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Characterization of Set of k-g-Inverses

P. Jenita

Department of Mathematics
KPR Institute of Engineering and Technology
KPR knowledge city, Arasur, Coimbatore, India

A. R. Meenakshi

Department of Mathematics
Karpagam college of Engineering
Coimbatore - 641 032, India

Abstract

In this paper, we have obtained the characterization of the set of all k-g-inverses of a fuzzy matrix and characterized the set of various k-g-inverses associated with a k-regular fuzzy matrix.

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1. Introduction

A matrix over $\mathcal{F}$=[0,1] is called a fuzzy matrix with operations (+,•) defined as a+b=max{a,b} and a•b=min{a,b}for all a, b$\in\mathcal{F}$. Let $\mathcal{F}_n$ be the set of all n×n fuzzy matrices over $\mathcal{F}$. A^T denotes the transpose of A. If a solution exists for the matrix equation AXA=A, then A sis called a regular fuzzy matrix and such a solution called a generalized (g^-) inverse of A and is denoted as A^- [2]. A{1} denotes the set of all g-inverses of a regular matrix A. Recently, Meenakshi and Jenita [4] have introduced the concept of k-regular fuzzy matrix analogous to that of generalized inverse of a complex matrix [1] and as a generalization of a regular fuzzy matrix [2,3].

Definition 1.1[4]:

A matrix $A \in \mathcal{F}_n$, is said to be right k-regular if there exists a matrix $X \in \mathcal{F}_n$ such that $A^kXA = A^k$, for some positive integer k. X is called a right k-g-inverse of A.
Let $A\{1_r^k\} = \{X / A^kXA = A^k\}$.

Definition 1.2[4]:

A matrix $A \in \mathcal{F}_n$, is said to be left k-regular if there exists a matrix $Y \in \mathcal{F}_n$ such that $AYA^k = A^k$, for some positive integer k. Y is called a left k-g-inverse of A.
Let $A\{1_\ell^k\} = \{Y / AYA^k = A^k\}$.

In general, right k-regular is different from left k-regular. Hence a right k-g-inverse need not be a left k-g-inverse [4]. Hence forth we call a right k-regular (or) left k-regular matrix as a k-regular matrix. Let $A\{1^k\} = A\{1_r^k\} \cup A\{1_\ell^k\}$.

Definition 1.3[5]:

A matrix $A \in \mathcal{F}_n$, is said to have a $\{3^k\}$ inverse if there exists a matrix $X \in \mathcal{F}_n$ such that $(A^kX)^T = A^kX$, for some positive integer k. X is called the $\{3^k\}$ inverse of A.
Let $A\{3^k\} = \{X / (A^kX)^T = A^kX\}$.

Definition 1.4[5]:

A matrix $A \in \mathcal{F}_n$, is said to have a $\{4^k\}$ inverse if there exists a matrix $X \in \mathcal{F}_n$ such that
$(XA^k)^T = XA^k$, for some positive integer k. X is called the $\{4^k\}$ inverse of A.
Let $A\{4^k\} = \{X / (XA^k)^T = XA^k\}$.

Theorem 1.1[5]:

For $A \in \mathcal{F}_n$ and for any $G \in \mathcal{F}_n$, if $A^kX = A^kG$, where X is a $\{1_r^k, 3^k\}$ inverse of A then, G is a $\{1_r^k, 3^k\}$ inverse of A.

Theorem 1.2[5]:

For $A \in \mathcal{F}_n$, X is a $\{1_r^k, 3^k\}$ inverse of A and G is a $\{1_\ell^k, 3\}$ inverse of A then, $A^kX = A^kG$.

Theorem 1.3[5]:

For $A \in \mathcal{F}_n$ and for any $G \in \mathcal{F}_n$, if $XA^k = GA^k$, where X is a $\{1_\ell^k, 4^k\}$ inverse of A then, G is a $\{1_\ell^k, 4^k\}$ inverse of A.

Theorem 1.4[5]:

For $A\in\mathcal{F}_n$, X is a $\{1_\ell^k, 4^k\}$ inverse of A and G is a $\{1_r^k, 4\}$ inverse of A then, $XA^k=GA^k$,

In particular for k=1, Theorems (1.1) to (1.4) reduces to the following:

Theorem 1.5[3]:

For $A\in\mathcal{F}_{mn}$, the set A{1,3} consists of all solutions for X of AX=AG, where G is a {1,3} inverse of A and the set A{1,4} consists of all solutions for X of XA=GA, where G is a {1,4} inverse of A.

2. Characterization of set of k-g-inverses

Lemma 2.1:

For $A\in\mathcal{F}_n$, if G and G^* are right k-g-inverses of A such that $G^*\geq G$, then G+H is a right k-g-inverse of A for some $H\in\mathcal{F}_n$ such that $G^*\geq G+H\geq G$.

Proof:

Since G and G^* are right k-g-inverses of A with $G^*\geq G$, let $G^*-G=H$. Then $G^*\geq H$ and $G^*\geq G+H\geq G$ ____________(2.1).

Pre multiplying by A^k and post multiplying by A in Equation (2.1), we get $A^kG^*A\geq A^k(G+H)A\geq A^kGA\Rightarrow A^k\geq A^k(G+H)A\geq A^k \Rightarrow A^k=A^k(G+H)A$. Thus G+H is a k-g-inverse of A.

Lemma 2.2:

For $A\in\mathcal{F}_n$, if G and G^* are left k-g-inverses of A such that $G^*\geq G$, then G+K is a left k-g-inverse of A for some $K\in\mathcal{F}_n$ such that $G^*\geq G+K\geq G$.

Proof:

Proof is similar to Lemma (2.1) and hence omitted.

Theorem 2.1:

Let $A\in\mathcal{F}_n$ and G be a particular right k-g-inverse of A.

Then $A_G\{1_r^k\} = \{G + H/ \text{ for all } H\in\mathcal{F}_n \text{ such that } A^k \geq A^kHA\}$ ______(2.2)

is the set of all right k-g-inverses of A dominating G.

Proof:

Let $\wp$ denote the set on the R.H.S of (2.2).

Suppose $G^* \in A_G\{1_r^k\}$, then $G^*\geq G$. Let $G^*-G=H$.

By Lemma (2.1), $G^* \geq G+H \geq G$ and G+H is a right k-g-inverse of A dominating G.
Then, $A^k(G+H)A = A^k \Rightarrow A^kGA+A^kHA = A^k$
$\Rightarrow A^k+A^kHA = A^k$
$\Rightarrow A^k \geq A^kHA.$
Hence $G+H \in \wp$. Thus for each $G^* \in A_G\{1_r^k\}$, there exists a unique element in $\wp$.
Conversely, for any $G^* \in \wp$, $G^*=G+H \geq G$ with $A^k \geq A^kHA$, then

$$\begin{aligned} A^kG^*A &= A^k(G+H)A \\ &= A^kGA+A^kHA \\ &= A^k+A^kHA \\ &= A^k. \end{aligned}$$

Thus $G^* \in A_G\{1_r^k\}$. Hence the theorem.

Theorem 2.2:

Let $A \in \mathcal{F}_n$ and G be a particular left k-g-inverse of A.
Then $A_G\{1_\ell^k\} = \{G + K/ \text{ for all } K \in \mathcal{F}_n \text{ such that } A^k \geq AKA^k\}$
is the set of all left k-g-inverses of A dominating G.

Proof:

This can be proved in the same manner as in Theorem (2.1) and hence omitted.

Theorem 2.3:

For $A \in \mathcal{F}_n$ and $G \in A\{1_r^k,3\}$,

$$A_G\{1_r^k,3^k\} = \{G + H/ \text{ for all fuzzy matrix } H \in \mathcal{F}_n \text{ such that } A^kG \geq A^kH\} \quad \text{\_\_\_\_\_\_(2.3)}$$

is the set of all $\{1_r^k,3^k\}$ inverses of A dominating G.

Proof:

Let $\wp$ denote the set on the R.H.S of (2.3).
Suppose $G^* \in A_G\{1_r^k,3^k\}$, then $G^* \geq G$. Let $G^*-G=H$.
Since $A_G\{1_r^k,3^k\} \subseteq A_G\{1_r^k\}$, by Lemma (2.1), $G^* \geq G+H \geq G \Rightarrow A^kG^* = A^k(G+H) \geq A^kG$.
By Theorem (1.2), $G^* \in A_G\{1_r^k,3^k\}$ and $G \in A\{1_r^k,3\} \Rightarrow A^kG^* = A^kG$
$\Rightarrow A^k(G+H) = A^kG$
$\Rightarrow A^kG + A^kH = A^kG$
$\Rightarrow A^kG \geq A^kH$.
Hence $G+H \in \wp$. Thus for each $G^* \in A_G\{1_r^k,3^k\}$, there exists a unique element in $\wp$.
Conversely, for any $G^* \in \wp$, $G^*=G+H \geq G$ with $A^kG \geq A^kH$, then

$A^k G^* = A^k(G+H) = A^k G + A^k H = A^k G$. Since $G \in A\{1_r^k,3\} \Rightarrow G \in A\{1_r^k,3^k\}$.
Therefore, by Theorem (1.1), $G^* \in A_G\{1_r^k,3^k\}$. Hence the theorem.

Theorem 2.4:

For $A \in \mathcal{F}_n$ and $G \in A\{1_\ell^k,4\}$,

$A_G\{1_\ell^k,4^k\} = \{G + K/ \text{ for all fuzzy matrix } K \in \mathcal{F}_n \text{ such that } GA^k \geq KA^k\}$ ______(2.4)

is the set of all $\{1_\ell^k,4^k\}$ inverses of A dominating G.

Proof:

Let $\wp$ denote the set on the R.H.S of (2.4).

Suppose $G^* \in A_G\{1_\ell^k,4^k\}$, then $G^* \geq G$. Let $G^*-G=K$.

Since $A_G\{1_\ell^k,4^k\} \subseteq A_G\{1_\ell^k\}$, by Lemma (2.2), $G^* \geq G+K \geq G \Rightarrow G^*A^k = (G+K)\ A^k \geq GA^k$.

By Theorem (1.4), $G^* \in A_G\{1_\ell^k,4^k\}$ and $G \in A\{1_\ell^k,4\} \Rightarrow G^*A^k = GA^k$

$$\Rightarrow (G+K)A^k = GA^k$$
$$\Rightarrow GA^k + KA^k = GA^k$$
$$\Rightarrow GA^k \geq KA^k.$$

Hence $G+K \in \wp$. Thus for each $G^* \in A_G\{1_\ell^k,4^k\}$, there exists a unique element in $\wp$.
Conversely, for any $G^* \in \wp$, $G^*=G+K \geq G$ with $GA^k \geq KA^k$, then
$G^*A^k = (G+K)A^k = GA^k + KA^k = GA^k$. Since $G \in A\{1_\ell^k,4\} \Rightarrow G \in A\{1_\ell^k,4^k\}$.
Therefore, by Theorem (1.3), $G^* \in A_G\{1_\ell^k,4^k\}$. Hence the theorem.

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