

## $H_\infty$ Model Reduction of 2D Markovian Jump System with Roesser Model

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**Abstract:** This paper extends the results obtained for one-dimensional Markovian jump systems to investigate the problem of  $H_\infty$  model reduction for a class of linear discrete time 2D Markovian jump systems with state delays in Roesser model which is time-varying and mode-independent. The reduced-order model with the same randomly jumping parameters is proposed which can make the error systems stochastically stable with a prescribed  $H_\infty$  performance. A sufficient condition in terms of linear matrix inequalities (LMIs) plus matrix inverse constraints are derived for the existence of a solution to the reduced-order model problems. The cone complementarity linearization (CCL) method is exploited to cast them into nonlinear minimization problems subject to LMI constraints. A numerical example is given to illustrate the design procedures.

### Introduction

With the development of modern industry and economy, more and more multivariable systems and multidimensional signal need to be handled. Such as multi-dimensional digital image processing, multivariable network realization, meteorological satellite image analysis, which are mostly appear as 2D discrete system model. For these profound engineering backgrounds, in recent years, 2D discrete systems have received much attention, and many important results are available in the literatures [1, 2]. In many engineering applications, a lot of complex physical systems are frequently described by high or even infinite order mathematical models which usually bring troubles for our research and analysis of system performance. An effective method to solve the problems is to use a low order model to approximate the original system without significant error and is convenient to implement, thus model reduction plays an important role in the process of control system design. Furthermore, the problem of model reduction for 2D systems has received considerable attention due to their importance in practical applications where a lower-order system is usually desired to represent a high-order system. For example, Du researched the reduced-order approximation of 2D digital filters using the LMI approach in [3].

On a different direction, a considerable research effort has been recently devoted to the analysis of the Markovian jump system whose structures are subject to random abrupt changes may due to component or interconnections failures, sudden environment changes, change of the operating point of a linearized model of a nonlinear, and so on. The application of Markovian jump systems can be found in many physical systems, such as manufacturing systems, target tracking, and power system [4]. And some problems of stability, controller design and filtering related to these systems also have been extensively studied by numerous scholars, see for instance [5,6], and the references therein. Since delay usually occur in many physical and engineering systems and causes instability and poor performance of systems, time-delay systems have been studied extensively on the subject of control and model reduction over the years. For example, in [7] Qing Wang and James Lam addressed the model approximation for discrete-time Markovian jump systems with mode-dependent time delays. However, the aforementioned results are only concerned with one-dimensional systems, and to the best of the authors' knowledge, few effort has been made toward investigating the problems arising in 2D jump systems.

In this paper, we extends the results obtained for one-dimensional Markovian jump systems to investigate the problem of  $H_\infty$  model reduction for a class for linear discrete time 2D Markovian jump systems with state delays in Roesser model which are time-varying and mode-independent.

The jump parameters are modeled by a finite-state Markov process. A reduced-order model with the same randomly jumping parameters is proposed which can make the error systems stochastically stable with a prescribed  $H_\infty$  performance. Then sufficient conditions in terms of LMIs plus matrix inverse constraints are derived for the existence of a solution to the reduced-order model problems. Since these obtained conditions are not expressed as strict LMIs, the CCL method is exploited to cast them into nonlinear minimization problems subject to LMI constraints, which can be readily solved by standard numerical soft ware. A numerical example is given to illustrate the design procedures.

### Problem formulation

Consider the following 2D discrete time-delays system in the Roesser with Markovian jump parameters:

$$\begin{aligned} \begin{bmatrix} \mathbf{x}^h(i+1, j) \\ \mathbf{x}^v(i, j+1) \end{bmatrix} &= \mathbf{A}(r_{i,j}) \begin{bmatrix} \mathbf{x}^h(i, j) \\ \mathbf{x}^v(i, j) \end{bmatrix} + \mathbf{A}_{d1}(r_{i,j}) \mathbf{x}^h(i-d_1, j) + \mathbf{A}_{d2}(r_{i,j}) \mathbf{x}^v(i, j-d_2) + \mathbf{B}(r_{i,j}) \mathbf{w}(i, j) \\ z(i, j) &= \mathbf{C}(r_{i,j}) \begin{bmatrix} \mathbf{x}^h(i, j) \\ \mathbf{x}^v(i, j) \end{bmatrix} + \mathbf{C}_{d1}(r_{i,j}) \mathbf{x}^h(i-d_1, j) + \mathbf{C}_{d2}(r_{i,j}) \mathbf{x}^v(i, j-d_2) + \mathbf{D}(r_{i,j}) \mathbf{w}(i, j) \end{aligned} \quad (1)$$

where  $n=n_1+n_2$ ,  $\mathbf{x}^h(i, j) \in R^{n_1}$ ,  $\mathbf{x}^v(i, j) \in R^{n_2}$  represent the horizontal and vertical states respectively;  $\mathbf{w}(i, j) \in R^m$  is the disturbance input which is a square-integrable and norm bounded stochastic vector function over  $L_2\{[0, \infty), [0, \infty)\}$ ;  $z(i, j) \in R^m$  is the controlled output;  $d_1$  and  $d_2$  are constant positive integers representing delays along horizontal direction and vertical direction, respectively.  $\mathbf{A}(r_{i,j}) \in R^{n \times n}$ ,  $\mathbf{A}_{d1}(r_{i,j}) \in R^{n \times n_1}$ ,  $\mathbf{A}_{d2}(r_{i,j}) \in R^{n \times n_2}$ ,  $\mathbf{B}(r_{i,j}) \in R^{n \times m}$ ,  $\mathbf{C}(r_{i,j}) \in R^{l \times n}$ ,  $\mathbf{C}_{d1}(r_{i,j}) \in R^{l \times n_1}$ ,  $\mathbf{C}_{d2}(r_{i,j}) \in R^{l \times n_2}$ ,  $\mathbf{D}(r_{i,j}) \in R^{l \times m}$  are matrix functions of the time-varying parameter  $r_{i,j}$ .

The parameter  $r_{i,j}$  takes values in a finite set  $S=\{1, 2, \dots, s\}$ , with transition probabilities  $\Pr\{r_{i+1,j}=n | r_{i,j}=m\} = \Pr\{r_{i,j+1}=n | r_{i,j}=m\} = p_{mn}$ , where  $p_{mn} \geq 0$ , and for any  $m \in S$ ,  $\sum_{n=1}^s p_{mn} = 1$ .

For each possible value  $m \in S$ , to simplify the notation, when the system operates at the  $m$ th mode, that is  $r_{i,j}=m$ , the matrix  $\mathbf{A}(r_{i,j})$  will be denoted by  $A_m$ ,  $\mathbf{A}_{d1}(r_{i,j})$  will be denoted by  $A_{d1m}$ , and so on.

In this note, we denote the system state as  $x(i, j)=[x^h(i, j) \ x^v(i, j)]^T$ . The boundary conditions  $(\mathbf{X}_0, \mathbf{R}_0)$  are defined as follows:

$$\begin{aligned} \mathbf{X}_0 &= \begin{bmatrix} \mathbf{x}^{h^T}(-d_1, 0) & \mathbf{x}^{h^T}(-d_1, 1) & \mathbf{x}^{h^T}(-d_1, 2) & \dots & \mathbf{x}^{h^T}(1-d_1, 0) & \mathbf{x}^{h^T}(1-d_1, 1) & \mathbf{x}^{h^T}(1-d_1, 2) & \dots \\ \mathbf{x}^{v^T}(0, 0) & \mathbf{x}^{v^T}(0, 1) & \mathbf{x}^{v^T}(0, 2) & \dots & \mathbf{x}^{v^T}(0, -d_2) & \mathbf{x}^{v^T}(1, -d_2) & \mathbf{x}^{v^T}(2, -d_2) & \dots \\ \mathbf{x}^{v^T}(0, 1-d_2) & \mathbf{x}^{v^T}(1, 1-d_2) & \mathbf{x}^{v^T}(2, 1-d_2) & \dots & \mathbf{x}^{v^T}(0, 0) & \mathbf{x}^{v^T}(0, 1) & \mathbf{x}^{v^T}(0, 2) & \dots \end{bmatrix}^T \\ \mathbf{R}_0 &= \{r_{0,0}, r_{0,1}, r_{0,2}, \dots, r_{0,0}, r_{1,0}, r_{2,0}, r_{3,0}\} \end{aligned}$$

In this paper, our purpose is to find a mean-square asymptotically stable  $\tilde{n}$ th-order 2D jump system:

$$\begin{aligned} \begin{bmatrix} \tilde{\mathbf{x}}^h(i+1, j) \\ \tilde{\mathbf{x}}^v(i, j+1) \end{bmatrix} &= \tilde{\mathbf{A}}_m \begin{bmatrix} \tilde{\mathbf{x}}^h(i, j) \\ \tilde{\mathbf{x}}^v(i, j) \end{bmatrix} + \tilde{\mathbf{A}}_{d1m} \tilde{\mathbf{x}}^h(i-d_1, j) + \tilde{\mathbf{A}}_{d2m} \tilde{\mathbf{x}}^v(i, j-d_2) + \tilde{\mathbf{B}}_m \mathbf{w}(i, j) \\ \tilde{z}(i, j) &= \tilde{\mathbf{C}}_m \begin{bmatrix} \tilde{\mathbf{x}}^h(i, j) \\ \tilde{\mathbf{x}}^v(i, j) \end{bmatrix} + \tilde{\mathbf{C}}_{d1m} \tilde{\mathbf{x}}^h(i-d_1, j) + \tilde{\mathbf{C}}_{d2m} \tilde{\mathbf{x}}^v(i, j-d_2) + \tilde{\mathbf{D}}_m \mathbf{w}(i, j) \end{aligned} \quad (2)$$

where  $\tilde{n}=\tilde{n}_1+\tilde{n}_2$ ,  $\tilde{\mathbf{x}}^h(i, j) \in R^{\tilde{n}_1}$ ,  $\tilde{\mathbf{x}}^v(i, j) \in R^{\tilde{n}_2}$ ,  $\tilde{\mathbf{A}}_m \in R^{\tilde{n} \times \tilde{n}}$ ,  $\tilde{\mathbf{A}}_{d1m} \in R^{\tilde{n} \times \tilde{n}_1}$ ,  $\tilde{\mathbf{A}}_{d2m} \in R^{\tilde{n} \times \tilde{n}_2}$ ,  $\tilde{\mathbf{B}}_m \in R^{\tilde{n} \times m}$ ,  $\tilde{\mathbf{C}}_m \in R^{l \times \tilde{n}}$ ,  $\tilde{\mathbf{C}}_{d1m} \in R^{l \times \tilde{n}_1}$ ,  $\tilde{\mathbf{C}}_{d2m} \in R^{l \times \tilde{n}_2}$ ,  $\tilde{\mathbf{D}}_m \in R^{l \times m}$  with  $\tilde{n} < n$ ;  $r_{i,j}=m$  is the same Markov chain as in (1), such that the error system:

$$\begin{aligned} \bar{\mathbf{x}}^h(i, j) &= \bar{\mathbf{A}}(r_{i,j}) \bar{\mathbf{x}}(i, j) + \bar{\mathbf{A}}_{d1}(r_{i,j}) \bar{\mathbf{x}}_{d1}^h(i, j) + \bar{\mathbf{A}}_{d2}(r_{i,j}) \bar{\mathbf{x}}_{d2}^v(i, j) + \bar{\mathbf{B}}(r_{i,j}) \mathbf{w}(i, j) \\ \bar{\mathbf{z}}(i, j) &= \bar{\mathbf{C}}(r_{i,j}) \bar{\mathbf{x}}(i, j) + \bar{\mathbf{C}}_{d1}(r_{i,j}) \bar{\mathbf{x}}_{d1}^h(i, j) + \bar{\mathbf{C}}_{d2}(r_{i,j}) \bar{\mathbf{x}}_{d2}^v(i, j) + \bar{\mathbf{D}}(r_{i,j}) \mathbf{w}(i, j) \end{aligned} \quad (3)$$

where  $\bar{x}'(i,j) = \begin{bmatrix} x'(i,j) \\ \tilde{x}'(i,j) \end{bmatrix}$ ,  $\bar{x}(i,j) = \begin{bmatrix} x(i,j) \\ \tilde{x}(i,j) \end{bmatrix}$ ,  $\bar{z}(i,j) = z(i,j) - \tilde{z}(i,j)$ ,  $\bar{x}_{d_1}^h(i,j) = \begin{bmatrix} x^h(i-d_1,j) \\ \tilde{x}^h(i-d_1,j) \end{bmatrix}$ ,  $\bar{x}_{d_2}^v(i,j) = \begin{bmatrix} x^v(i,j-d_2) \\ \tilde{x}^v(i,j-d_2) \end{bmatrix}$ ,  
 $x'(i,j) = \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}$ ,  $\tilde{x}'(i,j) = \begin{bmatrix} \tilde{x}^h(i+1,j) \\ \tilde{x}^v(i,j+1) \end{bmatrix}$ ,  $x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$ ,  $\tilde{x}(i,j) = \begin{bmatrix} \tilde{x}^h(i,j) \\ \tilde{x}^v(i,j) \end{bmatrix}$ ,  $\bar{A}_m = \begin{bmatrix} A_m & 0 \\ 0 & \tilde{A}_m \end{bmatrix}$ ,  
 $\bar{A}_{d_1m} = \begin{bmatrix} A_{d_1m} & 0 \\ 0 & \tilde{A}_{d_1m} \end{bmatrix}$ ,  $\bar{B}_m = \begin{bmatrix} B_m \\ \tilde{B}_m \end{bmatrix}$ ,  $\bar{A}_{d_2m} = \begin{bmatrix} A_{d_2m} & 0 \\ 0 & \tilde{A}_{d_2m} \end{bmatrix}$ ,  $\bar{C}_m = [C_m \quad -\tilde{C}_m]$ ,  $\bar{C}_{d_1m} = [C_{d_1m} \quad -\tilde{C}_{d_1m}]$ ,  
 $\bar{C}_{d_2m} = [C_{d_2m} \quad -\tilde{C}_{d_2m}]$ ,  $\bar{D}_m = D_m - \tilde{D}_m$ ,

is mean-square asymptotically stable and has  $H_\infty$  performance.

Before presenting the main objective of this paper, we first introduce the following definitions for the error systems (3), which will be essential for our derivation.

**Definition 1:** The error 2D jump system (3) with  $w_{i,j} = 0$  is said to be mean-square asymptotically stable if  $\lim_{i+j \rightarrow \infty} E\{\|\bar{x}_{i,j}\|^2\} = 0$  for every boundary condition  $(X_0, R_0)$  satisfying Assumption 1.

**Definition 2:** For a given scalar  $\gamma > 0$ , the error 2D jump system (3) is said to be mean-square asymptotically stable with an  $H_\infty$  disturbance attenuation level  $\gamma$  if it is mean-square asymptotically stable and under zero boundary condition  $X_0 = 0$ ,  $\|z\|_2^2 < \gamma^2 \|w\|_2^2$  for all non-zero  $w = \{w_{ij}\}$ ,

where  $\|z\|_2^2 = E\left\{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |z_{i,j}|^2\right\}$ ,  $\|w\|_2^2 = E\left\{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |w_{i,j}|^2\right\}$ .

**Lemma 1:** Given a symmetric matrix  $\Omega$  and two matrices  $\Psi$  and  $T$ , consider the problem of finding some matrix  $G$  such that

$$\Omega + \Psi G \Upsilon + (\Psi G \Upsilon)^T < 0$$

Then (4) is solvable for  $G$  if and only if

$$\Psi^\perp \Omega \Psi^{\perp T} < 0, \quad \Upsilon^{T \perp} \Omega \Upsilon^{T \perp T} < 0 \tag{4}$$

**Main results:**

In this section, we give a solution to the  $H_\infty$  model reduction problem formulated previously, by using matrix inequality approach. To this end, we first present the following result for the error system (3) to be asymptotically stable with  $H_\infty$  performance constraints which will play a key role in solving the aforementioned problem.

**Theorem 1:** Consider the error system (3), for a given a scalar  $\gamma > 0$ , the error system (3) is asymptotically stable with  $H_\infty$  performance, if there exist positive definite symmetric matrices  $P_m = \text{diag}\{P_m^h, P_m^v, \tilde{P}_m^h, \tilde{P}_m^v\} \in R^{(n+\tilde{n}) \times (n+\tilde{n})}$ ,  $Q = \text{diag}\{Q^h, Q^v, \tilde{Q}^h, \tilde{Q}^v\} \in R^{(n+\tilde{n}) \times (n+\tilde{n})}$ ,  $P = (P_1, P_2, \dots, P_s)$ ,  $m \in S$ , such that the following linear matrix inequality holds for:

$$\begin{bmatrix} -\bar{P}_m & \bar{P}_m \bar{A}_m & \bar{P}_m \bar{A}_{d_1m} & \bar{P}_m \bar{A}_{d_2m} & \bar{P}_m \bar{B}_m & 0 \\ * & -\bar{P}_m + Q & 0 & 0 & 0 & \bar{C}_m^T \\ * & * & -\bar{Q}^h & 0 & 0 & \bar{C}_{d_1m}^T \\ * & * & * & -\bar{Q}^v & 0 & \bar{C}_{d_2m}^T \\ * & * & * & * & -\gamma^2 I & \bar{D}_m^T \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{5}$$

where  $\bar{P}_m = \sum_{n=1}^s p_{mn} P_n$ ,  $P_n = \text{diag}\{P^h(r_{i+1,j}), P^v(r_{i,j+1}), \tilde{P}^h(r_{i+1,j}), \tilde{P}^v(r_{i,j+1})\}$ ,  $\bar{Q}^h = \text{diag}\{Q^h, \tilde{Q}^h\}$ ,  $\bar{Q}^v = \text{diag}\{Q^v, \tilde{Q}^v\}$ .

**Proof :** Consider system (3), let  $w(t)=0$  and the mode be  $m$  at time  $t$ , that is  $r(i,j)=m \in S$ . We can verify that  $\{(\bar{x}(i,j), r(t,j)), t \geq 0\}$  is a Markov process with initial state  $(X(0), R(0))$ . Now, we define a stochastic Lyapunov functional  $V(\cdot)$  as follows:

$$V(\bar{x}(i,j), r(i,j) = m) \equiv V_{h0}(x^h(i,j), m) + V_{v0}(x^v(i,j), m) + V_{h1}(\tilde{x}^h(i,j), m) + V_{v1}(\tilde{x}^v(i,j), m),$$

where:

$$V_{h0}(\mathbf{x}^h(i, j), m) = \mathbf{x}^{h^T}(i, j) \mathbf{P}_m^h \mathbf{x}^h(i, j) + \sum_{\theta=1}^{d_1} \mathbf{x}^{h^T}(i-\theta, j) \mathbf{Q}^h \mathbf{x}^h(i-\theta, j), V_{v0}(\mathbf{x}^v(i, j), m) = \mathbf{x}^{v^T}(i, j) \mathbf{P}_m^v \mathbf{x}^v(i, j) + \sum_{\theta=1}^{d_2} \mathbf{x}^{v^T}(i, j-\theta) \mathbf{Q}^v \mathbf{x}^v(i, j-\theta),$$

$$V_{h1}(\tilde{\mathbf{x}}^h(i, j), m) = \tilde{\mathbf{x}}^{h^T}(i, j) \tilde{\mathbf{P}}_m^h \tilde{\mathbf{x}}^h(i, j) + \sum_{\theta=1}^{d_1} \tilde{\mathbf{x}}^{h^T}(i-\theta, j) \tilde{\mathbf{Q}}^h \tilde{\mathbf{x}}^h(i-\theta, j), V_{v1}(\tilde{\mathbf{x}}^v(i, j), m) = \tilde{\mathbf{x}}^{v^T}(i, j) \tilde{\mathbf{P}}_m^v \tilde{\mathbf{x}}^v(i, j) + \sum_{\theta=1}^{d_2} \tilde{\mathbf{x}}^{v^T}(i, j-\theta) \tilde{\mathbf{Q}}^v \tilde{\mathbf{x}}^v(i, j-\theta).$$

Define  $\Phi(t) = [\bar{\mathbf{x}}^T(i, j) \quad \bar{\mathbf{x}}_{d_1}^{h^T}(i, j) \quad \bar{\mathbf{x}}_{d_2}^{v^T}(i, j)]^T$ , applying the Schur complement we can get:

$$\Delta V(\bar{\mathbf{x}}(i, j), m) = E\{V(\bar{\mathbf{x}}(i, j), n)\} - V(\bar{\mathbf{x}}(i, j), m)$$

$$= \bar{\mathbf{x}}^T(i, j) \bar{\mathbf{P}}_m \bar{\mathbf{x}}(i, j) - \bar{\mathbf{x}}^T(i, j) (\mathbf{P}_m + \mathbf{Q}) \bar{\mathbf{x}}(i, j) - \bar{\mathbf{x}}_{d_1}^{h^T}(i, j) \bar{\mathbf{Q}}^h \bar{\mathbf{x}}_{d_1}^h(i, j) - \bar{\mathbf{x}}_{d_2}^{v^T}(i, j) \bar{\mathbf{Q}}^v \bar{\mathbf{x}}_{d_2}^v(i, j) = \Phi^T(i, j) \Pi_m \Phi(i, j)$$

where  $\Pi_m = \begin{bmatrix} -\mathbf{P}_m + \mathbf{Q} + \bar{\mathbf{A}}_m^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_m & \bar{\mathbf{A}}_m^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_{d_1 m} & \bar{\mathbf{A}}_m^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_{d_2 m} \\ * & -\bar{\mathbf{Q}}^h + \bar{\mathbf{A}}_{d_1 m}^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_{d_1 m} & \bar{\mathbf{A}}_{d_1 m}^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_{d_2 m} \\ * & * & -\bar{\mathbf{Q}}^v + \bar{\mathbf{A}}_{d_2 m}^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_{d_2 m} \end{bmatrix}.$

The inequality (5) guarantees  $\Pi_m < 0$ . Then we have  $\Delta V(\bar{\mathbf{x}}(i, j), m) < 0$ . Define  $d = \max\{d_1, d_2\}$ , then for all  $r(i, j) = m \in S$ , we have:

$$\frac{E\{V(\bar{\mathbf{x}}(i, j), n)\} - V(\bar{\mathbf{x}}(i, j), m)}{V(\bar{\mathbf{x}}(i, j), m)} \leq -\frac{\Phi^T(i, j) (-\Pi_m) \Phi(i, j)}{\bar{\mathbf{x}}^T(i, j) \mathbf{P}_m \bar{\mathbf{x}}(i, j) + d[\bar{\mathbf{x}}_{d_1}^T(i, j) \bar{\mathbf{Q}}^h \bar{\mathbf{x}}_{d_1}^h + \bar{\mathbf{x}}_{d_2}^T(i, j) \bar{\mathbf{Q}}^v \bar{\mathbf{x}}_{d_2}^v]} \leq \min\left\{\frac{\lambda_{\min}(-\Pi_m)}{\lambda_{\max} \mathbf{P}_m + d \varepsilon^2 \lambda_{\max} \mathbf{Q}}\right\} = \alpha - 1$$

From  $\Delta V(\bar{\mathbf{x}}(i, j), m) = E\{V(\bar{\mathbf{x}}(i, j), n)\} - V(\bar{\mathbf{x}}(i, j), m) < 0$ , we can obtain:  $0 < \alpha = 1 - \min\left\{\frac{\lambda_{\min}(-\Pi_m)}{\lambda_{\max} \mathbf{P}_m + d \varepsilon^2 \lambda_{\max} \mathbf{Q}}\right\} < 1$ .

Hence, we have:  $E\{V(\bar{\mathbf{x}}(i, j), n)\} \leq \alpha V(\bar{\mathbf{x}}(i, j), m)$ .

Taking expectation of both sides yields:

$$E\left\{\bar{\mathbf{x}}^T(i, j) \mathbf{P}_m \bar{\mathbf{x}}(i, j) + \sum_{\theta=1}^{d_1} \left[\mathbf{x}^{h^T}(i+1-\theta, j) \mathbf{Q}^h \mathbf{x}^h(i+1-\theta, j) + \tilde{\mathbf{x}}^{h^T}(i+1-\theta, j) \tilde{\mathbf{Q}}^h \tilde{\mathbf{x}}^h(i+1-\theta, j)\right] + \sum_{\theta=1}^{d_2} \left[\mathbf{x}^{v^T}(i, j+1-\theta) \mathbf{Q}^v \mathbf{x}^v(i, j+1-\theta) + \tilde{\mathbf{x}}^{v^T}(i, j+1-\theta) \tilde{\mathbf{Q}}^v \tilde{\mathbf{x}}^v(i, j+1-\theta)\right]\right\}$$

$$\leq \alpha E\left\{\bar{\mathbf{x}}^T(i, j) \mathbf{P}_m \bar{\mathbf{x}}(i, j) + \sum_{\theta=1}^{d_1} \left[\mathbf{x}^{h^T}(i-\theta, j) \mathbf{Q}^h \mathbf{x}^h(i-\theta, j) + \tilde{\mathbf{x}}^{h^T}(i-\theta, j) \tilde{\mathbf{Q}}^h \tilde{\mathbf{x}}^h(i-\theta, j)\right] + \sum_{\theta=1}^{d_2} \left[\mathbf{x}^{v^T}(i, j-\theta) \mathbf{Q}^v \mathbf{x}^v(i, j-\theta) + \tilde{\mathbf{x}}^{v^T}(i, j-\theta) \tilde{\mathbf{Q}}^v \tilde{\mathbf{x}}^v(i, j-\theta)\right]\right\}$$

Define  $i+j=k$ , by using this relationship iteratively and performing superposition of the two sides of inequalities from  $j=0$  to  $j=k+1$ , it is easy to get:

$$E\left\{\sum_{j=0}^{k+1} \left[\mathbf{x}^{h^T}(k+1-j, j) \mathbf{P}^h(r_{k+1-j, j}) \mathbf{x}^h(k+1-j, j) + \mathbf{x}^{v^T}(k+1-j, j) \mathbf{P}^v(r_{k+1-j, j}) \mathbf{x}^v(k+1-j, j) + \tilde{\mathbf{x}}^{h^T}(k+1-j, j) \tilde{\mathbf{P}}^h(r_{k+1-j, j}) \tilde{\mathbf{x}}^h(k+1-j, j) + \tilde{\mathbf{x}}^{v^T}(k+1-j, j) \tilde{\mathbf{P}}^v(r_{k+1-j, j}) \tilde{\mathbf{x}}^v(k+1-j, j) + \sum_{\theta=1}^{d_1} \left[\mathbf{x}^{h^T}(k+1-j-\theta, j) \mathbf{Q}^h \mathbf{x}^h(k+1-j-\theta, j) + \mathbf{x}^{v^T}(k+1-j-\theta, j) \mathbf{Q}^v \tilde{\mathbf{x}}^v(k+1-j-\theta, j)\right] + \sum_{\theta=1}^{d_2} \left[\tilde{\mathbf{x}}^{h^T}(k+1-j, j-\theta) \tilde{\mathbf{Q}}^h \tilde{\mathbf{x}}^h(k+1-j, j-\theta) + \tilde{\mathbf{x}}^{v^T}(k+1-j, j-\theta) \tilde{\mathbf{Q}}^v \tilde{\mathbf{x}}^v(k+1-j, j-\theta)\right]\right]\right\}$$

$$\leq \alpha E\left\{\sum_{j=0}^{k+1} \alpha^j \left[\mathbf{x}^{h^T}(0, k+1-j) \mathbf{P}^h(r_{0, k+1-j}) \mathbf{x}^h(0, k+1-j) + \mathbf{x}^{v^T}(k+1-j, 0) \mathbf{P}^v(r_{k+1-j, 0}) \mathbf{x}^v(k+1-j, 0) + \tilde{\mathbf{x}}^{h^T}(0, k+1-j) \tilde{\mathbf{P}}^h(r_{0, k+1-j}) \tilde{\mathbf{x}}^h(0, k+1-j) + \tilde{\mathbf{x}}^{v^T}(k+1-j, 0) \tilde{\mathbf{P}}^v(r_{k+1-j, 0}) \tilde{\mathbf{x}}^v(k+1-j, 0) + \sum_{\theta=1}^{d_1} \left[\mathbf{x}^{h^T}(-\theta, k+1-j) \mathbf{Q}^h \mathbf{x}^h(-\theta, k+1-j) + \mathbf{x}^{v^T}(k+1-j, -\theta) \mathbf{Q}^v \tilde{\mathbf{x}}^v(k+1-j, -\theta)\right] + \sum_{\theta=1}^{d_2} \left[\tilde{\mathbf{x}}^{h^T}(-\theta, k+1-j) \tilde{\mathbf{Q}}^h \tilde{\mathbf{x}}^h(-\theta, k+1-j) + \tilde{\mathbf{x}}^{v^T}(k+1-j, -\theta) \tilde{\mathbf{Q}}^v \tilde{\mathbf{x}}^v(k+1-j, -\theta)\right]\right]\right\}$$

Denoting  $\beta = \frac{\max_{m \in S} \lambda_{\max}(\mathbf{P}_m + \mathbf{Q})}{\min_{m \in S} \lambda_{\min}(\mathbf{P}_m + \mathbf{Q})}$ . Based on  $\bar{\mathbf{x}}^T(i, j) \mathbf{P}_m \bar{\mathbf{x}}(i, j) \leq V(\bar{\mathbf{x}}(i, j), m)$ , we have:

$$\begin{aligned} & \mathbb{E} \left\{ \sum_{j=0}^{k+1} \left[ |\mathbf{x}_{k+1-j, j}^h|^2 + |\mathbf{x}_{k+1-j, j}^v|^2 + |\tilde{\mathbf{x}}_{k+1-j, j}^h|^2 + |\tilde{\mathbf{x}}_{k+1-j, j}^v|^2 \right] \right\} \\ & \leq \beta \sum_{j=0}^{k+1} \alpha^j \mathbb{E} \left[ |\mathbf{x}_{0, k+1-j}^h|^2 + |\mathbf{x}_{k+1-j, 0}^v|^2 + |\tilde{\mathbf{x}}_{0, k+1-j}^h|^2 + |\tilde{\mathbf{x}}_{k+1-j, 0}^v|^2 + \sum_{\theta=1}^{d_1} \left[ |\mathbf{x}_{-\theta, k+1-j}^h|^2 + |\tilde{\mathbf{x}}_{-\theta, k+1-j}^h|^2 \right] \right. \\ & \quad \left. + \sum_{\theta=1}^{d_2} \left[ |\mathbf{x}_{k+1-j, -\theta}^v|^2 + |\tilde{\mathbf{x}}_{k+1-j, -\theta}^v|^2 \right] \right] \end{aligned} \quad (6)$$

Command  $\chi_k = \sum_{j=0}^k \left[ |\mathbf{x}_{k-j, j}^h|^2 + |\mathbf{x}_{k-j, j}^v|^2 + |\tilde{\mathbf{x}}_{k-j, j}^h|^2 + |\tilde{\mathbf{x}}_{k-j, j}^v|^2 \right]$ , based on the inequality (6), sums up  $\chi_k$

from  $k = 0$  to  $k = S$  and perform some simple arithmetic yields:

$$\sum_{j=0}^S \mathbb{E} \{ \chi_k \} \leq \beta \frac{1 - \alpha^S}{1 - \alpha} \mathbb{E} \left\{ \sum_{k=0}^S \left[ |\mathbf{x}_{0, k}^h|^2 + |\mathbf{x}_{k, 0}^v|^2 + |\tilde{\mathbf{x}}_{0, k}^h|^2 + |\tilde{\mathbf{x}}_{k, 0}^v|^2 + \sum_{\theta=1}^{d_1} \left[ |\mathbf{x}_{-\theta, k}^h|^2 + |\tilde{\mathbf{x}}_{-\theta, k}^h|^2 \right] + \sum_{\theta=1}^{d_2} \left[ |\mathbf{x}_{k, -\theta}^v|^2 + |\tilde{\mathbf{x}}_{k, -\theta}^v|^2 \right] \right] \right\}.$$

Then, under Assumption 1, the right side of the above inequality is bounded, which means  $\lim_{k \rightarrow \infty} \mathbb{E} \{ \chi_k \} = 0$ , that is  $\lim_{i+j \rightarrow \infty} \mathbb{E} \{ |\bar{\mathbf{x}}_{i, j}|^2 \} = 0$ , then by Definition 1, the error system (3) is mean-square asymptotically stable.

Assume zero boundary condition and consider the following index:

$$J = \mathbb{E} \{ \mathbf{z}^T(i, j) \mathbf{z}(i, j) - \gamma^2 \mathbf{w}(i, j)^T \mathbf{w}(i, j) + \Delta V(\bar{\mathbf{x}}(i, j), m) \}$$

Define  $\Gamma(i, j) = \left[ \bar{\mathbf{x}}^T(i, j) \quad \bar{\mathbf{x}}_{d_1}^T(i, j) \quad \bar{\mathbf{x}}_{d_2}^T(i, j) \quad \mathbf{w}^T(i, j) \right]^T$ , puts the expression of  $\mathbf{z}(i, j)$  in  $J$  Then:

$$J = \mathbb{E} \{ \mathbf{z}^T(i, j) \mathbf{z}(i, j) - \gamma^2 \mathbf{w}(i, j)^T \mathbf{w}(i, j) + V(\bar{\mathbf{x}}(i, j), n) - V(\bar{\mathbf{x}}(i, j), m) \} = \Gamma^T(i, j) \bar{\Pi}_m \Gamma(i, j)$$

where 
$$\bar{\Pi}_m = \begin{bmatrix} -\mathbf{P}_m + \mathbf{Q} + \bar{\mathbf{A}}_m^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_m & \bar{\mathbf{A}}_m^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_{d1m} & \bar{\mathbf{A}}_m^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_{d2m} & \bar{\mathbf{A}}_m^T \bar{\mathbf{P}}_m \bar{\mathbf{B}}_m \\ * & -\bar{\mathbf{Q}}^h + \bar{\mathbf{A}}_{d1m}^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_{d1m} & \bar{\mathbf{A}}_{d1m}^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_{d2m} & \bar{\mathbf{A}}_{d1m}^T \bar{\mathbf{P}}_m \bar{\mathbf{B}}_m \\ * & * & -\bar{\mathbf{Q}}^v + \bar{\mathbf{A}}_{d2m}^T \bar{\mathbf{P}}_m \bar{\mathbf{A}}_{d2m} & \bar{\mathbf{A}}_{d2m}^T \bar{\mathbf{P}}_m \bar{\mathbf{B}}_m \\ * & * & * & -\gamma^2 \mathbf{I} + \bar{\mathbf{B}}_m^T \bar{\mathbf{P}}_m \bar{\mathbf{B}}_m \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{C}}_m^T \\ \bar{\mathbf{C}}_{d1m}^T \\ \bar{\mathbf{C}}_{d2m}^T \\ \bar{\mathbf{D}}_m^T \end{bmatrix} \begin{bmatrix} \bar{\mathbf{C}}_m^T \\ \bar{\mathbf{C}}_{d1m}^T \\ \bar{\mathbf{C}}_{d2m}^T \\ \bar{\mathbf{D}}_m^T \end{bmatrix} < 0.$$

Using Schur complement, the inequality (5) guarantees  $\bar{\Pi}_m < 0$ . Then we have  $J < 0$ , which means for every  $r(i, j) = m \in S$  we have:

$$\mathbb{E} \{ V(\bar{\mathbf{x}}(i, j), n) - V(\bar{\mathbf{x}}(i, j), m) \} < \mathbb{E} \{ V(\bar{\mathbf{x}}(i, j), m) - \mathbf{z}^T(i, j) \mathbf{z}(i, j) + \gamma^2 \mathbf{w}(i, j)^T \mathbf{w}(i, j) \}.$$

Considering the zero boundary condition, by using this relationship iteratively and performing superposition of the two sides of inequalities from  $j = 0$  to  $j = k + 1$ , we can get:

$$\begin{aligned} & \mathbb{E} \left\{ \sum_{j=0}^{k+1} \left[ |\mathbf{x}_{k+1-j, j}^h|^2 + |\mathbf{x}_{k+1-j, j}^v|^2 + |\tilde{\mathbf{x}}_{k+1-j, j}^h|^2 + |\tilde{\mathbf{x}}_{k+1-j, j}^v|^2 \right] \right\} \leq \beta \sum_{j=0}^k \alpha^j \mathbb{E} \left[ |\mathbf{x}_{0, k-j}^h|^2 + |\mathbf{x}_{k-j, 0}^v|^2 + |\tilde{\mathbf{x}}_{0, k-j}^h|^2 + |\tilde{\mathbf{x}}_{k-j, 0}^v|^2 + \sum_{\theta=1}^{d_1} \left[ |\mathbf{x}_{-\theta, k-j}^h|^2 + |\tilde{\mathbf{x}}_{-\theta, k-j}^h|^2 \right] \right. \\ & \quad \left. + \sum_{\theta=1}^{d_2} \left[ |\mathbf{x}_{k-j, -\theta}^v|^2 + |\tilde{\mathbf{x}}_{k-j, -\theta}^v|^2 \right] - \mathbf{z}^T(k-j, j) \mathbf{z}(k-j, j) + \gamma^2 \mathbf{w}(k-j, j)^T \mathbf{w}(k-j, j) \right] \end{aligned}$$

Summing up the two sides of the above inequalities from  $k = 0$  to  $k = S$ , we have:

$$\mathbb{E} \left\{ \sum_{k=0}^S \sum_{j=0}^k \mathbf{z}^T(k-j, j) \mathbf{z}(k-j, j) \right\} < \mathbb{E} \left\{ \sum_{k=0}^S \sum_{j=0}^k \gamma^2 \mathbf{w}(k-j, j)^T \mathbf{w}(k-j, j) \right\} - \mathbb{E} \left\{ \sum_{j=0}^{S+1} \left[ \bar{\mathbf{x}}_{S+1-j, j}^T \mathbf{P}(r_{S+1-j, j}) \bar{\mathbf{x}}_{S+1-j, j} \right] \right\},$$

Therefore, when  $k \rightarrow \infty$  we have:

$$\mathbb{E} \left\{ \sum_{k=0}^{\infty} \sum_{j=0}^k \mathbf{z}^T(k-j, j) \mathbf{z}(k-j, j) \right\} < \mathbb{E} \left\{ \sum_{k=0}^{\infty} \sum_{j=0}^k \gamma^2 \mathbf{w}(k-j, j)^T \mathbf{w}(k-j, j) \right\}.$$

That is,  $\| \mathbf{z} \|_2^2 < \gamma^2 \| \mathbf{w} \|_2^2$  for all non-zero  $\mathbf{w} = \{w_{ij}\} \in L_2 \{[0, \infty), [0, \infty)\}$ , and the proof is concluded.  $\blacklozenge$

**Theorem 2:** Consider the mean-square asymptotically stable 2D jump system (1). Given a constant  $\gamma > 0$ , there exists a reduced  $\tilde{n}$ th-order system (2) solves the  $H_\infty$  model reduction problem if there exists a block-diagonal matrix  $\mathbf{X} = (\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2, \dots, \bar{\mathbf{X}}_S)$  with  $\bar{\mathbf{X}}_m > 0$ ,  $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_S)$  with  $\mathbf{P}_m > 0$ ,  $\mathbf{P}_m = \text{diag}\{\mathbf{P}_m^h, \mathbf{P}_m^v, \tilde{\mathbf{P}}_m^h, \tilde{\mathbf{P}}_m^v\}$ ,  $\mathbf{Q} = \text{diag}\{\mathbf{Q}^h, \mathbf{Q}^v, \tilde{\mathbf{Q}}^h, \tilde{\mathbf{Q}}^v\}$ ,  $m \in S$ , such that the following linear matrix inequality holds for

$$\begin{bmatrix} -E\bar{X}_m E^T & EA_{0m} & EA_{d10m} & EA_{d20m} & EB_{0m} \\ * & -P_m + Q & 0 & 0 & 0 \\ * & * & -\bar{Q}^h & 0 & 0 \\ * & * & * & -\bar{Q}^v & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0, \begin{bmatrix} -\bar{X}_m & A_{0m} & A_{d10m} & A_{d20m} & 0 \\ * & -P_m + Q & 0 & 0 & C_{0m}^T \\ * & * & -\bar{Q}^h & 0 & C_{d10m}^T \\ * & * & * & -\bar{Q}^v & C_{d20m}^T \\ * & * & * & * & -I \end{bmatrix} < 0, \bar{X}_m \sum_{n=1}^s p_{mn} P_n = I$$

Furthermore, if matrix  $\bar{X}_m$ ,  $P_m$ ,  $Q$  are the solutions of above inequalities, then a reduced order model can be written as:

$$G_m = -R_m^{-1} \Psi_m^T A_m \Upsilon_m^T (\Upsilon_m A_m \Upsilon_m^T)^{-1} + R_m^{-1} V_m^{1/2} L_m (\Upsilon_m A_m \Upsilon_m^T)^{-1/2}, A_m = (\Psi_m R_m^{-1} \Psi_m^T - Q_m)^{-1}, V_m = R_m - \Psi_m^T (A_m - A_m \Upsilon_m^T (\Upsilon_m A_m \Upsilon_m^T)^{-1} \Upsilon_m A_m) \Psi_m \quad (7)$$

where  $R_m > 0$  such that  $A_m > 0$  and  $L_m$  is any matrix satisfying  $\|L_m\| < 1$ , and

$$A_{0m} = \begin{bmatrix} A_m & 0 \\ 0 & 0 \end{bmatrix}, A_{d10m} = \begin{bmatrix} A_{d1m} & 0 \\ 0 & 0 \end{bmatrix}, A_{d20m} = \begin{bmatrix} A_{d2m} & 0 \\ 0 & 0 \end{bmatrix}, B_{0m} = \begin{bmatrix} B_m \\ 0 \end{bmatrix}, G_m = \begin{bmatrix} \tilde{D}_m & \tilde{C}_m & \tilde{C}_{d1m} & \tilde{C}_{d2m} \\ \tilde{B}_m & \tilde{A}_m & \tilde{A}_{d1m} & \tilde{A}_{d2m} \end{bmatrix},$$

$$C_{0m} = [C_m \ 0], C_{d10m} = [C_{d1m} \ 0], C_{d20m} = [C_{d2m} \ 0], D_{0m} = D_m, \Upsilon_m = [0 \ H \ M \ W \ N \ 0]$$

$$\Omega_m = \begin{bmatrix} -\bar{P}_m & \bar{P}_m A_{0m} & \bar{P}_m A_{d10m} & \bar{P}_m A_{d20m} & \bar{P}_m B_{0m} & 0 \\ * & -P_m + Q & 0 & 0 & 0 & C_{0m}^T \\ * & * & -\bar{Q}^h & 0 & 0 & C_{d10m}^T \\ * & * & * & -\bar{Q}^v & 0 & C_{d20m}^T \\ * & * & * & * & -\gamma^2 I & D_{0m}^T \\ * & * & * & * & * & -I \end{bmatrix}, \Psi_m = \begin{bmatrix} \bar{P}_m F \\ 0 \\ 0 \\ 0 \\ 0 \\ S \end{bmatrix}, H = \begin{bmatrix} 0_{m,n} & 0_{m,\bar{n}} \\ 0_{\bar{n},n} & I_{\bar{n}} \\ 0_{\bar{n}_1,n} & 0_{\bar{n}_1,\bar{n}} \\ 0_{\bar{n}_2,n} & 0_{\bar{n}_2,\bar{n}} \end{bmatrix}, M = \begin{bmatrix} 0_{m,n_1} & 0_{m,\bar{n}_1} \\ 0_{\bar{n}_1,n_1} & 0_{\bar{n}_1,\bar{n}_1} \\ 0_{\bar{n}_1,n_1} & I_{\bar{n}_1} \\ 0_{\bar{n}_2,n_1} & 0_{\bar{n}_2,\bar{n}_1} \end{bmatrix},$$

$$W = \begin{bmatrix} 0_{m,n_2} & 0_{m,\bar{n}_2} \\ 0_{\bar{n}_1,n_2} & 0_{\bar{n}_1,\bar{n}_2} \\ 0_{\bar{n}_1,n_2} & 0_{\bar{n}_1,\bar{n}_2} \\ 0_{\bar{n}_2,n_2} & I_{\bar{n}_2} \end{bmatrix}, N = \begin{bmatrix} I_m \\ 0_{\bar{n},m} \\ 0_{\bar{n}_1,m} \\ 0_{\bar{n}_2,m} \end{bmatrix}, F = \begin{bmatrix} 0_{n,l} & 0_{n,\bar{n}} \\ 0_{\bar{n},l} & I_{\bar{n}} \end{bmatrix}, S = [-I_l \ 0_{l,\bar{n}}] \quad (8)$$

**Proof:** The proof is omitted because of the limited space. ◆

It should be noted that the obtained conditions in Theorem 2 are not LMI conditions due to the equations. However, with the result of a cone complementarily linearization algorithm, we can solve this feasibility problem by formulating it into a linear optimization problem subject to LMI constraints.

### Model reduction algorithm

For  $X = (\bar{X}_1, \bar{X}_2, \dots, \bar{X}_s)$  with  $\bar{X}_m > 0$  and  $P = (P_1, P_2, \dots, P_s)$  with  $P_m > 0$ ,  $m \in S$ , define a convex set by a set of LMIs as:

$$\xi_{(X,P)}^d = \{(X, P): \bar{X}_m > 0, P_m > 0, \text{ for all } m \in S \}$$

It can be seen from Theorem 2, the  $H_\infty$  reduced order models for 2D jump linear systems (1) can be obtained if there exist  $X = (\bar{X}_1, \bar{X}_2, \dots, \bar{X}_s)$  and  $P = (P_1, P_2, \dots, P_s)$  such that:

$$(X, P) \in \xi_{(X,P)}^d, \bar{X}_m \sum_{n=1}^s p_{mn} P_n = I, \text{ for all } m \in S \quad (9)$$

is feasible.

The CCL algorithm is based on the fact that for any matrices  $X > 0$  and  $P > 0$ ; if the LMI  $\begin{bmatrix} P & I \\ I & X \end{bmatrix} \geq 0$  is feasible, then  $\text{Trace}(XP) \geq n$ , and  $\text{Trace}(XP) = n$  if and only if  $XP = I$ . Hence, a feasible solution of (9) can be obtained from the solution of the following nonconvex optimization problem.

We may see that if the optimal solution satisfying  $\sum_{n=1}^s p_{mn} \text{Trace}(\bar{X}_m P_n) = n + \bar{n}$  then (9) is solved;

Hence, the  $H_\infty$  model reduction problem is now changed to a problem of finding a global solution of the minimization problem. Although it is still not possible to always find the global optimal

solution, the proposed nonlinear minimization problem is easier to solve by CCL algorithm than the original nonconvex feasibility problem.

With the above expressions, the following algorithm is proposed to solve the 2D jump system  $H_\infty$  model reduction problem:

**Step1:** Choose the initial values for the matrix pair  $(P_0, X_0)$ , the order of the reduced-order  $\tilde{n}$  and the  $H_\infty$  norm bound  $\gamma$ ;

**Step2:** Define the linear function

$$f_k(X, P) = \sum_{m=1}^s \sum_{n=1}^s p_{mn} \text{Trace}(\bar{X}_{mk} P_n + P_{nk} \bar{X}_m);$$

**Step3:** Find  $(X_{k+1}, P_{k+1})$  solving the following convex programming:

$$\min_{(X, P) \in \mathcal{C}(X, P_n)} \{f_k(X, P)\};$$

**Step4:** If  $f_k$  converges, then exit; otherwise, set  $k = k + 1$ ; and go to step 2.

**Step5:** Construct a reduced-order model based on (7).

It can be seen that step 1 is a simple LMI problem, and step 3 is a convex programming with LMI constraints. From the explanation in [7],  $f_k$  is decreasing and bounded below by  $2s(n + \tilde{n})$ , which implies that the  $H_\infty$  model reduction problem is solvable for a given  $\gamma > 0$ .

### Numerical example

For mode 1, the system matrices are given by:

$$A_1 = \begin{bmatrix} 0.5 & 0.01 & 0.01 & 0 \\ 0 & 0.6 & 0 & 0.01 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.4 \end{bmatrix}, A_{d11} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, A_{d21} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \\ 0.02 & 0 \\ 0 & 0.03 \end{bmatrix}, B_1 = \begin{bmatrix} 0.1 \\ 0.7 \\ 1.3 \\ 0.5 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1.2 & 0.4 & 0.6 & 0.9 \\ 0.4 & 0.5 & 0.6 & 0.1 \end{bmatrix}, C_{d11} = \begin{bmatrix} 0.12 & 0.04 \\ 0.04 & 0.05 \end{bmatrix}, C_{d21} = \begin{bmatrix} 0.06 & 0.09 \\ 0.06 & 0.01 \end{bmatrix}, D_1 = \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix}.$$

For mode 2, the system matrices are given by:

$$A_2 = \begin{bmatrix} -0.3 & 0.01 & 0 & 0 \\ 0 & -0.7 & 0 & 0.02 \\ 0.03 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & -0.4 \end{bmatrix}, A_{d12} = \begin{bmatrix} 0.01 & 0.01 \\ 0 & -0.02 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, A_{d22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.02 & 0 \\ 0 & -0.01 \end{bmatrix}, B_2 = \begin{bmatrix} 0.3 \\ 0.7 \\ 1.2 \\ 0.1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1.1 & 0.5 & 0.7 & 1.9 \\ 0.1 & 0.3 & 0.4 & 0.4 \end{bmatrix}, C_{d12} = \begin{bmatrix} 0.012 & 0.004 \\ 0.004 & 0.005 \end{bmatrix}, C_{d22} = \begin{bmatrix} 0.006 & 0.009 \\ 0.006 & 0.001 \end{bmatrix}, D_2 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}.$$

Applying the model reduction algorithm mentioned above, making  $\gamma=5.8943$ ,  $L_1 = L_2 = [L_{11} \ L_{12}]$ , where:  $L_{11} = I_5$ ,  $L_{12} = [I_2 \ I_2 \ [0 \ 1]^T]^T$ , we can figure out the following results:

$$R_1 = \begin{bmatrix} 0.1057 & -0.1216 & 0.0002 & 0.0000 & 0.0009 \\ -0.1216 & 0.2952 & 0.0001 & -0.0003 & 0.0026 \\ 0.0002 & 0.0001 & 0.4247 & -0.0000 & -0.0023 \\ 0.0000 & -0.0003 & -0.0000 & 0.4247 & -0.0000 \\ 0.0009 & 0.0026 & -0.0023 & -0.0000 & 0.4250 \end{bmatrix}, R_2 = \begin{bmatrix} 7.7719 & 0.0625 & -0.0001 & 0.0010 & -0.0000 \\ 0.0625 & 7.5170 & 0.0008 & -0.0001 & 0.0013 \\ -0.0001 & 0.0008 & 5.5630 & 0.0002 & -0.1084 \\ 0.0010 & -0.0001 & 0.0002 & 5.5634 & 0.0014 \\ -0.0000 & 0.0013 & -0.1084 & 0.0014 & 5.5616 \end{bmatrix}$$

Using the algorithms to solve the non-convex feasibility problem in theorem 2, reduced-order system model parameters can be calculated:

$$\tilde{A}_1 = \begin{bmatrix} 0.5186 & -0.0285 & -0.0031 \\ 0.0046 & 0.1051 & 0.0041 \\ -0.0016 & 0.0001 & 0.1351 \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} -0.2509 & -0.0249 & 0.0003 \\ 0.7945 & 0.1477 & 0.0045 \\ -0.0487 & 0.0005 & 0.1659 \end{bmatrix}, \tilde{A}_{d11} = \begin{bmatrix} 0.0060 & 0.0002 \\ -0.0128 & 0.0005 \\ 0.0002 & -0.0003 \end{bmatrix},$$

$$\begin{aligned} \tilde{A}_{d21} &= \begin{bmatrix} 0.0626 \\ -0.0299 \\ 0.1335 \end{bmatrix}, \quad \tilde{A}_{d22} = \begin{bmatrix} -0.0013 & 0.0005 \\ -0.0045 & -0.0028 \\ 0.0001 & -0.0016 \end{bmatrix}, \quad \tilde{A}_{d23} = \begin{bmatrix} 0.0688 \\ -0.0302 \\ 0.2063 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} -0.0288 \\ 0.1075 \\ -0.004 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} -0.0306 \\ 0.1820 \\ -0.0010 \end{bmatrix}, \\ \tilde{C}_{d11} &= \begin{bmatrix} 0.1342 & -0.0000 \\ 0.0004 & 0.1336 \end{bmatrix}, \quad \tilde{C}_1 = \begin{bmatrix} 0.0079 & 0.0031 & 0.0004 \\ -0.0035 & 0.0004 & -0.0005 \end{bmatrix}, \quad \tilde{C}_{d12} = \begin{bmatrix} 0.0000 \\ -0.0002 \end{bmatrix}, \quad \tilde{C}_{d22} = \begin{bmatrix} -0.0001 \\ -0.0012 \end{bmatrix}, \\ \tilde{D}_2 &= \begin{bmatrix} 0.2064 \\ 0.2038 \end{bmatrix}, \quad \tilde{C}_2 = \begin{bmatrix} 0.0391 & 0.0009 & -0.0001 \\ -0.0810 & -0.0016 & -0.0012 \end{bmatrix}, \quad \tilde{C}_{d12} = \begin{bmatrix} 0.1662 & 0.0001 \\ -0.0008 & 0.2061 \end{bmatrix}, \quad \tilde{D}_1 = \begin{bmatrix} 0.1333 \\ 0.1338 \end{bmatrix}. \end{aligned}$$

The simulation results imply that the desired goal is well achieved.

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