# Compositional Belief Merging

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### Abstract

Belief merging aims at extracting a coherent and informative view from a set of belief bases. A first requirement for belief merging operators is to obey basic rationality conditions. Another expected property is to preserve as much information as possible from the input bases. In this paper, we show how new merging operators, called compositional operators, can be defined from existing ones. Such operators aim at offering a higher discriminative power than the merging operators on which they are based, without leading to a complexity shift or losing rationality postulates. We identify some sufficient conditions for ensuring that rationality is fully preserved by composition.

## Introduction

Belief merging (Baral, Kraus, and Minker 1991; Revesz 1997; Lin and Mendelzon 1999; Konieczny and Pino Pérez 2002; Konieczny, Lang, and Marquis 2004; Benferhat et al. 2002; Everaere, Konieczny, and Marquis 2010) aims at extracting a coherent and informative view from a (usually conflicting) set of belief bases.

Mainly developed in AI, belief merging is close to voting methods (Arrow 1963; Arrow, Sen, and Suzumura 2002) as developed in social choice: in both cases the objective is somehow to find a point of view which best reflects the data provided by the group. An important requirement for a voting method is to provide a unique winner whatever the input profile, i.e., a voting method must achieve a maximal discrimination among the candidates. In belief merging one does not ask for such a strong requirement, especially because it would violate some expected postulates. Thus when the bases are jointly consistent with the given integrity constraints, the expected result of the merging process consists of the conjunction of the bases with the constraints. This result is often not a complete base, and indeed, in such a situation, there is no reason that the merging operator "magically" completes the resulting base. Thus, within belief merging, it would not really make sense to use arbitrary tie-breaking rules, as used in voting methods, in order to improve arbitrarily the discriminative power of the approach. Nevertheless, the design of merging operators offering a good discriminative power without questioning rationality is important since the resulting merged base is expected to be as informative as possible.

In order to make things more concrete, consider the following merging scenario. A murder has been committed in a house and four witnesses saw somebody leaving the house after the crime time. The evidence reported by the witnesses was unfortunately highly conflicting:

- The first witness saw a woman, young (say, less than thirty years old) and wearing a hat.
- The second witness saw a man, not that young (say, more than thirty years old) and wearing a hat.
- The third witness saw a man, young but not wearing a hat.
- The fourth witness saw a man, not that young but not wearing a hat.

What can be concluded from these pieces of evidence? Of course, a consensual answer is not expected here since there are several rational ways to merge conflicting data. One of them consists in accepting the facts which are supported by as many sources of information as possible. On this example, three witnesses over four agree that the person who left the house were a man, but there is no agreement on the other facts (precisely half of the people saw a young person, and half of the people also saw a person wearing a hat.) Here the conclusion is just that the suspect is a man. A second merging principle consists in according to each description as much credit as the number of sources of information it satisfies. Since each testimony conflicts with the other ones, the corresponding conclusion is just the disjunction of all testimonies (note that all the sources of information agree on this disjunction.)

In formal terms, the first conclusion can be derived by taking advantage of the distance-based merging operator[1] given by the Hamming distance and sum as an aggregation function, while the second conclusion can be drawn using the distance-based merging operator given by the drastic distance and sum as an aggregation function. Both operators can be viewed as capturing rational ways to merge in the sense that they are Integrity Constraints (IC) merging operators (i.e., they satisfy the expected postulates for merging.)

But what if the two merging principles are to be jointly applied? In this case, the expected conclusion is that the sus-



[1] See Section *IC Merging* for a formal definition.

pect is a man who is not that young or does not wear a hat. This conclusion is more informative than both conclusions derived so far. But is it possible to draw it using a rational merging operator?

The main objective of this paper is to determine how to model rational merging operators enabling to derive such refined conclusions. In particular, we are interested in determining whether (and under which conditions) existing IC merging operators can be composed in such a way that the resulting operator is rational as well. To this end, we define compositional merging operators and study their properties. Composing merging operators typically leads to new merging operators. As to postulates, we show that all rationality postulates but (IC4) are preserved by composition. We also identify a number of sufficient conditions on the merging operators used in the composition for ensuring (IC4). We show that the complexity bounds of the inference problem for compositional merging operators is the same as the one for the merging operators on which they are based. Especially, the increase of inferential power offered by composition does not lead to a complexity shift.

## Preliminaries

We consider a propositional language $\mathcal{L}$ defined from a finite set of propositional variables $\mathcal{P}$ and the usual connectives.

An interpretation (or state of the world) $\omega$ is a total function from $\mathcal{P}$ to $\{0,1\}$. $\Omega$ is the set of all interpretations. An interpretation is usually denoted by a bit vector whenever a strict total order on $\mathcal{P}$ is specified. An interpretation $\omega$ is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $[\phi]$ denotes the set of models of formula $\phi$, i.e., $[\phi] = \{\omega \in \Omega \mid \omega \models \phi\}$.

A *base* $K$ denotes the set of beliefs of an agent, it is a finite set of propositional formulae, interpreted conjunctively (i.e., viewed as the conjunction of its elements).

A *profile* $E$ denotes a group of $n$ agents that are involved in the merging process; formally $E$ is given by a muti-set $\{K_1, \ldots, K_n\}$ of bases. $\bigwedge E$ denotes the conjunction of all elements of $E$, and $\sqcup$ denotes the multi-set union. Two multi-sets $E = \{K_1, \ldots, K_n\}$ and $E = \{K'_1, \ldots, K'_n\}$ are equivalent, noted $E \equiv E'$, iff there exists a permutation $\pi$ over $\{1, \ldots, n\}$ such that for each $i \in 1, \ldots, n$, we have $K_i \equiv K'_{\pi(i)}$.

A *merging operator* $\triangle$ is a function which associates with a profile $E$ and an integrity constraint $\mu$ (a formula from $\mathcal{L}$) a (merged) base $\triangle_\mu(E)$, and which *satisfies the postulates (IC0) and (IC1)*, recalled in the next section. $\triangle(E)$ is an abbreviation for $\triangle_\top(E)$.

A merging operator $\Delta^1$ is said to be as *discriminative* as a merging operator $\Delta^2$, noted $\Delta^1 \models \Delta^2$, iff for any profile $E$ and integrity constraint $\mu$, we have $\Delta^1_\mu(E) \models \Delta^2_\mu(E)$.

Finally, whenever $\leq$ denotes a pre-order, $\simeq$ denotes the corresponding indifference relation (i.e. the equivalence relation given by $\leq \cap \geq$), and the symbol $<$ denotes the strict part of $\leq$. $min(S, \leq)$ is the set of all $\omega \in S$ such that $\nexists \omega' \in S, \omega' < \omega$.

## IC Merging

In the following, the rationality postulates for merging pointed out in (Konieczny and Pino Pérez 2002) are considered:

**Definition 1** *A merging operator* $\triangle$ *is an* IC merging operator *iff it satisfies the following properties:*

**(IC0)** $\triangle_\mu(E) \models \mu$

**(IC1)** *If $\mu$ is consistent, then $\triangle_\mu(E)$ is consistent*

**(IC2)** *If $\bigwedge E$ is consistent with $\mu$, then $\triangle_\mu(E) \equiv \bigwedge E \wedge \mu$*

**(IC3)** *If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\triangle_{\mu_1}(E_1) \equiv \triangle_{\mu_2}(E_2)$*

**(IC4)** *If $K_1 \models \mu$ and $K_2 \models \mu$, then $\triangle_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\triangle_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent*

**(IC5)** $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2) \models \triangle_\mu(E_1 \sqcup E_2)$

**(IC6)** *If $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ is consistent, then $\triangle_\mu(E_1 \sqcup E_2) \models \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$*

**(IC7)** $\triangle_{\mu_1}(E) \wedge \mu_2 \models \triangle_{\mu_1 \wedge \mu_2}(E)$

**(IC8)** *If $\triangle_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(E) \models \triangle_{\mu_1}(E)$*

IC merging operators can be equivalently characterized in terms of syncretic assignments:

**Definition 2** *A* syncretic assignment *is a function mapping each profile $E$ to a total pre-order $\leq_E$ over $\Omega$ such that for any profiles $E, E_1, E_2$ and for any belief bases $K, K'$ the following conditions hold:*

1. *If $\omega \models \bigwedge E$ and $\omega' \models \bigwedge E$, then $\omega \simeq_E \omega'$*
2. *If $\omega \models \bigwedge E$ and $\omega' \not\models \bigwedge E$, then $\omega <_E \omega'$*
3. *If $E_1 \equiv E_2$, then $\leq_{E_1} = \leq_{E_2}$*
4. $\forall \omega \models K\ \exists \omega' \models K'\ \omega' \leq_{\{K,K'\}} \omega$
5. *If $\omega \leq_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$, then $\omega \leq_{E_1 \sqcup E_2} \omega'$*
6. *If $\omega <_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$, then $\omega <_{E_1 \sqcup E_2} \omega'$*

**Proposition 1 ((Konieczny and Pino Pérez 2002))** *A merging operator $\triangle$ is an IC merging operator iff there exists a syncretic assignment that maps each profile $E$ to a total pre-order $\leq_E$ over $\Omega$ such that $[\triangle_\mu(E)] = \min([\mu], \leq_E)$.*

The proof of this representation theorem shows that a similar characterization result can be obtained if one removes the fairness property (IC4) from the set of postulates and condition 4 from the conditions of the syncretic assignment (see (Konieczny and Pino Pérez 2002) for details.) Let us call a *weak-syncretic assignment* an assignment that satisfies conditions 1, 2, 3, 5, 6 in Definition 2. Thus, as a corollary, we have:

**Corollary 1** *A merging operator $\triangle$ satisfies postulates (IC0)-(IC3) and (IC5-IC8) (it is then called a* wIC merging operator*) iff there exists a weak-syncretic assignment that maps each profile $E$ to a total pre-order $\leq_E$ over $\Omega$ such that $[\triangle_\mu(E)] = \min([\mu], \leq_E)$.*

A convenient way to define IC merging operators consists in using a distance and an aggregation function:

**Definition 3** *A* (pseudo-)distance *between interpretations is a function $d : \Omega \times \Omega \to \mathbb{R}^+$ such that for any $\omega_1, \omega_2 \in \Omega$:*

- $d(\omega_1, \omega_2) = d(\omega_2, \omega_1)$

| | $[K_1] = \{011\}$ | | $[K_2] = \{101\}$ | | $[K_3] = \{110\}$ | | $[K_4] = \{100\}$ | | $E = \{K_1, K_2, K_3, K_4\}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $d_D$ | $d_H$ | $d_D$ | $d_H$ | $d_D$ | $d_H$ | $d_D$ | $d_H$ | $d^{d_D,\Sigma}$ | $d^{d_H,\Sigma}$ |
| 000 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 1 | 4 | 7 |
| 001 | 1 | 1 | 1 | 1 | 1 | 3 | 1 | 2 | 4 | 7 |
| 010 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 2 | 4 | 7 |
| 011 | 0 | 0 | 1 | 2 | 1 | 2 | 1 | 3 | **3** | 7 |
| **100** | 1 | 3 | 1 | 1 | 1 | 1 | 0 | 0 | **3** | **5** |
| **101** | 1 | 2 | 0 | 0 | 1 | 2 | 1 | 1 | **3** | **5** |
| **110** | 1 | 2 | 1 | 2 | 0 | 0 | 1 | 1 | **3** | **5** |
| 111 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 4 | **5** |

Table 1: Merging belief bases with $\Delta^{d_D,\Sigma}\bullet\Delta^{d_H,\Sigma}$

- $d(\omega_1, \omega_2) = 0$ *iff* $\omega_1 = \omega_2$

Usual distances considered in merging (Konieczny and Pino Pérez 2002) are the Hamming distance $d_H$: $d_H(\omega_1, \omega_2)$ is the number of propositional letters on which the two interpretations differ (this corresponds to the 1-norm distance, also referred to as the Manhattan distance) and the drastic distance $d_D$, defined as $d_D(\omega_1, \omega_2) = 0$ if $\omega_1 = \omega_2$, and $= 1$ otherwise (this corresponds to the infinity-norm distance, also known as Chebyshev distance.)

**Definition 4** *An* aggregation function $f$ *is a function mapping for any positive integer* $n$*, each* $n$*-uple of non-negative real numbers into a non-negative real number such that for any* $x_1, \ldots, x_n, x, y \in \mathbb{R}^+$*:*

- *if* $x \leq y$*, then* $f(x_1, ..., x, ..., x_n) \leq f(x_1, ..., y, ..., x_n)$
- $f(x_1, \ldots, x_n) = 0$ *iff* $x_1 = \ldots = x_n = 0$
- $f(x) = x$

Standard aggregation functions are sum ($\Sigma$), max, leximax (Gmax), leximin (Gmin), etc. (see (Konieczny and Pino Pérez 2002; Everaere, Konieczny, and Marquis 2010) for definitions.)

**Definition 5** *Let* $d$ *and* $f$ *be a distance between interpretations and an aggregation function respectively. The* distance-based merging operator $\triangle^{d,f}$ *is defined by* $[\triangle^{d,f}_{\mu}(E)] = \min([\mu], \leq_E)$*, where the total pre-order* $\leq_E$ *on* $\Omega$ *is defined in the following way (with* $E = \{K_1, \ldots, K_n\}$*):*

- $d(\omega, K) = \min_{\omega' \models K} d(\omega, \omega')$
- $d(\omega, E) = f(d(\omega, K_1), \ldots, d(\omega, K_n))$
- $\omega \leq_E \omega'$ *iff* $d(\omega, E) \leq d(\omega', E)$

Usual distance-based merging operators are IC merging operators (see (Konieczny, Lang, and Marquis 2004) for more details.)

## Compositional Belief Merging

Let us start with the definition of compositional belief merging operators:

**Definition 6** *Given two merging operators* $\Delta^1$ *and* $\Delta^2$*, we define the* compositional merging operator $\Delta^2\bullet\Delta^1$ *by*

$$\Delta^2\bullet\Delta^1_{\mu}(E) = \Delta^2_{\Delta^1_{\mu}(E)}(E)$$

*where* $E$ *is any profile and* $\mu$ *any integrity constraint.*

Clearly enough, this definition makes sense since $\Delta^2\bullet\Delta^1$ always satisfies (IC0) and (IC1) as soon as both $\Delta^1$ and $\Delta^2$ satisfy it. $\bullet$ can thus be viewed as a composition law for belief merging operators.

Given $n$ merging operators $\Delta^1, \ldots, \Delta^n$, $\Delta^n\bullet\ldots\bullet\Delta^1$ denotes the compositional merging operator given by $\Delta^n\bullet(\Delta^{n-1}\bullet(\ldots\bullet\Delta^1)...)$. Parentheses can be freely omitted in this sequence. Indeed, it is easy to show that $\bullet$ is associative:

**Proposition 2** *Given three merging operators* $\Delta^1$*,* $\Delta^2$ *and* $\Delta^3$*, we have* $\Delta^1\bullet(\Delta^2\bullet\Delta^3) = (\Delta^1\bullet\Delta^2)\bullet\Delta^3$.

For each $i \in 1, \ldots, n$, $\Delta^i\bullet\ldots\bullet\Delta^1$ is referred to as a suboperator of $\Delta^n\bullet\ldots\bullet\Delta^1$.

We now make precise the extent to which compositional merging operators allow for preserving more information from the input profile:

**Proposition 3** *For any merging operators* $\Delta^1, \ldots, \Delta^n$*, for any* $i$ *s.t.* $1 \leq i \leq n$*, we have* $\Delta^n\bullet\ldots\bullet\Delta^1 \models \Delta^i\bullet\ldots\bullet\Delta^1$.

This property shows that compositional merging operators are at least as discriminative as their suboperators. In particular, as expected, the compositional merging operator $\Delta^n\bullet\ldots\bullet\Delta^1$ is always at least as discriminative as the first merging operator $\Delta^1$ used in its definition.

Let us illustrate this property by providing a formal example, which is a counterpart of the informal example discussed in the introduction (we consider three propositional letters meaning respectively ”man”, ”young”, ”wearing a hat”; these letters are always considered in this order.)

**Example 1** *Let us consider the profile* $E = \{K_1, K_2, K_3, K_4\}$ *with* $[K_1] = \{011\}$, $[K_2] = \{101\}$, $[K_3] = \{110\}$, $[K_4] = \{100\}$ *which correspond respectively to the testimonies of witnesses* 1 *to* 4*. Here* $\mu = \top$ *(i.e., there is no integrity constraint). Computations are summarized in Table 1, where, for each interpretation* $\omega$ *(first column), the next columns give successively the (drastic and the Hamming) distance between* $\omega$ *and the four bases of* $E$*. The last two columns give respectively the distances between* $\omega$ *and* $E$ *according to the two operators* $\Delta^{d_D,\Sigma}$ *and* $\Delta^{d_H,\Sigma}$*. The minimal distances in those two columns are bold faced. The models of* $\Delta^{d_D,\Sigma}\bullet\Delta^{d_H,\Sigma}(E)$ *are given by the colored raws. In this example* $[\Delta^{d_D,\Sigma}\bullet\Delta^{d_H,\Sigma}(E)] = \{100, 101, 110\}$*, whereas* $[\Delta^{d_D,\Sigma}(E)] = \{011, 100, 101, 110\}$*, and* $[\Delta^{d_H,\Sigma}(E)] = \{110, 101, 110, 111\}$*. Thus this example shows that neither* $\Delta^{d_D,\Sigma}$ *nor* $\Delta^{d_H,\Sigma}$ *gives the same* *merged base as $\Delta^{d_D,\Sigma}\bullet\Delta^{d_H,\Sigma}$ for this profile and integrity constraint. Furthermore, we have $\Delta^{d_D,\Sigma}\bullet\Delta^{d_H,\Sigma}(E) \models \Delta^{d_D,\Sigma}(E)$ and $\Delta^{d_D,\Sigma}\bullet\Delta^{d_H,\Sigma}(E) \models \Delta^{d_H,\Sigma}(E)$. None of the converse entailments holds.*

Let us now give a semantical definition of compositional merging operators, when the merging operators used in their definition are wIC merging operators. Given $n$ binary relations $\leq^1, \ldots, \leq^n$, let $lex(\leq^1, \ldots, \leq^n)$ denote the binary relation given by $x\ lex(\leq^1, \ldots, \leq^n)\ y$ iff $(\forall i \in 1, \ldots, n, x \simeq^i y)$ or $(\exists k \in 1, \ldots, n, \forall 1 \leq i < k, x \simeq^i y$ and $x <^k y)$. It is easy to show that $lex(\leq^1, \ldots, \leq^n)$ is a total pre-order when $\leq^1, \ldots, \leq^n$ are total pre-orders.

**Proposition 4** *Let $\Delta^1, \ldots, \Delta^n$ be $n$ wIC merging operators. Let $E$ be any profile and $\mu$ be any integrity constraint. Let us note $\leq_E^i$ the pre-order associated with $E$ by the weak-syncretic assignment of operator $\Delta_i$. Then*

$$[\Delta^n\bullet \ldots \bullet\Delta^1_\mu(E)] = \min([\mu], \leq_E)$$

*where $\leq_E = lex(\leq_E^1, \ldots, \leq_E^n)$.*

This semantical characterization shows that the models of the merged base consists precisely of the models of $\mu$, which are in sequence, minimal with respect to $\leq_E^i$, when $i$ varies from 1 to $n$. In formal terms, we have: $[\Delta^n\bullet \ldots \bullet\Delta^1_\mu(E)] = \min(\ldots \min(\min([\mu], \leq_E^1), \leq_E^2), \ldots, \leq_E^n)$.

We now state some properties making precise some logical connections between the compositional merging operator $\Delta^n\bullet \ldots \bullet\Delta^1$ and the merging operators $\Delta^1, \ldots, \Delta^n$ used in its definition. Of course, nothing prevents such merging operators $\Delta^1, \ldots, \Delta^n$ from being themselves compositional operators. Since $\bullet$ is associative, the only important point is to preserve the ordering of the non-compositional operators at work in the composition sequence. Indeed, this ordering has a strong impact on the resulting merged base in the general case:

**Example 2** *Consider $K_1 = \{00\}$, $K_2 = \{11\}$ and the profile $E = \{K_1, K_2, K_2\}$. Assume that there is no integrity constraint ($\mu = \top$). Then $[\Delta^{d_H,Gmax}\bullet\Delta^{d_H,\Sigma}(E)] = \{11\}$ while $[\Delta^{d_H,\Sigma}\bullet\Delta^{d_H,Gmax}(E)] = \{10, 01\}$.*

It is easy to show that $\bullet$ has a neutral element, the "trivial" merging operator $\Delta^t$ given by $\Delta^t_\mu(E) \equiv \mu$, and that composition of merging operators satisfying (IC2) has a neutral element as well: the drastic merging operator $\Delta^d$ such that $\Delta^d_\mu(E) \equiv \bigwedge E \wedge \mu$ if consistent, and $\Delta^d_\mu(E) \equiv \mu$ otherwise.

The following proposition furnishes some sufficient conditions for removing merging operators in composition sequences:

**Proposition 5** *Let $\Delta^1$ and $\Delta^2$ be two wIC merging operators.*

1. $\Delta^2\bullet\Delta^1 = \Delta^2$ *iff* $\Delta^2 \models \Delta^1$.
2. $\Delta^2\bullet\Delta^1 = \Delta^1$ *if* $\Delta^1 \models \Delta^2$.

This proposition shows that in any composition sequence of wIC merging operators, one can always remove from two *successive* operators the less discriminative one. Formally:

**Proposition 6** *Let $\Delta^1$, $\Delta^2$ be two wIC merging operators. We have $\Delta^1\bullet\Delta^2\bullet\Delta^1 = \Delta^2\bullet\Delta^1$.*

Consequently, since $\bullet$ is associative, in every composition sequence defining a compositional merging operator, keeping only the very last occurrence of any multi-occurrent operator leads to an equivalent merging operator.

We now successively consider the properties offered by compositional merging operators, from the rationality point of view, from the point of view of inferential power, and finally from a computational perspective.

**Proposition 7** *If $\Delta^1, \Delta^2, \ldots, \Delta^n$ are wIC merging operators, then $\Delta^n\bullet\Delta^{n-1}\bullet \ldots \bullet\Delta^1$ is a wIC merging operator.*

This is a very interesting property since it ensures that deriving new belief merging operators by composing existing ones preserves their rationality.

It is worthwhile noting that a similar property does not hold for "full" IC operators, i.e., (IC4) may be lost by composition. Indeed, let us consider two propositional letters $a$ and $b$ (taken in this order), two bases $K_1$ and $K_2$ such that $[K_1] = \{00\}$ and $[K_2] = \{01, 10\}$. Let us also consider two (pseudo-)distances $d_1$ and $d_2$ given by $d_1(\omega, \omega')= d_D(\omega_a, \omega'_a)+2*d_D(\omega_b, \omega'_b)$ and $d_2(\omega, \omega')= 2*d_D(\omega_a, \omega'_a)+ d_D(\omega_b, \omega'_b)$, where $\omega_x$ is the restriction of the interpretation $\omega$ over the propositional letter $x$. The distance-based merging operators given by $d_1$ (resp. $d_2$) and $\Sigma$ as aggregation function are IC merging operators (Konieczny, Lang, and Marquis 2004). However $[\Delta^{d_2,\Sigma}\bullet\Delta^{d_1,\Sigma}_\top(\{K_1, K_2\})] = \{00\}$. Thus, the resulting merged base is consistent with $K_1$ but it is not consistent with $K_2$, which shows that (IC4) is not satisfied.

The preservation of (IC4) can be guaranteed nevertheless in some restricted cases:

**Proposition 8** *If $\Delta$ is an IC merging operator, then $\Delta^{d_D,\Sigma}\bullet\Delta$ and $\Delta\bullet\Delta^{d_D,\Sigma}$ are IC merging operators.*

The latter proposition shows in particular that the compositional belief merging operator $\Delta^{d_D,\Sigma}\bullet\Delta^{d_H,\Sigma}$ considered in Example 1 is an IC merging operator, which addresses an issue considered in the introduction. Furthermore, it turns out that this proposition is not as restricted as it appears at a first glance. Indeed, when considering the drastic distance $d_D$, distance-based merging typically amounts to using $\Sigma$ as aggregation function:

**Definition 7** *An aggregation function $f$ satisfies* symmetry *if for any permutation $\pi$ over $\{1, \ldots, n\}$,*
$f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$.
*An aggregation function $f$ satisfies* strict monotony *if*
$x < y \implies f(x_1, \ldots, x, \ldots, x_n) < f(x_1, \ldots, y, \ldots, x_n)$.

**Proposition 9** *For any aggregation function $f$ satisfying symmetry and strict monotony, $\Delta^{d_D,f} = \Delta^{d_D,\Sigma}$.*

Another way to guarantee that (IC4) is preserved consists in taking advantage of distance-based merging operators defined from the same distance:

**Proposition 10** *Let $f_1, \ldots, f_n$ be $n$ aggregation functions, and let $d$ be a distance such that the operators $\Delta^{d,f_i}$ are IC merging operators. Then $\Delta^{d,f_n}\bullet\Delta^{d,f_{n-1}}\bullet \ldots \bullet\Delta^{d,f_1}$ is an IC merging operator.*

Let us now focus on the discrimination issue for compositional merging. The main question concerns the possibility of selecting a single interpretation via a merging sequence:

**Definition 8** *$\Delta$ is said to be a* maxichoice merging operator *if $\Delta_\mu(E)$ is a complete formula (i.e., it has a single model) when $\mu$ is consistent and $\bigwedge E \wedge \mu$ is inconsistent.*

Maxichoice operators exhibit a full discrimination among the models of $\mu$ whenever $\bigwedge E \wedge \mu$ is inconsistent. Thus, the maxichoice requirement can prove useful if one wants to select exactly one solution (for instance, for a decision making purpose) from conflicting bases. Maxichoice operators also play the role of tie-breaking rules when occurring as initials in composition sequences, since no further refinement is possible once a maxichoice merging operator has ben considered.

A valuable family of maxichoice merging operators is composed of imposed operators:

**Definition 9** *Let $\prec$ be a total strict order on $\Omega$. The* imposed merging operator $\Delta$ *induced by $\prec$ is given by the assignment with associates every profile $E$ with the total pre-order $\leq_E$ over $\Omega$ such that:*

- *If $\omega \models \bigwedge E$ and $\omega' \models \bigwedge E$ then $\omega \simeq_E \omega'$*
- *If $\omega \models \bigwedge E$ and $\omega' \not\models \bigwedge E$ then $\omega <_E \omega'$*
- *If $\omega \not\models \bigwedge E$ and $\omega' \not\models \bigwedge E$ then ($\omega <_E \omega'$ iff $\omega \prec \omega'$)*

*For any $\mu$, $[\Delta_\mu(E)] = \min([\mu], \leq_E)$.*

**Proposition 11** *Every imposed merging operator is a maxichoice merging operator.*

It is interesting to note that imposed merging operators exhibit good logical properties:

**Proposition 12** *Every imposed merging operator is a wIC merging operator.*

It is also easy to show that imposed merging operators are not IC merging operators (they do not satisfy (IC4) in general).

As to the discriminative power, we can prove that any wIC merging operator can be arbitrarily refined, until reaching a maxichoice wIC merging operator, via a sequence of compositions:

**Proposition 13** *Every wIC merging operator $\Delta^1$ can be refined to a maxichoice wIC merging operator, i.e., there exists a sequence of wIC merging operators $\Delta^n \bullet \ldots \bullet \Delta^2$ such that $\Delta^n \bullet \ldots \bullet \Delta^2 \bullet \Delta^1$ is a maxichoice wIC merging operator.*

Finally, we have identified some complexity bounds for compositional merging operators:

**Proposition 14** *If $\Delta^1, \ldots, \Delta^n$ are wIC merging operators such that for every $i \in 1, \ldots, n$ and every profile $E$, deciding whether $\omega_1 \leq^i_E \omega_2$ can be done in polynomial time for any pair of interpretations $\omega_1$, $\omega_2$, then the inference problem for $\Delta^n \bullet \ldots \bullet \Delta^1$ is in $\Pi^p_2$ and is $\mathsf{D}^\mathsf{p}$-hard.*

Since these complexity bounds also hold when the sequence is reduced to a single operator $\Delta^1$, the gain in discriminative power offered by composing belief merging operators does not imply a complexity shift.

## Conclusion

In this paper, we have considered composition as a way to define new belief merging operators from existing ones. Composition leads to merging operators with an improved discriminative power, without questioning the most central logic properties.

In (Gauwin, Konieczny, and Marquis 2007) iterated merging operators have been defined as a way to improve the discriminative power of merging operators. In a nutshell, the idea was to use a single merging operator on a given profile in an iterative way, the merged base being used at each iteration to revise each base of the profile. It turns out that the resulting operators exhibit bad merging properties, but can prove useful for modeling negotiation (or conciliation) processes. Contrastingly, the compositional merging operators defined in this paper are based on several merging operators and offer better merging properties.

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