On Path Partitions and Colourings in Digraphs

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Abstract. We provide a new proof of a theorem of Saks which is an extension of Greene's Theorem to acyclic digraphs, by reducing it to a similar, known extension of Greene and Kleitman's Theorem. This suggests that the Greene-Kleitman Theorem is stronger than Greene's Theorem on posets. We leave it as an open question whether the same holds for all digraphs, that is, does Berge's conjecture concerning path partitions in digraphs imply the extension of Greene's theorem to all digraphs (conjectured by Aharoni, Hartman and Hoffman)?

1 Introduction

Dilworth's well known theorem [7] states that in a partially ordered set the size of a maximum antichain equals the size of a minimum chain partition. Greene and Kleitman [13] generalized Dilworth's theorem to a min-max theorem for the maximum cardinality of the union of k antichains ($k \in \mathbb{N}$). Previously, Greene [12] had proved a similar min-max theorem where the role of chains and antichains is interchanged. Linial [15] conjectured that the theorems of Greene-Kleitman and Greene can be extended to all digraphs by replacing the equality by an inequality. Later, Berge [3] made a stronger conjecture than Linial's extending the Greene-Kleitman theorem to all digraphs, and Aharoni, Hartman and Hoffman (AHH) [1] made a similar conjecture which extends Greene's theorem to all digraphs, and is stronger than Linial's conjecture. Both conjectures of Berge and Aharoni-Hartman-Hoffman were proved for all acyclic digraphs (see [15], [6],[17], [5] and [1]). For $k = 1$ Berge's conjecture holds by the Gallai-Milgram theorem [11], and the Aharoni-Hartman-Hoffman conjecture holds by the Gallai-Roy theorem [10,16]. Recently, Berger and Hartman [4] proved Berge's conjecture for $k = 2$. For other values of k (except for the extreme upper values), all of the conjectures mentioned above are open. For a survey of the subject see [14].

The purpose of this paper is reduce the Aharoni-Hartman-Hoffman conjecture for acyclic digraphs to Berge's conjecture. Furthermore, a polynomial algorithm is given proving the AHH Conjecture, based on an oracle for Berge's Conjecture. If the same holds for all digraphs, then it will be sufficient to prove Berge's conjecture, and the rest of the conjectures will follow.

2 Preliminaries and the Main Result

Let $G = (V, E)$ be a directed graph and let $|V| = n$. If L is a collection of subsets of V, we set $\bigcup L := \{x; x \in A \text{ for some } A \in L\}$. The cardinality of the set X is

denoted by $|X|$.
A *path* P in G is a sequence of distinct vertices $(v_1, v_2, \dots, v_l)$ such that $(v_i, v_{i+1}) \in E$, for $i = 1, 2, \dots, l-1$. Let the *cardinality* of P be $|P| = l$. If a path P is of cardinality one, then we say it is *trivial*.
For positive integers q, k, a *q-path system* is a family $\mathcal{P}^q := \{P_1, P_2, \dots, P_q\}$ of q pairwise disjoint paths, a *k-colouring* is a family $\mathcal{C}^k := \{C_1, C_2, \dots, C_k\}$ of k pairwise disjoint independent sets, also called *colour classes*.

Denote $\lambda_q := \max |\bigcup \mathcal{P}^q|$ and $\alpha_k := \max |\bigcup \mathcal{C}^k|$ where the maximum is taken over all q-path systems and k-colourings, respectively, and $|\bigcup \mathcal{P}^q|$ ($|\bigcup \mathcal{C}^k|$) denote the number of vertices covered by $\mathcal{P}^q$ ($\mathcal{C}^k$). A q-path system with $|\bigcup \mathcal{P}^q| = \lambda_q$ is called *optimal*.

A family $\mathcal{P}$ of paths is called a *path partition* of G if all its members are pairwise disjoint, and $\cup \mathcal{P} = V$. The k-norm $|\mathcal{P}|_k$ of a path partition $\mathcal{P} = \{P_1, \dots, P_m\}$ is defined by $|\mathcal{P}|_k := \sum_{i=1}^{m} min\{|P_i|, k\}$. Denote by $\mathcal{P}^0$ the set of trivial paths in $\mathcal{P}$, and by $\mathcal{P}^{\geq k}$ ($\mathcal{P}^{>k}$) the sets of paths in $\mathcal{P}$ of cardinality at least (more than) k. A partition which minimizes $|\mathcal{P}|_k$ is called *k-optimal*. Let $\pi_k(G) = min|\mathcal{P}|_k$ where the minimum is taken over all possible path partitions in G. If k is the cardinality of the smallest non-trivial path on $\mathcal{P}$, then clearly, $|\mathcal{P}|_k = k|\mathcal{P}^{\geq k}| + |\mathcal{P}^0|$.

Similarly, a *colouring* $\mathcal{C}$ is a family of pairwise disjoint independent sets where $\cup \mathcal{C} = V$. The q-norm $|\mathcal{C}|_q$ of a colouring $\mathcal{C} = \{C_1, \dots, C_m\}$ is defined by $|\mathcal{C}|_q := \sum_{i=1}^{m} min\{|C_i|, q\}$. Let $\chi_q(G) = min|\mathcal{C}|_q$ where the minimum is taken over all possible colourings in G.

Theorem 1 (Greene-Kleitman[13]) *If G is a graph of a poset, then $\alpha_k(G) = \pi_k(G)$ for all $1 \leq k \leq \lambda_1$.*

Theorem 2 (Greene[12]) *If G is a graph of a poset, then $\lambda_q(G) = \chi_q(G)$ for all $1 \leq q \leq \alpha_1$.*

The inequality $\alpha_k(G) \leq \pi_k(G)$ is trivial since any path P (chain) in a poset induces a clique which can meet a k-colouring by at most $\min\{|P_i|, k\}$ vertices. Similarly, every independent set (antichain) C in a poset can meet a q-path system by at most $\min\{|C_i|, q\}$ vertices, implying the inequality $\lambda_q(G) \leq \chi_q(G)$. The other direction of the inequalities is less trivial and was conjectured by Linial to be true for all digraphs:

Conjecture 3 (Linial[15]) *Let G be a digraph, and k, q positive integers. Then*

1. $\alpha_k(G) \geq \pi_k(G)$
2. $\lambda_q(G) \geq \chi_q(G)$

For any graph G, a k-colouring $\mathcal{C}^k$ is *orthogonal* to a path partition if each $P_i \in \mathcal{P}$ meets exactly $min\{|P_i|, k\}$ different colour classes in $\mathcal{C}^k$. Similarly, a colouring $\mathcal{C}$ is *orthogonal* to a q-path system $\mathcal{P}^q$ if each $C_i \in \mathcal{C}$ meets exactly $min\{|C_i|, q\}$ different paths in $\mathcal{P}^q$.

Conjecture 4 (Berge's Strong Path Partition Conjecture [3]) *Let G be a digraph, k a positive integer, and $\mathcal{P}$ a k-optimal path partition. Then there exists a k-colouring $\mathcal{C}^k$ orthogonal to $\mathcal{P}$.*

This conjecture implies Conjecture 3-(1). The following is an equivalent conjecture to Conjecture 4:

Conjecture 5 (Equivalent to Conjecture 4) *Let G be a digraph, k a positive integer, and let $\mathcal{P}$ be some path partition in G. Then either there exists a k-colouring $\mathcal{C}^k$ orthogonal to $\mathcal{P}$, or there exists a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_k < |\mathcal{P}|_k$*

The following conjecture extends Greene's Theorem to all digraphs and implies part 2 of Conjecture 3, in a similar way that Berge's Conjecture extends the Greene-Kleitman Theorem:

Conjecture 6 (Aharoni, Hartman, Hoffman (AHH) [1]) *Let G be a digraph, q a positive integer, and $\mathcal{P}^q$ an optimal q-path system. Then there exists a colouring $\mathcal{C}$ orthogonal to $\mathcal{P}^q$.*

Conjecture 7 (Equivalent to Conjecture 6) *Let G be a digraph, q a positive integer, and $\mathcal{P}^q$ some q-path system in G. Then either there exists a colouring $\mathcal{C}$ orthogonal to $\mathcal{P}^q$, or there exists a q-path system $\mathcal{P}'^q$ with $|\mathcal{P}'^q| > |\mathcal{P}^q|$.*

Conjecture 4 implies Conjecture 3-(1) and it holds for $k = 1$ by the Gallai-Milgram [11] theorem. Conjecture 6 implies Conjecture 3-(2) and it holds for $q = 1$ by the Gallai-Roy [10,16] theorem.

The following definition of Frank [9] helps us in uniting all Conjectures 3-6.

Definition 8 *A q-path system $\mathcal{P}^q = \{P_1, P_2, \ldots, P_q\}$ and a k-colouring $\mathcal{C}^k = \{C_1, C_2, \ldots, C_k\}$ are* orthogonal *if*

1. $V = (\cup\mathcal{P}^q) \bigcup (\cup\mathcal{C}^k)$
2. $|P_i \cap C_j| = 1$ *for* $1 \le i \le q, 1 \le j \le k$

For a q-path system $\mathcal{P}^q$, the *associated path partition* $\mathcal{P}$ is defined by $\mathcal{P} := \mathcal{P}^q \cup \{\{x\}; x \notin \cup\mathcal{P}^q\}$. Similarly, the *associated colouring* to a k-colouring $\mathcal{C}^k$ is defined by $\mathcal{C} := \mathcal{C}^k \cup \{\{x\}; x \notin \cup\mathcal{C}^k\}$.

Observation 9
1. *Let $\mathcal{P}^q$ be a q-path system orthogonal to $\mathcal{C}^k$, a k-colouring, for some integers q and k. Then $\mathcal{C}^k$ is orthogonal to the associated path partition $\mathcal{P} := \mathcal{P}^q \cup \{\{x\}; x \notin \cup\mathcal{P}^q\}$ and the associated colouring $\mathcal{C} := \mathcal{C}^k \cup \{\{x\}; x \notin \cup\mathcal{C}^k\}$ is orthogonal to $\mathcal{P}^q$.*
2. *Conversely, if $\mathcal{C}^k$ is orthogonal to some path partition $\mathcal{P}$, then $\mathcal{C}^k$ is orthogonal to $\mathcal{P}^q := \mathcal{P}^{\ge k}$. Note that $\mathcal{C}^k$ is also orthogonal to $\mathcal{P}^q$ where $\mathcal{P}^q$ consists of paths of cardinality more than k, and any number of paths of cardinality exactly k are included in $\mathcal{P}^q$.*
3. *Similarly, if a colouring $\mathcal{C}$ is orthogonal to some q-path system $\mathcal{P}^q$, then $\mathcal{P}^q$ is orthogonal to the k-colouring $\mathcal{C}^k := \mathcal{C}^{\ge q}$, where $\mathcal{C}^{\ge q}$ denotes the set of independent sets in $\mathcal{C}$ of size at least q.*

4. *Furthermore, if $\mathcal{P}^q$ and $\mathcal{C}^k$ are orthogonal then $\alpha_k(G) \geq \pi_k(G)$ and $\lambda_q(G) \geq \chi_q(G)$, implying Linial's conjectures for these values of k and q.*

Theorem 10 *Conjecture 7 can be reduced to Conjecture 5 for acyclic digraphs.*

Corollary 11 *Theorem 1 implies Theorem 2, i.e. Greene-Kleitman's Theorem implies Greene's Theorem.*

Proof of Corollary: Assume Theorem 1 holds. Then any optimal k-colouring $\mathcal{C}^k$ in a graph of a poset must be orthogonal to an optimal path partition $\mathcal{P}$ because in a poset each $P \in \mathcal{P}$ can meet at most $\min\{|P|, k\}$ vertices from $\mathcal{C}^k$. If some P would meet less than $\min\{|P|, k\}$ vertices from $\mathcal{C}^k$, we would have $\alpha_k(G) < \pi_k(G)$, contrary to Theorem 1. Hence Conjecture 4 holds for posets, implying by Theorem 10 that Conjecture 6 holds, and hence Greene's Theorem (Theorem 2) follows.

3 Proof of Theorem 10

3.1 Outline of Proof:

We assume that Conjecture 5 is true. Given any path partition $\mathcal{P}$, and positive integer k, we assume that we have some Oracle that either finds a k-colouring $\mathcal{C}^k$ orthogonal to $\mathcal{P}$, or finds a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_k < |\mathcal{P}|_k$. Let $q \geq 1$, and let $\mathcal{P}^q$ be a q-path system. We will show that Conjecture 7 holds for $\mathcal{P}^q$. Let $\mathcal{P}$ be the path partition associated with $\mathcal{P}^q$. We prove that either there exists a k, $1 \leq k \leq \min_{P \in \mathcal{P}^q} |P|$, and a k- colouring $\mathcal{C}^k$ orthogonal to $\mathcal{P}$, implying by Observation 9-(1) and (2) that $\mathcal{C} := \mathcal{C}^k \cup \{\{x\}; x \notin \cup \mathcal{C}^k\}$ is orthogonal to $\mathcal{P}^q$, or we find another q-path system $\mathcal{P}'^q$ with $|\mathcal{P}'^q| > |\mathcal{P}^q|$.

The proof is algorithmic and it uses a network constructed from G as in Frank [9]. We define a flow f which corresponds to the path partition $\mathcal{P}$ associated with $\mathcal{P}^q$. We begin with $k = 1$. If $\mathcal{P}$ is 1-optimal, then by the Gallai -Milgram Theorem (or Conjecture 4 for $k = 1$) there exists an independent set $\mathcal{C}^1$ orthogonal to it, implying that $\mathcal{C} := \mathcal{C}^1 \cup \{\{x\}; x \notin \cup \mathcal{C}^k\}$ is orthogonal to $\mathcal{P}^q$ (by Observation 9) and we are done. Otherwise, $\mathcal{P}$ is not 1-optimal, and the Oracle finds a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_1 < |\mathcal{P}|_1$. Let f' be the flow corresponding to $\mathcal{P}'$. Then $f' - f$ is a feasible flow in the residual network $N_{f'}$ corresponding to f'. Depending on f', we either increase k by one, or we show that $f' - f$ can be used to find a new flow f'' which satisfies two main conditions:

1. f'' corresponds to a path partition $\mathcal{P}''$ in G.
2. $\mathcal{P}''$ contains a q-path system which covers more vertices than $\mathcal{P}^q$, contradicting the optimality of $\mathcal{P}^q$.

We prove that if no k-colouring $\mathcal{C}^k$ exists which is orthogonal to $\mathcal{P}$, for all $1 \leq k \leq \min_{P \in \mathcal{P}^q} |P|$, then a new flow is found, yielding a k-path which covers more vertices than $\mathcal{P}^q$. This will imply Conjecture 7.

In general digraphs, the subgraph $\mathcal{P}''$, corresponding to f'', may contain cycles and the proof may fail.

3.2 Details of Proof:

We proceed to define the network N, the flow f corresponding to a path partition $\mathcal{P}$, the residual network N_f corresponding to f, and the criterion for either increasing k, or finding a flow f'' which contradicts the optimality of $\mathcal{P}^q$.

The network We describe the network N as in [9]. Assume $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$, and $k \geq 1$. Associate a network $N = (\overline{V}, \overline{E}, a, c, s, t)$ with G as follows:

Let $\overline{V} = \{s, t, v'_1, v'_2, \ldots, v'_n, v''_1, v''_2, \ldots, v''_n\}$, $\overline{E} = \{(s, v'_i); i = 1, 2, \ldots, n\} \cup \{(v''_i, t); i = 1, 2, \ldots, n\} \cup \{(v'_i, v''_j); (v_i, v_j) \in E\} \cup \{(v'_i, v''_i); 1 \leq i \leq n\} \cup \{(s, t)\}$.

All of the arc capacities $c(e)$ are equal to one, while the costs $a(e)$ are:

$$a(e) = \begin{cases} 1 \text{ if } e = (v'_i, v''_i) \\ k \text{ if } e = (s, t) \\ 0 \text{ otherwise} \end{cases}$$

We denote the value of a feasible flow f by $\mathrm{val}(f)$ and its cost by $\mathrm{cost}(f)$.

Path partitions and flows Since all the capacities are one we may assume that a feasible flow in N is integral. We define a *full* feasible flow as a flow which satisfies that for each $v_i \in V(G)$, at least one of the edges (s, v'_i) or (v''_i, t) has non-zero flow. For example, a maximal flow (i.e. a flow f s.t. there exists no other flow f', $f \leq f'$) is a full flow.

Assume we have a full feasible flow f in N of value v. We associate with it a partition $\mathcal{P} = \mathcal{P}(f)$ of $V(G)$ into paths defined as follows: If $f(v'_i, v''_j) = 1$, $i \neq j$, then $(v_i, v_j) \in E[\mathcal{P}]$, and if $f(v'_i, v''_i) = 1$ then (v_i) is a trivial path in $\mathcal{P}$. Since f is full each vertex in $V(G)$ is covered by $\mathcal{P}$, and since all the capacities are one, $\mathcal{P}$ is indeed a collection of disjoint paths. If G is not acyclic, then $\mathcal{P}$ is a collection of disjoint paths and cycles.

If $\mathcal{P} = \mathcal{P}(f)$ is a path partition, and k is less than or equal to the smallest non-trivial path in $\mathcal{P}$, then

$$|\mathcal{P}|_k = k|\mathcal{P}^{\geq k}| + |\mathcal{P}^0| = k|\mathcal{P}^{>1}| + |\mathcal{P}^0| = k(n - \mathrm{val}(f)) + \mathrm{cost}(f) \qquad (1)$$

Conversely, given a path partition $\mathcal{P}$ in G, we associate with it the flow $f := f_{\mathcal{P}}$ defined by: If $(v_i) \in \mathcal{P}^0$ then $f(s, v'_i) = f(v'_i, v''_i) = f(v''_i, t) = 1$. For each $(v_i, v_j) \in E[\mathcal{P}]$ define $f(s, v'_i) = f(v'_i, v''_j) = f(v''_j, t) = 1$. The flow in all other edges is defined as zero. It is easy to check that $f_{\mathcal{P}}$ is a full feasible flow with $\mathrm{val}(f) = n - |\mathcal{P}^{>1}|$, and $\mathrm{cost}(f) = |\mathcal{P}^0|$. If $k \leq \min_{P \in \mathcal{P}^{>1}} |P|$, then the equation in formula (1) holds.

The residual network For a given flow f in N, the *residual network* N_f is defined to be:

$$N_f = (\overline{V}, \overline{E_f}, a_f, c_f, s, t)$$

where

$$\overline{E_f} := \{e \in \overline{E}; f(e) < c(e)\} \cup \{\overleftarrow{e}; e \in \overline{E} \text{ and } f(e) > 0\}.$$

Here if $e = (u, v)$ then $\overleftarrow{e} = (v, u)$ is its reverse. The *residual capacity* $c_f : \overline{E_f} \to \mathbb{R}^+$ is defined as $c_f \equiv 1$. The cost function $a_f : E_f \to \mathbb{R}$ is defined as: $a_f(e) := a(e)$ for every $e \in \overline{E}$, and $a_f(\overleftarrow{e}) := -a(e)$ for every $e \in \overline{E}$.

Lemma 12 (see[2]) *1. If f is a feasible flow in a network N, and g is a feasible flow in the residual network N_f, then $f + g$ is a feasible flow in the original network N defined as follows: $(f + g)(e) = f(e) + g(e) - g(\overleftarrow{e})$ for every $e \in \overline{E}$. (If $e \notin \overline{E_f}$, $\overleftarrow{e} \notin \overline{E_f}$, we let $g(e) = 0$, $g(\overleftarrow{e}) = 0$, respectively). The flow $f + g$ satisfies $\mathrm{val}(f + g) = \mathrm{val}(f) + \mathrm{val}(g)$ and $\mathrm{cost}(f + g) = \mathrm{cost}(f) + \mathrm{cost}(g)$.*

2. Similarly, if f, f' are two feasible flows in a network N, then $f' - f$ is a feasible flow in the residual network N_f of value $\mathrm{val}(f') - \mathrm{val}(f)$ and cost $\mathrm{cost}(f') - \mathrm{cost}(f)$. The flow $f' - f$ in N_f is defined as follows: If $e \in E \cap E_f$ then $(f' - f)(e) = (f'(e) - f(e))^+$ where $x^+ = max\{x, 0\}$. Similarly, if $e \in E$, and $\overleftarrow{e} \in E_f$, then $(f' - f)(\overleftarrow{e}) = (f(e) - f('(e))^+$.

3. Since all the capacities are one, $f' - f$ can be represented as the sum of flows along $s - t$ paths and cycles in N_f, each having flow value of $0, 1$ or -1.

For a full feasible flow f define

$$w_k(f) := k \cdot \mathrm{val}(f) - \mathrm{cost}(f) \tag{2}$$

If f is the flow corresponding to a path partition $\mathcal{P}$, then by Equation (1), $|\mathcal{P}|_k = kn - w_k(f)$.

Lemma 13 *Let $\mathcal{P}$ and $\mathcal{P}'$ be path partitions with $|\mathcal{P}'|_k < |\mathcal{P}|_k$, and assume that all non-trivial paths in $\mathcal{P}$ and $\mathcal{P}'$ are of cardinality at least k. Let f and f' be feasible flows in N corresponding to $\mathcal{P}$ and $\mathcal{P}'$, respectively. Let $f' - f = f_1 + f_2 + ... + f_m$, where each f_i is an $s - t$-flow in N_f of value $0, 1$ or -1. Then there exists some f_{i_0} ($1 \leq i_0 \leq m$) with $w_k(f_{i_0}) > 0$.*

Proof: From $|\mathcal{P}'|_k - |\mathcal{P}|_k < 0$, it follows from (1) and (2) that $w_k(f') - w_k(f) > 0$. By Lemma 12,

$$w_k(f') - w_k(f) = k \cdot \mathrm{val}(f') - \mathrm{cost}(f') - (k \cdot \mathrm{val}(f) - \mathrm{cost}(f)) = k \cdot \mathrm{val}(f' - f) - \mathrm{cost}(f' - f) =$$

$$= w_k(f' - f) = k \cdot \mathrm{val}(\Sigma_{i=1}^m f_i) - \mathrm{cost}(\Sigma_{i=1}^m f_i) = \Sigma_{i=1}^m w_k(f_i) > 0$$

Hence, there must be some f_{i_0}, $(1 \leq i_0 \leq m)$ with $w_k(f_{i_0}) > 0$. ■

The Algorithm and proof of correctness

0. **Input:** G , $q \geq 1$, $\mathcal{P}^q$
1. **Initialize:** $k \leftarrow 1$; $\mathcal{P} := \mathcal{P}^q \cup \{\{x\}; x \notin \cup \mathcal{P}^q\}$; $f \leftarrow f_{\mathcal{P}}$
2. **while** $(k \leq \min_{P \in \mathcal{P}^q} |P|)$ **do**
3. **begin**
4. **if** $(\exists \mathcal{C}^k$ orthogonal to $\mathcal{P})$ **then** **Stop**
5. Let $\mathcal{P}'$ with $|\mathcal{P}'|_k < |\mathcal{P}|_k$, and $w_k(f_{i_0}) > 0$. **If** $(\mathrm{val}(f_{i_0}) = 0$ or 1 $)$ **then** find improved $\mathcal{P}'^q$. **Stop**
6. **else** $(\mathrm{val}(f_{i_0}) = -1)$ **then** $k \leftarrow k + 1$
7. **end**

Remarks:

1. In line 4, the Oracle, as implied by Conjecture 5, finds a k-colouring $\mathcal{C}^k$ orthogonal to $\mathcal{P}$. By Observation 9, $\mathcal{C} = \mathcal{C}^k \cup \{\{x\}; x \notin \cup \mathcal{C}^k\}$ is orthogonal to $\mathcal{P}^q$, and we are done.
2. Otherwise, (line 5) the Oracle finds a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_k < |\mathcal{P}|_k$. We assume that all non-trivial paths in $\mathcal{P}'$ are of cardinality at least k. (Otherwise, we just break up paths of cardinality less than k into single vertices). Lemma 13 implies the existence of f_{i_0} with $\mathrm{val}(f_{i_0}) = 0, 1$ or -1 and $w_k(f_{i_0}) > 0$

Lemma 14 *Let $\mathcal{P}$ be a path partition and $f = f_{\mathcal{P}}$. If there exist flows f_{j_0} and f_{j_1} in N_f satisfying* $\mathrm{val}(f_{j_0}) = 1, \mathrm{val}(f_{j_1}) = -1$, $\mathrm{cost}(f_{j_0}) + \mathrm{cost}(f_{j_1}) < 0$*, then there exists a flow f' in N_f with* $\mathrm{val}(f') = 0$ *and* $\mathrm{cost}(f') < 0$.

Proof: If f_{j_0} and f_{j_1} are disjoint, then $f' = f_{j_0} + f_{j_1}$ is a flow satisfying $\mathrm{val}(f') = \mathrm{val}(f_{j_0}) + \mathrm{val}(f_{j_1}) = 0$ and $\mathrm{cost}(f') = \mathrm{cost}(f_{j_0}) + \mathrm{cost}(f_{j_1}) < 0$. Otherwise, $f_{j_0} + f_{j_1}$ is a collection of cycles (not necessarily disjoint!) of total negative cost. One of these cycles must have a negative cost, and corresponds to a flow f' in N_f with $\mathrm{val}(f') = 0$ and $\mathrm{cost}(f') < 0$. ■

We are now ready to prove the correctness of the algorithm:

Theorem 15 *Given a q-path system $\mathcal{P}^q$, the algorithm above either finds a colouring $\mathcal{C}$ orthogonal to it, or a q-path system $\mathcal{P}''^q$ with $|\bigcup \mathcal{P}''^q| > |\bigcup \mathcal{P}^q|$.*

Proof: In line 1, k is initialized to 1, the path partition $\mathcal{P}$ associated with $\mathcal{P}^q$ is constructed, and a flow f corresponding to $\mathcal{P}$ is defined in N. If the algorithm stops at line 4, then by Remark (1) above we are done. Otherwise, the Oracle finds a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_k < |\mathcal{P}|_k$. Let f_{i_0} be the flow (of value 0,1 or -1) as implied in Lemma 13. Let $f'' := f + f_{i_0}$, and $\mathcal{P}'' := \mathcal{P}(f'')$.

Case 1: Assume $\mathrm{val}(f_{i_0}) = 0$. Since $w_k(f_{i_0}) > 0$ it follows that $\mathrm{cost}(f_{i_0}) < 0$ and $f'' = f + f_{i_0}$ is a flow with $\mathrm{val}(f'') = \mathrm{val}(f)$ and $\mathrm{cost}(f'') < \mathrm{cost}(f)$. Then $|\mathcal{P}''^{>1}| = |\mathcal{P}^{>1}| = n - \mathrm{val}(f) \leq q$. However, $|\bigcup \mathcal{P}''^{>1}| = n - \mathrm{cost}(f'') > n - \mathrm{cost}(f) = |\bigcup \mathcal{P}^q|$. If $\mathcal{P}^q$ contains no trivial paths then $\mathcal{P}''^{>1}$ is a q-path system with $|\bigcup \mathcal{P}''^q| > |\bigcup \mathcal{P}^q|$, and we are done. Otherwise, we add the necessary number of trivial paths to $\mathcal{P}''^{>1}$ to make it a q-path system, and we are done again.

Case 2: If $\mathrm{val}(f_{i_0}) = -1$ then k is increased unless $k = \min_{P\in\mathcal{P}^q} |P|$. Assume $k = \min_{P\in\mathcal{P}^q} |P|$. From $w_k(f_{i_0}) > 0$ we deduce that $\mathrm{cost}(f_{i_0}) \leq -k-1$. If $k = 1$ then $\mathcal{P}$ contains t trivial paths, for some $t \geq 1$, and $q-t$ non-trivial paths. Now $|\mathcal{P}''^{>1}| = n - \mathrm{val}(f'') = n - (\mathrm{val}(f) + \mathrm{val}(f_{i_0})) = n - \mathrm{val}(f) + 1 = |\mathcal{P}^{>1}| + 1 = q - t + 1$. But $\mathrm{cost}(f_{i_0}) \leq -2$, implying that $\mathcal{P}''^{>1}$ in addition to $t-1$ trivial paths is a q-path system which covers at least one more vertex than $\mathcal{P}^q$.

Assume now that $k \geq 2$. Let $P = (v_1, v_2, \ldots, v_k)$ be the shortest path in $\mathcal{P}^q$. Then $f_{i_1} = (s, v'_k, v''_k, v'_{k-1}, v''_{k-1}, \ldots, v'_1, v''_1, t)$ is a flow in N_f with $\mathrm{val}(f_{i_1}) = 1$ and $\mathrm{cost}(f_{i_1}) = k$. By applying Lemma 14 on the flows f_{i_1} and f_{i_0} we are guaranteed the existence of a flow f' in N_f with $\mathrm{val}(f') = 0$ and $\mathrm{cost}(f') < 0$. We let $f'' := f + f'$. As was shown in Case 1, $\mathcal{P}''^q = \mathcal{P}''^{>1}$ is a q-path system which covers more vertices than $\mathcal{P}^q$.

Case 3.1: If $\mathrm{val}(f_{i_0}) = 1$ and $k = 1$, then $\mathrm{cost}(f_{i_0}) \leq 0$. Then $|\mathcal{P}''^{>1}| = n - \mathrm{val}(f'') = n - \mathrm{val}(f) - 1 \leq q-1$, and $\mathcal{P}''^{>1}$ covers $n - \mathrm{cost}(f'') \geq n - \mathrm{cost}(f) = |\bigcup \mathcal{P}^q|$ vertices. If we add any path from $G - \bigcup \mathcal{P}''^{>1}$ to the family $\mathcal{P}''^{>1}$, we have a q-path system which covers more vertices than $\mathcal{P}^q$, and we are done.

Case 3.2: Finally, if $\mathrm{val}(f_{i_0}) = 1$ and $k \geq 2$, then k was increased to its current value because there exists a flow f_{i_1} with $\mathrm{val}(f_{i_1}) = -1$ and $w_{k-1}(f_{i_1}) > 0$. From $w_k(f_{i_0}) > 0$, and $w_{k-1}(f_{i_1}) > 0$ we deduce that $\mathrm{cost}(f_{i_0}) < k$ and $\mathrm{cost}(f_{i_1}) < -(k-1)$. By Lemma 14, there exists a flow f' with $\mathrm{val}(f') = 0$ and $\mathrm{cost}(f') < 0$. The rest follows as in Case 1. This completes the proof.

■

4 When G is not Acyclic

In an arbitrary digraph, which is not necessarily acyclic, we have no guarantee that $f'' := f + f_{i_0}$ (where f_{i_0} was defined in Lemma 13) corresponds to a path partition. It may correspond to a partition of paths and cycles. However, we do know that $f' = f + \Sigma_{i=1}^{m} f_i$ does correspond to a path partition (the partition $\mathcal{P}'$!). Perhaps f_{i_0} can be replaced by a collection of flows, thus yielding the following conjecture:

Conjecture 16 *Let G be digraph, $k \geq 1$. Assume $\mathcal{P}$ and $\mathcal{P}'$ are path partitions in G and $|\mathcal{P}'|_k < |\mathcal{P}|_k$. We assume that all non-trivial paths in $\mathcal{P}$ and $\mathcal{P}'$ are of cardinality at least k. Let f (f') be the corresponding flow $f_{\mathcal{P}}$ (respectively $f'_{\mathcal{P}}$) in our network N. Let $f' - f = \sum_{i=1}^{m} f_i$ be a decomposition of $f' - f$ such that for*

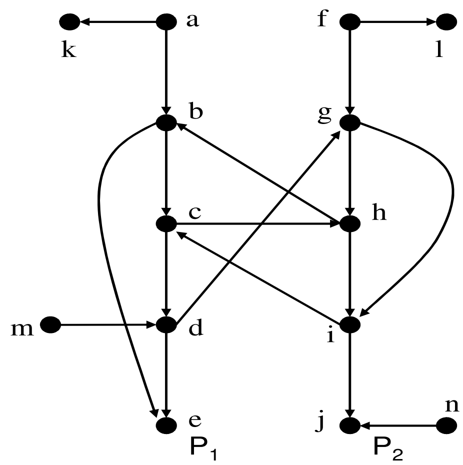

Fig. 1. G. $\mathcal{P}^2 = \{P_1, P_2\}$

any i, $f_i \in N_f$ *and* $\mathrm{val}(f_i) \in \{0, -1, 1\}$. *Then there exists a subset* $I \subseteq \{1, ..., m\}$ *such that the flow* $S = \sum_{i \in I} f_i$ *satisfies:*
(i) $S \in N_f$
(ii) $\mathcal{P}(f + S)$ *is acyclic*
(iii) $w_k(S) > 0$
(iv) $\mathrm{val}(S) \in \{0, -1, 1\}$

Mercier[8] has found a counterexample to this conjecture. Consider the graph in Figure 1. Let $k = 1$, $\mathcal{P} = \{(a, b, c, d, e), (f, g, h, i, j), (k), (l), (m), (n)\}$ and $\mathcal{P}' = \{(a, k), (f, l), (n, j), (m, d, g, i, c, h, b, e)\}$.
Let $f_1 = (t, n'', n', j'', i', c'', b', e'', d', g'', f', l'', l', s)$, and $f_2 = (t, m'', m', d'', c', h'', g', i'', h', b'', a', k'', k', s)$. It is easy to verify that $f' - f = f_1 + f_2$, where f' and f correspond to the path partitions $\mathcal{P}'$ and $\mathcal{P}$, respectively. Since both f_1 and f_2 are flows in N_f from t to s, it follows that $\mathrm{val}(f_1 + f_2) = -2$. However, $\mathcal{P}(f + f_i)$ contains a cycle, for each $i = 1, 2$. This contradicts Conjecture 16.

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