of the form $(N, m, 0)$, for all $m = 1, \ldots$.
2. In any state (n, m, x) the following action sets $A(n, m, x)$ are available.
   (i) $A(0, 0, x) = \{a_0\}$.
   (ii) $A(n, m, x) = \{a_0, \ldots, a_p\}$ for $n > 0$.
3. When an action $a \in A(n, m, x)$ is taken in state (n, m, x) the following transitions are possible.
   (i) If $n = 0$, then starting from state $(0, 0, x)$ the next state is $(0, 0, x)$ with probability 1.
   (ii) If $n > 0$ then depending on whether or not the buyer wins the current auction the next state is $(n - 1, m', x + 1)$ with probability $p_m(a) q_{mm'}(N - n)$ or state $(n - 1, m', x)$ with $\bar{p}_m(a) q_{mm'}(N - n)$.
4. When an action $a \in A(n, m, x)$ is taken in state (n, m, x) the expected reward $r_a(n, m, x)$ is as follows.
   (i) $r_a(0, 0, x) = \sum_{d=0}^{x} (rd + s(x - d)) f_D(d) + \sum_{d=x+1}^{\infty} (rx - \delta(d - x)) f_D(d)$
   (ii) $r_a(n, m, x) = -a\, p_m(a)$ if $n > 0$.

Let $a_{n,m,x}^*$ denote the optimal action in the state (n, m, x). Let $v(n, m, x)$ denote the value function in state (n, m, x) and $w(n, m, x; a)$ denote the expected future reward when action a is taken in state (n, m, x) and an optimal policy is followed thereafter. Note that $v(n, m, x) = w(n, m, x; a_{n,m,x}^*)$.

For $n \ge 1$, let

$$u(n, m, x) = E(v(n, Z_{N-n}, x) | Z_{N-(n+1)} = m)$$

$$= \sum_{m'=1}^{\infty} q_{mm'}(N - n) v(n - 1, m', x).$$

The dynamic programming equations are

$$v(n, m, x) = \max_{a \in A} \{w(n, m, x; a)\} \tag{1}$$

where for $n \ge 1$,

$$w(n, m, x; a) = r_a(n, m, x) + p_m(a) u(n - 1, m, x + 1) + \bar{p}_m(a) u(n - 1, m, x)$$

and

$$w(0, 0, x; a) = r_a(0, 0, x).$$

3. The structure of the optimal bidding policy

In this section we derive structural properties of the optimal bidding policy under the following assumptions.

Assumption A. For any fixed m, $p_m(a)$ is an increasing function of a.

Assumption B. For any fixed a, $p_m(a)$ is a decreasing function of m.

Assumption C. There exists a function G with $\sum_{i=-\infty}^{\infty} G(i) = 1$ such that:

$$q_{mm'}(n) = \begin{cases} G(m' - m) & \text{if } m' > 1, \\ \sum_{k=m-1}^{\infty} G(k) & \text{if } m' = 1. \end{cases} \tag{2}$$

Assumption D. The penalty function $\delta(x)$ is an increasing convex function of x and $\delta(x) = 0$ if $x \leq 0$.

We first state and prove the following.

Lemma 3.1. *The expected reward function in state* $(0, 0, x)$, r_a $(0, 0, x)$ *is an increasing function of x i.e.*

$$r_a(0, 0, x) \leq r_a(0, 0, x + 1). \tag{3}$$

Proof. The proof is evident from the fact that the difference $r_a(0, 0, x + 1) - r_a(0, 0, x)$ can be simplified to

$$\sum_{x+1}^{\infty} rp_D(d) + \sum_{d=0}^{x} sp_D(d) + \sum_{d=x+1}^{\infty} (\delta(d - x) - \delta(d - x - 1))p_D(d),$$

which is non-negative because $\delta(x)$ is an increasing function. □

The following Theorems 3.1 and 3.2, contain the main results of the paper.

Theorem 3.1. *Under Assumptions* A–C *the following relationships hold.*

$$v(n, m, x) \leq v(n, m, x + 1) \quad \forall n \geq 0, \tag{4}$$
$$v(n, m, x) \leq v(n + 1, m, x) \quad \forall n \geq 0, \tag{5}$$
$$v(n, m, x) \geq v(n, m + 1, x) \quad \forall n > 0. \tag{6}$$

Proof. The proof is by induction on n. We first show that Ineq. (4) holds. For $n = 0$ the inequality $v(0, 0, x) \leq v(0, 0, x + 1)$ is true from the definition of $v(0, 0, x)$ and Lemma 3.1. For $n = 1$ we show that $v(1, m, x) \leq v(1, m, x + 1)$ by contradiction. If we assume the contrary, i.e., $v(1, m, x) > v(1, m, x+1)$, it implies that $v(1, m, x) > w(1, m, x + 1; a^*_{1,m,x})$. The last inequality simplifies to

$$p_m(a^*_{1,m,x})(v(0, 0, x + 2) - v(0, 0, x + 1))$$
$$+ \bar{p}_m(a^*_{1,m,x})(v(0, 0, x + 1) - v(0, 0, x)) < 0,$$

which contradicts the previous step of the induction. Now, we assume that Ineq. (4) is true for $n - 1$ and show that it holds for n. The induction assumption of Ineq. (4) implies that $u(n-1, m, x) \leq u(n-1, m, x)$. From this and the definition of $w(n, m, x; a)$ we can conclude that $w(n, m, x; a) \leq w(n, m, x + 1; a) \ \forall a$, which in turn implies the following:

$$v(n, m, x) = \max_{a \in A} w(n, m, x; a) \leq \max_{a \in A} w(n, m, x + 1; a)$$
$$= v(n, m, x + 1).$$

This completes the induction.

Next, we prove that Ineq. (5) holds. For $n = 0$ we show that the opposite inequality

$$v(0, 0, x) > v(1, m, x), \tag{7}$$

leads to a contradiction. Indeed, the dynamic programming equations Eq. (1) imply that

$$v(1, m, x) = p_m(a^*_{1,m,x})(v(0, 0, x + 1) + a^*_{1,m,x})$$
$$+ \bar{p}_m(a^*_{1,m,x})v(0, 0, x). \tag{8}$$

Thus, $v(1, m, x)$ is a convex combination of $v(0, 0, x)$ and $v(0, 0, x + 1) + a^*_{1,m,x}$. From this and Ineq. (7) it follows that

$$v(0, 0, x) > v(0, 0, x + 1) + a^*_{1,m,x}, \tag{9}$$

which in turn implies that $w(1, m, x; a_0) = v(0, 0, x) \geq v(1, m, x)$. The last inequality implies that $a^*_{1,m,x} = a_0$; this and Ineq. (9) imply that $v(0, 0, x) > v(0, 0, x + 1)$ which contradicts Ineq. (4), for $n = 0$.

The induction assumption in this case is $v(n - 2, m, x) \leq v(n-1, m, x)$. From the induction assumption and Assumption C it follows that $u(n - 2, m, x) \leq u(n - 1, m, x)$. From this fact and the definition of $w(n, m, x; a)$ we can conclude that $w(n-1, m, x; a) \leq w(n, m, x; a) \ \forall a$. The last inequality implies that

$$v(n, m, x) = \max_{a \in A} w(n, m, x; a) \leq \max_{a \in A} w(n - 1, m, x; a)$$
$$= v(n - 1, m, x),$$

which completes the induction.

We now show that Ineq. (6) holds. For $n = 1$ we show that the opposite inequality $v(1, m, x) < v(1, m + 1, x)$ leads to a contradiction. The last inequality implies that $w(1, m, x; a^*_{1,m+1,x}) < v(1, m + 1, x)$. From this and Ineq. (7) it follows that

$$v(0, 0, x) > v(0, 0, x + 1) + a^*_{1,m,x}. \tag{10}$$

The above implies that $v(0, 0, x) > v(0, 0, x + 1)$ which contradicts Ineq. (4), for $n = 0$.

We complete the induction of Ineq. (6) along similar lines. We assume that it holds for $n - 1$ and show that it holds for n. As above we show that the opposite inequality $v(n, m, x) < v(n, m + 1, x)$ leads to a contradiction. The last inequality implies that $w(n, m, x; a^*_{n,m+1,x}) < v(n, m+1, x)$. Simplifying the previous inequality leads to

$$u(n, m, x + 1) + a^*_{n,m,x+1} < u(n, m, x). \tag{11}$$

From the definition of $v(n+1, m, x)$ the last inequality implies that

$$w(n, m, x; a_0) = v(n, m, x) \geq v(n + 1, m, x). \tag{12}$$

The last inequality implies that $a^*_{n,m,x+1} = a_0$. This and Ineq. (11) imply that $u(n, m, x+1) < u(n, m, x)$, which is a contradiction. □

Theorem 3.2. *Under Assumptions* A–C *the following relationships hold.*

$$a^*_{n,m,x} \geq a^*_{n,m,x+1} \quad \text{for } n \geq 0, \tag{13}$$
$$a^*_{n,m,x} \geq a^*_{n+1,m,x} \quad \text{for } n > 0, \tag{14}$$
$$a^*_{n,m,x} \leq a^*_{n,m+1,x} \quad \text{for } n \geq 0. \tag{15}$$

Proof. The proof is by induction on n. First we prove Ineq. (13) are true. For $n = 0$ the inequality is obviously true because $a^*_{0,0,x} = a_0$ for all x.

To complete the induction of Ineq. (13) we assume that $a^*_{n-1,m,x} \geq a^*_{n-1,m,x+1}$ and prove that $a^*_{n,m,x} \geq a^*_{n,m,x+1}$. To prove the last inequality we assume that $a^*_{n,m,x+1} > a^*_{n,m,x}$ and show that this produces a contradiction. From the definitions of $v(n, m, x)$ and $w(n, m, x; a)$ we have $v(n, m, x) < w(n, m, x; a^*_{n,m,x+1})$ and

$v(n, m, x+1) < w(n, m, x+1; a^*_{n,m,x})$. Simplifying and combining the results of the last two inequalities we obtain

$$u(n-1, m, x+1) - u(n-1, m, x)$$
$$> u(n-1, m, x+2) - u(n-1, m, x+1).$$

This can be rewritten as

$$\sum_{m'} q_{mm'}[2v(n-1, m', x+1) - v(n-1, m', x)$$
$$- v(n-1, m', x+2)] > 0.$$

The above inequality implies the following:

$$\sum_{m'} q_{mm'}[v(n-1, m', x+1) - w(n-1, m', x; a^*_{n-1,m',x+1})]$$
$$> \sum_{m'} q_{mm'}[w(n-1, m', x+2; a^*_{n-1,m',x+1})$$
$$- v(n-1, m', x+1)]. \tag{16}$$

Notice that the induction assumption implies the following for all $m \geq 1$

$$u(n-2, m, x) + u(n-2, m, x+2) > 2u(n-2, m, x+1). \tag{17}$$

Simplifying Ineq. (16) using Assumption C and Ineq. (17) leads to the inequality

$$\sum_{m'} q_{mm'}[u(n-2, m', x+1) - u(n-2, m', x+2)]$$
$$> \sum_{m'} q_{mm'}[u(n-2, m', x+1) - u(n-2, m', x+2)]$$

which is a contradiction because both sides of the strict inequality are identical.

We next show that Ineq. (14) holds. For $n = 1$ we need to prove that $a^*_{1,m,x} \geq a^*_{2,m,x}$. To prove this we assume $a^*_{1,m,x} < a^*_{2,m,x}$ and show that it produces a contradiction. We know that $v(1, m, x) > w(1, m, x; a^*_{2,m,x})$ and $v(2, m, x) > w(2, m, x; a^*_{1,m,x})$. Simplifying the inequalities and combining the results leads to $v(0, 0, x+1) - v(0, 0, x) < u(1, m, x+1) - u(1, m, x)$ or equivalently

$$v(0, 0, x+1) - v(0, 0, x)$$
$$< \sum_{m'} q_{mm'}[v(1, m', x+1) - v(1, m', x)].$$

The above inequality implies that

$$v(0, 0, x+1) - v(0, 0, x)$$
$$< \sum_{m'} q_{mm'}[v(1, m', x+1) - w(1, m', x; a^*_{1,m',x+1})].$$

The above inequality simplifies to

$$v(0, 0, x+1) - v(0, 0, x) < v(0, 0, x+1) - v(0, 0, x)$$

which is a contradiction. For the next step in the induction we assume that $a^*_{n-1,m,x} \leq a^*_{n,m,x}$ and prove that $a^*_{n,m,x+1} \leq a^*_{n+1,m,x}$. To prove the last inequality we assume that $a^*_{n,m,x+1} > a^*_{n+1,m,x}$ and show that it produces a contradiction. From the definitions of $v(n, m, x)$ and $w(n, m, x; a)$ we have $v(n, m, x) > w(n, m, x; a^*_{n+1,m,x})$ and $v(n+1, m, x) > w(n+1, m, x; a^*_{n,m,x})$. Simplifying and combining the results of the last two inequalities we obtain

$$u(n, m, x+1) - u(n, m, x) < u(n-1, m, x+1)$$
$$- u(n-1, m, x).$$

The above inequality is equivalent to the following inequality

$$\sum_{m'} q_{mm'}[v(n, m', x+1) - v(n, m', x)]$$
$$< \sum_{m'} q_{mm'}[v(n-1, m', x+1) - v(n-1, m', x)].$$

The above inequality implies the following:

$$\sum_{m'} q_{mm'}[w(n, m', x+1; a^*_1) - v(n, m', x)]$$
$$< \sum_{m'} q_{mm'}[v(n-1, m', x+1) - w(n-1, m', x; a^*_2)]$$

where $a^*_1 = a^*_{n,m',x}$ and $a^*_2 = a^*_{n-1,m',x+1}$.

Simplifying the above inequality using Assumption C and the induction assumption leads to the inequality

$$\sum_{m'} q_{mm'}[u(n-2, m', x+1) - u(n-2, m', x)]$$
$$< \sum_{m'} q_{mm'}[u(n-2, m', x+1) - u(n-2, m', x)],$$

which is a contradiction because both sides of the strict inequality are identical.

We next prove Ineq. (15). For $n = 0$ the inequality is obviously true because $a^*_{0,0,x} = a_0$ for all x. For $n = 1$, we assume that $a^*_{1,m,x} > a^*_{1,m+1,x}$ and show that it leads to a contradiction. The last inequality implies that $v(1, m, x) > w(1, m, x; a^*_{1,m+1,x})$ and $v(1, m+1, x) > w(1, m+1, x; a^*_{1,m,x})$. Simplifying the inequalities we obtain $v(0, 0, x+1) - v(0, 0, x) > T_{1,m}$ and $v(0, 0, x+1) - v(0, 0, x) < T_{1,m+1}$ where

$$T_{n,m} = \frac{a^*_{n,m,x} p_m(a^*_{n,m,x}) - a^*_{n,m+1,x} p_m(a^*_{n,m+1,x})}{p_m(a^*_{n,m,x}) - p_m(a^*_{n,m+1,x})}.$$

This is a contradiction because we have $T_{1,m} \geq T_{1,m+1}$, from Assumption B together with $a^*_{1,m,x} > a^*_{1,m+1,x}$.

To complete the induction of Ineq. (15) assume it holds for $n-1$. To prove it holds for n, assume that $a^*_{n,m+1,x} < a^*_{n,m,x}$ and show that this produces a contradiction. Since $a^*_{n,m,x}$ is the optimal action in state (n, m, x) we have that $v(n, m, x) > w(n, m, x, a^*_{n,m+1,x})$. This simplifies to:

$$u(n-1, m, x+1) - u(n-1, m, x) > T_{n,m}. \tag{18}$$

Similarly in state $(n, m+1, x)$ we have $v(n, m+1, x) > w(n, m+1, x, a^*_{n,m,x})$ which simplifies to

$$u(n-1, m+1, x+1) - u(n-1, m+1, x) < T_{n,m+1}. \tag{19}$$

We next show that the following inequality is true

$$T_{n,m+1} < T_{n,m}. \tag{20}$$

Indeed, from the definitions of $T_{n,m}$ and $T_{n,m+1}$ Ineq. (20) simplifies to the following inequality

$$p_m(a^*_{n,m,x}) p_m(a^*_{n,m-1,x})$$
$$< p_{m-1}(a^*_{n,m,x}) p_{m-1}(a^*_{n,m-1,x}),$$

which is true under Assumption B and the assumption $a^*_{n,m,x} > a^*_{n,m+1,x}$.

Now, inequalities (18)–(20) together imply that

$$u(n-1, m+1, x+1) - u(n-1, m+1, x)$$
$$< u(n-1, m, x+1) - u(n-1, m, x).$$

This implies that

$$\sum_{i=-m+1}^{\infty} \{G(i)[v(n-1, m+i+1, x+1) \\ - v(n-1, m+i+1, x)]\} \\ + G(-m)(v(n-1, 1, x+1) - v(n-1, 1, x)) \\ < \sum_{i=-m+1}^{\infty} \{G(i)[v(n-1, m+i, x+1) - v(n-1, m+i, x)]\}.$$

From Theorem 3.1 we have $v(n-1, 1, x+1) - v(n-1, 1, x) \geq 0$. Hence, the above inequality implies that

$$\sum_{m'=1}^{\infty} q_{mm'}[v(n-1, m'+1, x+1) - v(n-1, m'+1, x)] \\ < \sum_{m'=1}^{\infty} q_{mm'}[v(n-1, m', x+1) - v(n-1, m', x)].$$

From the above inequality we obtain the following

$$\sum_{m'=1}^{\infty} q_{mm'}[w(n-1, m'+1, x+1; a_1^*) - v(n-1, m', x+1)] \\ < \sum_{m'=1}^{\infty} q_{mm'}[v(n-1, m'+1, x) - w(n-1, m', x; a_2^*)], \quad (21)$$

where $a_1^* = a_{n-1,m',x+1}^*$ and $a_2^* = a_{n-1,m'+1,x}^*$.

Simplifying the induction assumption, as above, we obtain the following

$$u(n-2, m+1, x+1) - u(n-2, m, x+1) \\ > u(n-2, m+1, x) - u(n-2, m, x). \quad (22)$$

From Assumption B we have $p_m(a) \geq p_{m+1}(a)$, so, let $p_m(a_1^*) = p_{m+1}(a_1^*) + \epsilon_1$ and $p_m(a_2^*) = p_{m+1}(a_2^*) + \epsilon_2$, for some non-negative ϵ_1, ϵ_2.

Simplifying Ineq. (21) using Assumption C and Ineq. (22) leads to the inequality

$$\sum_{m'=1}^{\infty} q_{mm'}[u(n-2, m'+1, x+1) - u(n-2, m', x+1)] \\ + \epsilon_1[-a_1^* + u(n-2, m'+1, x+1) - u(n-2, m'+1, x)] \\ + \epsilon_2[-a_2^* + u(n-2, m', x+2) - u(n-2, m', x+1)] \\ < \sum_{m'=1}^{\infty} q_{mm'}[u(n-2, m'+1, x+1) - u(n-2, m', x+1)].$$

From the proof of Ineq. (6) we know that the terms multiplying ϵ_1 and ϵ_2 are positive. Hence the above inequality implies

$$\sum_{m'=1}^{\infty} q_{mm'}[u(n-2, m'+1, x+1) - u(n-2, m', x+1)] \\ < \sum_{m'=1}^{\infty} q_{mm'}[u(n-2, m'+1, x+1) - u(n-2, m', x+1)],$$

which is a contradiction because both sides of the strict inequality are identical. □

4. Comparison with traditional newsvendor model

In this section we compare the expected total reward of the newsvendor auction model with that of the traditional newsvendor model when in the latter the acquisition cost c changes. As the value of c increases the optimal order quantity
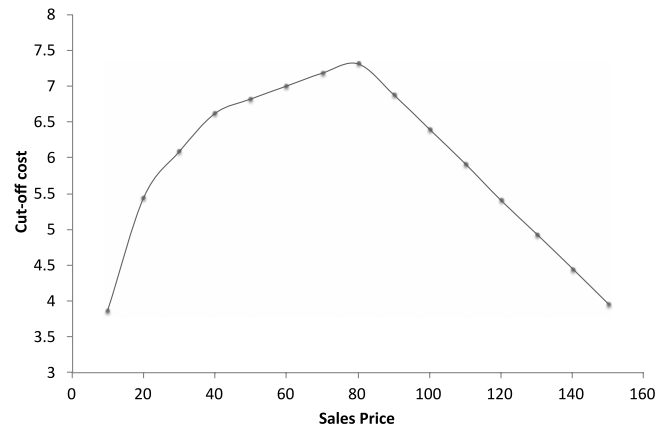

Fig. 1. Cutoff cost vs the sales price.

of the traditional newsvendor model will decrease and with it the expected total reward will also decrease. At some value of c the expected total reward of the classic-newsvendor will become smaller than that of the auction-newsvendor and it will stay smaller. We call this the cutoff price. Next we present a graph detailing the above results.

In Fig. 1 we present a graph of the values of the cost at which the total expected reward of the newsvendor auction models and traditional newsvendor model are the same (*i.e.*, the cutoff price), for increasing values of the sales price r. For both the newsvendor auction model and the traditional newsvendor the salvage value is chosen to be a fifth of the expected sales price *i.e.*, $s = r/5$, the demand distribution is $f_D(d) = 1/9$, for $d = \{0, 1, \ldots, 8\}$ and the penalty for lost sales $\delta(x) = x$. For the auction model we assume $N = 10$ and $A = \{1 \cdots 10\}$. The number of opponents in each auction is assumed to be four, *i.e.*, $m = 4$. We also assume that the probability of winning $p_4(a) = 1/10$.

The graph presented in Fig. 1 is that of a concave function which becomes linear as the value of r increases. This is apparent from the structure of the newsvendor model because the total expected reward is a convex decreasing function of the cost of acquiring the item and a linear increasing function of the reward. Further, as the value of r increases the cutoff point curve becomes linear because the expected revenue dominates the expected cost.

5. Discussion

This model can be extended in several ways and we are currently studying the following cases:

(i) One could consider the multi-period or an infinite horizon with the objective of maximization of expected discounted profit over the horizon. The dynamic programming equations in this case would be as follows:

$$v(n, m, x) = \max_{a \in A} \{w(n, m, x; a)\} \quad (23)$$

where for $n \geq 1$

$$w(n, m, x; a) = r_a(n, m, x) + p_m(a)u(n-1, m, x+1) \\ + \bar{p}_m(a)u(n-1, m, x)$$

and

$$w(0, 0, x; a) = r_a(0, 0, x) + \beta \sum_m q_m(1)v(N, m, 0),$$

where β is the discount factor.

(ii) Items can be sold after each auction.

(iii) The condition that the probability $p_m(a)$ of winning, for fixed m and a, is constant through all the auctions can be relaxed.

(iv) Further, the case in which the probability of winning $p_m(a)$ is not known but it can be estimated during auctions using

methods such those used by Burnetas and Katehakis [1] and Katehakis and Robbins [8].

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