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Positive Sensitivity Analysis In Linear Programming With Bounded Variables

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Abstract

The present paper discusses positive sensitivity analysis (PSA) in linear programming with bounded variables. Positive sensitivity analysis is a sensitivity analysis method for linear programming that finds the range of perturbations within which the components of a given optimal solution which are strictly between their bounds remain strictly between bounds and which are at their lower and upper bounds remain at lower and upper bounds respectively. Its main advantage is that it is applicable to both an optimal basic and non-basic optimal solution. In this paper, we examine how the range of PSA varies according to the optimal solution used for PSA and discuss the relationship between the ranges of PSA using different optimal solutions. We also discuss the relationship between PSA and sensitivity analysis using optimal basis and the relationship between PSA and sensitivity analysis using the optimal partition. We show that the sensitivity analysis using optimal partition is a special case of PSA. In order to study these relationships and the properties, some results on duality have been discussed and existence of strictly complementary solution has been established for linear programming with bounded variables.

Keywords: Linear programming with bounded variables, sensitivity analysis, optimal partition sensitivity analysis, strictly complementary solution.

1. Introduction

The merits of linear programming are nowadays well established and linear programming is widely accepted as a useful tool in Operations Research and Management Science. A large number of companies are using this way of modeling to solve various kinds of practical problems. Applications include transportation problems, production planning, investment decision problems, blending problems, location and allocation problems, among many other.

A linear programming problem consists of linear relationships between decision variables. The variables correspond to the decision to be made. A linear objective function is specified which has to be maximized (e.g. profit) or minimized (e.g. cost). The possible decision variables are restricted to a certain area by various constraints and the bounds on the decision variables. In most of the cases, for solving the linear programming problem, the so-called Simplex method is used. The method, due to Dantzig (1951) has been implemented in a large variety of codes which are successfully used in practice. Not long after the publication of Dantzig's primal simplex method its dual version, developed by Lemke (1954). It has long been known that the dual simplex algorithm is a better alternative to the primal simplex algorithm for solving certain types of linear programming problems. The dual simplex method for linear programming with bounded variables has been been developed by Wagner (1958) which is further studied by Maros (2003a, 2003b). In 1984, Karmarker proposed his projective algorithm for linear programming, which he showed to be a polynomial time algorithm. This gave the impulse to an enormous amount of research on interior point methods. Another important topic in the field of linear programming is duality theory. Associated with every linear programming there always exists another linear programming problem which is based upon the same data and having the same solution. The original problem is called the primal problem while the associated one is called its dual problem. The dual problem was formulated by Neumann (see Murty, 1976) for linear programming with non-negative decision variables. Duality theorems were proved by D. Gale (1960), Kuhn and Tucker (1956) and Goldman et.al. (1956). Most of the packages available for solving linear programming do not only solve the linear programming problem but also provide the option to ask for information on the sensitivity of the solution to certain changes in the data. This is referred to as sensitivity analysis or postoptimal analysis. This information can be of tremendous importance in practice, where parameter values may be estimates. Questions of the type "What if $\cdots$" are frequently encountered and implementation of a specific solution may be difficult. Sensitivity analysis serves as a tool for obtaining information about the bottlenecks and degrees of freedom in the problem. For instance, when the cost of an activity or the available amount of resources is changed, we often need information about how the total cost of the current decision is altered, in order to obtain a new optimal decision for the new situation. In this case, sensitivity analysis can be applied.

The method of sensitivity analysis in simplex method is well developed on the foundation of optimal basis, it requires little computational effort. This method has been introduced in numerous papers and text books so far (see, for example: Dantzig, 1963; Gal, 1979) and has been used in many linear programming codes. However in case of degeneracy, it may yield incomplete information due to alternative optimal bases (Evans and Baker, 1982; Knolmayer, 1984; Jansen et.al., 1997). On the other hand, most interior-point methods produce a solution which converges to an optimal solution relatively interior to the optimal face. Some additional computation enables us to get an exact optimal basic or non-basic solution (Tapia and Zhang, 1991; Mehrotra and Ye, 1993; Bixby and Salzman, 1994). Interior-point method for linear programming with bounded variables has been developed by Castro (1995). Dahiya and Verma (2005) studied sensitivity analysis using optimal bases for the linear programming problem with bounded variables. However, since sensitivity analysis using an optimal basis cannot be applied to an optimal non basic solution, other methods for sensitivity analysis have been suggested: positive sensitivity analysis(PSA), sensitivity analysis using optimal partition and $\varepsilon-$ sensitivity analysis(Yang, 1990; Adler and Monteiro, 1992; Kim et.al., 1999; Yilidmir et.al. 2001). Yang(1990) introduced PSA for optimal solutions including optimal non basic solutions based on Sung and Park's (1988) definition for the linear programming problem with non-negative decision variables.Positive sensitivity analysis is defined to find the characteristic region within which variables having a zero and having a positive value in an optimal solution remain zero and positive in the perturbed problem, respectively. Adler and Monteiro (1992) developed a method of parametric analysis on the right-hand side by introducing the optimal partition. Monteiro and Mehrotra (1996) presented a parametric analysis by generalizing Adler and Monteiro's method, and Greenberg(2000) developed a method of sensitivity analysis using the optimal partition when cost coefficients and right-hand sides change simultaneously. To use Yang's and Adler and Monteiro's method, we need an optimal solution or the optimal partition, which requires additional computation for interior-point methods. Kim et.al. (1999) developed a practical sensitivity analysis method, $\varepsilon-$ sensitivity analysis, which can be directly applied to interior-point solutions produced by interior-point methods. Park et.al. (2004) studied PSA in linear programming that was useful for establishing the relationship between PSA and sensitivity analysis using optimal bases and using optimal partition. This motivated us to carryon this study for the case when decision variables are bounded. The purpose of this paper is to present some properties of PSA and examine how the range of PSA varies according to the optimal solution used for PSA, and study the relationship between the ranges of PSA using different optimal solutions for linear programming with bounded variables and consequently making use of these properties of PSA, we study the relationship PSA and other sensitivity analysis methods . Some duality results relevant to this study are discussed in the present paper for linear programming with bounded variables.

This paper is organized as follows: in Section 2, we discuss existence of strictly complementary solution in linear programming problem with bounded variables. In Section 3, we introduce three types of sensitivity analyses for linear programming with bounded variables and some basic results about the relationship between PSA and other sensitivity analysis methods are presented. In section 4, we discuss the relationship between the ranges of PSA using different optimal solutions, and present a necessary and sufficient condition for the range of PSA to include a positive and negative value. In section 5, we study the relationship between PSA using an optimal basic solution and sensitivity analysis using an optimal basis when a given optimal basic solution is degenerate. In section 6, some concluding remarks are given.

2. Basic results in linear programming with bounded variables

Consider the linear programming problem,

$$(P) \quad \min c^T x$$

subject to

$$Ax = b \tag{2.1}$$

$$0 \le x \le u, \tag{2.2}$$

where $c, x, u \in R^n; b \in R^m; A = [a_1\ a_2 \cdots a_n]_{m\times n} = [a_{ij}]_{m\times n}$ is a matrix and $\rho(A) = m$, u is the vector of upper bounds on the decision variable vector x.

Notations:

c_j = j^{th} component of vector c,

$A_{.j} = a_j =$ Activity vector of x_j,

$J = \{1,2,\ldots,n\}$.

Given an index set σ of the variables, let A_σ denote the submatrix of A with columns that correspond to indices in σ. Similarly, we use z_σ to denote the subvector of z with components that correspond to indices in σ. For any vector x, let x_j denote the j^{th}

component of x.
Since rank of A is m, there exists an $(m\times m)$ non-singular submatrix $A_B=(a_{B_1}\ a_{B_2}\cdots a_{B_m})$ of A. Let $B=\{B_1, B_2,\cdots, B_m\}$.
Let $N_1=\{j\in J, j\notin B$ s.t. $x_j=0\}=$ Index set of non basic variables which are at their lower bounds,
Let $N_2=\{j\in J, j\notin B$ s.t. $x_j=u_j\}=$ Index set of non basic variables which are at their upper bounds,
$y_j=A_B^{-1}a_j\ \forall\ j\in J$,
So, the columns of A can be permuted so that $A=(A_B\ A_{N_1}\ A_{N_2})$ and we can write $Ax=b$ as $A_Bx_B+A_{N_1}x_{N_1}+A_{N_2}x_{N_2}=b$, where $x=(x_B\ x_{N_1}\ x_{N_2})$ and $x_{N_1}=0$. Then a solution to $Ax=b$ is given by

$$x_B=A_B^{-1}b-\sum_{j\in N_2}y_ju_j, x_{N_1}=0, x_{N_2}=u_{N_2}.$$

Let $c=(c_B\ c_{N_1}\ c_{N_2})$ be the corresponding partition of c s.t. $c^Tx=c_B^Tx_B+c_{N_1}^Tx_{N_1}+c_{N_2}^Tx_{N_2}$ and $z_j-c_j=c_B^Ty_j-c_j, \forall\ j\in J$.

Definitions:

(a) The $(m\times m)$ non-singular matrix A_B is called basis matrix.
(b) The solution $x_B=A_B^{-1}b-\sum_{j\in N_2}y_ju_j, x_{N_1}=0, x_{N_2}=u_{N_2}$ is called the basic solution of $Ax=b$.
(c) x_B is the vector of basic variables, x_{N_1} is the vector of non basic variables which are at their lower bounds and x_{N_2} is the vector of non basic variables which are at their upper bounds.

The problem (P) can be solved by treating upper bound restrictions as constraints but this increases the size of the problem. On the other hand, this problem can be solved by using the upper bound simplex technique without increasing the size of the problem (see: Murty, 1976; Dantzig, 1963). A basic feasible solution $x=(x_B\ x_{N_1}\ x_{N_2})$ is an optimal solution of (P) if

$$z_j-c_j=0\ \forall j\in B, z_j-c_j\le 0\ \forall\ j\in N_1 \text{ and } z_j-c_j\ge 0\ \forall\ j\in N_2.$$

Consider the dual of the problem (P) as follows:

$$(D)\quad \max\ (b^Tv-u^Tw)$$

subject to

$$A^Tv-w\le c \tag{2.3}$$

$$w\ge 0 \text{ and } v \text{ is unrestricted.} \tag{2.4}$$

The dual was formulated by Neuman (see Murty, 1976) for the linear programming problem with non-negative variables. Duality theorems and complementary slackness theorem were proved by Gale (1960), Kuhn and Tucker (1956) and Goldman and Tucker (1956). All these important results can be easily extended to the problem (P).

Theorem 1. (Weak Duality) *If x^* is a feasible solution of (P) and (v^*, w^*) is a feasible solution of (D), then*

$$c^T x^* \geq b^T v^* - u^T w^*.$$

Theorem 2. *If x^* and (v^*, w^*) are feasible solutions of (P) and (D) respectively, and $c^T x^* = b^T v^* - u^T w^*$ (or merely $c^T x^* \leq b^T v^* - u^T w^*$), then x^* and (v^*, w^*) are optimal solutions to (P) and (D) respectively.*

Theorem 3. (Basic Duality Theorem) *If x^* is an optimal solution to the primal (P), then there exists a feasible solution (v^*, w^*) to the dual (D) such that $c^T x^* = b^T v^* - u^T w^*$ and conversely, if (v^*, w^*) is an optimal solution to the dual (D), then there exists a feasible solution x^* to the primal (P) such that $c^T x^* = b^T v^* - u^T w^*$*

Theorem 4. (Complementary Slackness Theorem) *x^* and (v^*, w^*) are optimal solution to (P) and (D) respectively iff $x_j^* w_{s_j}^* = 0$ and $w_j^*(u_j - x_j^*) = 0 \ \forall \ j = 1, 2, \cdots, n.$*

According to Theorem 7 of Terlaky (2001), the optimal solutions are complementary in the general sense i.e. they are not only complementary w.r.t. their own slack vector, but complementary w.r.t. to slack vector for any other optimal solution as well. We have the following result:

Corollary 1. Let (x^*, v^*, w^*, w_s^*) be a pair of optimal solutions of (P) and (D) and $(x^{'}, v^{'}, w^{'}, w_s^{'})$ be another pair of optimal solutions of (P) and (D), then

$$x^{*T} w_s^* = 0, w^{*T}(u - x^*) = 0, x^{'T} w_s^{'} = 0, w^{'T}(u - x^{'}) = 0$$

and $$x^{*T} w_s^{'} = 0, x^{'T} w_s^* = 0, w^{*T}(u - x^{'}) = 0, w^{'T}(u - x^*) = 0.$$

Consider the primal (P) and the dual (D) problems in the standard forms

$$(P) \quad \min c^T x \qquad\qquad (D) \quad \max b^T v - u^T w$$
$$Ax = b \qquad\qquad A^T v - w + w_s = c$$
$$0 \leq x \leq u \qquad\qquad w, w_s \geq 0$$

Let $A_{B_{m\times m}}$ be the basis matrix associated with the optimal basic feasible solution $x^* = (x_B, \quad x_{N_1} = 0, \quad x_{N_2} = u_{N_2})^T$. Let (v^*, w^*, w_s^*) be the corresponding optimal solution of the dual problem. If we solve problem (P) by using simplex method by treating upper bounds as constraints, the values of dual variables are given by net evaluations corresponding to artificial variables and slack variables as discussed in Theorem 3. If we solve primal problem by upper bound simplex technique (Murty, 1976), then we can directly find the values of v^*, w^* and w_s^* from the optimal table of (P) by using the following relations which can be easily establised using the duality theory:

$$v^* = (c_B^T A_B^{-1})^T, \ w_B^* = w_{s_B}^* = 0,$$

$$w_{N_1}^* = 0, w_{N_2}^* = [(z_j - c_j)]_{N_2}, \ w_{s_{N_1}}^* = -[(z_j - c_j)]_{N_1}, w_{s_{N_2}}^* = 0.$$

Consider the linear programming problem with non-negative variables and its dual problem as follows

$$(Primal) \quad \min c^T x \qquad\qquad (Dual) \quad \max b^T v$$
$$Ax = b \qquad\qquad A^T v + w_s = c$$
$$x \geq 0 \qquad\qquad w_s \geq 0$$

According to Goldman and Tucker (1956), there exists at least one optimal solution pair (x^*, v^*, w_s^*) to primal and dual which is strictly complementary, that is,

$$x_j^* + w_{s_j}^* > 0 \ \forall \ j = 1,2,\cdots,n.$$

Similarly, for the case of bounded variables the strictly complementary solution in (P) and (D) is defined as that optimal solution pair (x^*, v^*, w^*, w_s^*) to (P) and (D) which satisfy

$$x_j^* + w_{s_j}^* > 0 \ and \ w_j^* + (u_j - x_j^*) > 0 \ \forall \ j = 1,2,\cdots,n.$$

The existence of such solution can be easily proved by using the application of following theorem given by Tucker (1956) to a skew-symmetric matrix.

Theorem 5. The self-dual system $KW \geq 0, W \geq 0$ has a solution W^* such that $KW^* + W^* > 0$ where K is a skew-symmetric matrix.

The application of Theorem 5 to the following skew-symmetric matrix

$$K = \begin{bmatrix} 0 & -A^T & A^T & I & c \\ A & 0 & 0 & 0 & -b \\ -A & 0 & 0 & 0 & b \\ -I & 0 & 0 & 0 & u \\ -c^T & b^T & -b^T & -u^T & 0 \end{bmatrix}$$

yields a column vector $(v_o^{'}, v_o^{''}, w_o, x_o, t_o) \geq 0$ such that

(i.a) $A^T v_o^{'} - A^T v_o^{''} - w_o \leq ct_o$, i.e. $A^T v_o - w_o \leq ct_o$ where $v_o = v_o^{'} - v_o^{''}$ is unrestricted.

(ii.a) $Ax_o = bt_o, x_o \leq ut_o$.

(iii.a) $b^T v_o - u^T w_o \geq c^T x_o$.

(iv.a) $c^T x_o < b^T v_o - u^T w_o + t_o$.

(v.a) $A^T v_o - w_o < ct_o + x_o^T$.

(vi.a) $-x_o + w_o^T > -ut_o$.

Now we investigate the cases when $t_o > 0$ and $t_o = 0$ separately.

Lemma 1. *Suppose* $t_o > 0$. *Then there are optimal vectors* x_o *and* (v_o, w_o) *for the dual problems* (P) *and* (D) *respectively such that*

$$c^T x_o = b^T v_o - u^T w_o, (-A^T v_o + w_o + c) + x_o^T > 0, (u - x_o) + w_o^T > 0.$$

Proof. Since $t_o > 0$, the non-negative vector $(x_o, v_o^{'}, v_o^{''}, w_o, t_o)$ can be "normalized" so that $t_o = 1$ and hence the vector (x_o, v_o, w_o, t_o) without affecting the validity of the homogeneous inequalities (i.a) to (vi.a). Then (i.a) and (ii.a) (with $t_o = 1$) show that x_o

and (v_o, w_o) are feasible solution to (P) and (D) respectively; (iii.a) and Theorem 2 show that x_o and (v_o, w_o) are optimal also and Theorem 1 shows that $c^T x_o = b^T v_o - u^T w_o$. The relations $x_o^T + (c - A^T v_o + w_o) > 0$ and $(u - x_o) + w_o^T > 0$ follows from (iv.a) and (vi.a) (with $t_o = 1$). Thus we can choose the normalized x_o and (v_o, w_o) to be the desired x_o and (v_o, w_o).

Using the definition of w_s and the above lemma, we have

$$x_o^T + w_{s_o} > 0 \ \text{and} \ w_o^T + (u - x_o) > 0,$$

which proves the existence of a strictly complementary optimal solution.

Lemma 2. *Suppose* $t_o = 0$. *Then one of the following possibilities hold:*

(a) At least one of the dual problem has no feasible solution.

(b) If the maximization problem has a feasible vector, then the set of its feasible vectors is unbounded and $b^T v - u^T w$ is unbounded on this set. Dually for the minimization problem.

(c) Neither problem has a feasible vector.

Proof. (a) Suppose (v, w) a feasible vector for the maximization problem, then

$$A^T v - w \leq c, \ \ w \geq 0. \tag{2.5}$$

Using (ii.a), with $t_o = 0$ and non-negativity of w, we have

$$v^T A x_o = 0, w^T x_o \leq 0 \ \text{i.e.} \ v^T A x_o - w^T x_o \geq 0.$$

From (iv.a),

$$c^T x_o < b^T v_o - u^T w_o \tag{2.6}$$

As $x_o \geq 0$, from (2.6), we have

$$v^T A x_o - w^T x_o \leq c^T x_o \tag{2.7}$$

$$0 \leq v^T A x_o - w^T x_o \leq c^T x_o < b^T v_o - u^T w_o \tag{2.8}$$

If possible, let the minimization problem has a feasible solution x, so $Ax = b, 0 \leq x \leq u$.

$$\Rightarrow \ v_o^T A x = b^t v_o \ \text{ and } \ -w_o^T x \geq -u^T w_o,$$

$$\Rightarrow \ v_o^T A x - w_o^T x \geq b^t v_o - u^T w_o \tag{2.9}$$

From (i.a) with $t_o = 0$, we have $A^T v_o - w_o \leq 0$.

$$\Rightarrow \ v_o^T A x - w_o^T x \leq 0 \tag{2.10}$$

Relations (2.9) and (2.10), together imply that

$$0 \geq v_o^T A x - w_o^T x \geq b^t v_o - u^T w_o \tag{2.11}$$

this clearly contradicts (2.8), and so (a) is proved.

(b) Let (v, w) be a feasible solution to the maximization problem, then

$$A^T v - w \leq c, \ \ w \geq 0 \tag{2.12}$$

Also form (i.a), with $t_o = 0$,

$$A^T v_o - w_o \leq 0. \tag{2.13}$$

Consider $(v+\lambda v_o, w+\lambda w_o)$ for all $\lambda \geq 0$. Clearly $w+\lambda w_o \geq 0$. From (2.12) and (2.13),

$$A^T(v+\lambda v_o)-(w+\lambda w_o) \leq c.$$

Thus the entire infinite ray consists of feasible vector , which proves the first assertion of (b). Furthermore, from (2.8), $b^T v_o - u^T w_o > 0$, we see that

$$b^T(v+\lambda v_o)-u^T(w+\lambda w_o) = (b^T v - u^T w)+\lambda(b^T v_o - u^T w_o)$$

can be made arbitrary large by choosing λ large enough. So the second assertion of (b) is proved.

Finally, (c) is an immediate consequence of (b). This completes the proof of Lemma.

Throughout this paper, we assume that (P) and (D) are feasible.

3. **Definition of three sensitivity analysis methods**

For sensitivity analysis on the cost coefficient c_k that is perturbed by the amount θ, we consider another linear programming problem (P_θ) and the corresponding dual (D_θ):

$$(P_\theta): \quad \min\,(c+\theta e_k)^T x \qquad\qquad (D_\theta): \quad \max\, b^T v - u^T w$$
$$Ax = b \qquad\qquad A^T v - w + w_s = c+\theta e_k$$
$$0 \leq x \leq u \qquad\qquad w, w_s \geq 0$$

where $e_k \in R^n$ is a vector such that the k^{th} element is one and the others are zero. Also, for the right-hand side vector b, its h^{th} element i.e. b_h is perturbed by the amount γ, we consider the linear programming (P_γ) and its corresponding dual (D_γ):

$$(P_\gamma): \quad \min\, c^T x \qquad\qquad (D_\gamma): \quad \max\,(b+\gamma e_h)^T v - u^T w$$
$$Ax = (b+\gamma e_h) \qquad\qquad A^T v - w + w_s = c$$
$$0 \leq x \leq u \qquad\qquad w, w_s \geq 0$$

where $e_h \in R^m$ is the vector such that the h^{th} element is one and the others are zero.

Let B, N_1, N_2 be the index set of the basic, non-basic variables at their lower bounds and non-basic variables at their upper bounds respectively.

If $(x_B, x_{N_1}, x_{N_2})^T = \left((A_B^{-1}b - \sum_{j\in N_2} y_j u_j)\ ,0_{N_1}, u_{N_2}\right)^T$ is an optimal solution to (P), (A_B, A_{N_1}, A_{N_2}) is called a primal-optimal basis. Also, if $v = (A_B^{-1})^T c_B = (c_B^T A_B^{-1})^T$ and

$$(w_{s_B}, w_{s_{N_1}}, w_{s_{N_2}})^T = (0_B, -(z_j - c_j)_{N_1}, 0_{N_2})^T = (0_B, c_{N_1} - A_{N_1}^T v, 0_{N_2})^T,$$

$$(w_B, w_{N_1}, w_{N_2}) = (0_B, 0_{N_1}, (z_j - c_j)_{N_2})^T = (0_B, 0_{N_1}, A_{N_2}^T v - c_{N_2})^T,$$

is an optimal solution to (D), then (A_B, A_{N_1}, A_{N_2}) is called a dual-optimal basis. If (A_B, A_{N_1}, A_{N_2}) is both a primal-optimal and dual-optimal basis, it is called an optimal basis. For a primal-optimal basis (A_B, A_{N_1}, A_{N_2}), let $T_{c_k}(A_B, A_{N_1}, A_{N_2})$ denote the following range of θ:

$$T_{c_k}(A_B, A_{N_1}, A_{N_2}) = \left\{ \theta \left| \begin{bmatrix} A_B^T \\ A_{N_1}^T \\ A_{N_2}^T \end{bmatrix} v - \begin{bmatrix} w_B \\ w_{N_1} \\ w_{N_2} \end{bmatrix} + \begin{bmatrix} w_{s_B} \\ w_{s_{N_1}} \\ w_{s_{N_2}} \end{bmatrix} = \begin{bmatrix} c_B + (\theta e_k)_B \\ c_{N_1} + (\theta e_k)_{N_1} \\ c_{N_2} + (\theta e_k)_{N_2} \end{bmatrix}, \right. \\ w_{s_B} = w_B = w_{N_1} = w_{s_{N_2}} = 0, w_{s_{N_1}}, w_{N_2} \geq 0 \right\} \tag{3.1}$$

That is, $T_{c_k}(A_B, A_{N_1}, A_{N_2})$ represents the range of θ within which a primal-optimal basis (A_B, A_{N_1}, A_{N_2}) is an optimal basis of (P_θ). Note that $T_{c_k}(A_B, A_{N_1}, A_{N_2})$ may be the empty set. Similarly, for a dual-optimal basis (A_B, A_{N_1}, A_{N_2}) , let $T_{b_h}(A_B, A_{N_1}, A_{N_2})$ denote the following range of γ:

$$T_{b_h}(A_B, A_{N_1}, A_{N_2}) = \left\{ \gamma \left| 0 \leq x_B = A_B^{-1}(b + \gamma e_h) - \sum_{j \in N_2} y_j u_j \leq u_B, x_{N_1} = 0, x_{N_2} = u_{N_2} \right. \right\} \tag{3.2}$$

That is, $T_{b_h}(A_B, A_{N_1}, A_{N_2})$ represents the range of γ within which a dual-optimal basis (A_B, A_{N_1}, A_{N_2}) is an optimal basis of (D_θ). Note that $T_{b_h}(A_B, A_{N_1}, A_{N_2})$ may also be the empty set.

The traditional sensitivity analysis using an optimal basis, called basic sensitivity analysis, is defined as follows:

Definition 3.1. (Basic Sensitivity Analysis, BSA) Let B be the index set of basic variables and N_1, N_2 be the index set of non-basic variables which are at their lower and upper bounds respectively of an optimal basis. BSA using (A_B, A_{N_1}, A_{N_2}) on a cost coefficient c_k is to find the range of θ within which (A_B, A_{N_1}, A_{N_2}) remains an optimal basis to (P_θ) and (D_θ). Similarly, BSA using (A_B, A_{N_1}, A_{N_2}) on a right-hand side b_h is to find the range of γ within which (A_B, A_{N_1}, A_{N_2}) remains an optimal basis to (P_γ) and (D_γ). By the definition of $T_{c_k}(A_B, A_{N_1}, A_{N_2})$ and $T_{b_h}(A_B, A_{N_1}, A_{N_2})$, the ranges found by BSA using (A_B, A_{N_1}, A_{N_2}) on c_k and b_h are represented as $T_{c_k}(A_B, A_{N_1}, A_{N_2})$ and $T_{b_h}(A_B, A_{N_1}, A_{N_2})$, respectively. To perform BSA, we need an optimal basis associated with an optimal basic solution. In fact, BSA can be applied only to an optimal basic solution.

Before defining positive sensitivity analysis (PSA), some notation is introduced. For an arbitrary vector x whose components are non-negative and have some upper bound restrictions on them, let $\eta(x), \underline{\eta}(x)$ and $\overline{\eta}(x)$ denote the set of indices of variables as follows: $\eta(x) = \{ j \mid 0 < x_j < u_j \}$, $\underline{\eta}(x) = \{ j \mid x_j = 0 \}$, $\overline{\eta}(x) = \{ j \mid x_j = u_j \}$.

In addition, $\pi(x) = (\eta(x), \underline{\eta}(x), \overline{\eta}(x))$ is called the induced partition of x.

Definition 3.2. (Positive Sensitivity Analysis, PSA) Let x^* be an optimal solution to (P). The PSA using x^* on c_k is to find the range of θ within which there exists an optimal solution to (P_θ) whose induced partition is equal to $\pi(x^*)$. Similarly, the PSA using x^* on b_h is to find the range of γ within which there exists an optimal solution to (P_γ) whose induced partition is equal to $\pi(x^*)$.

Given an optimal solution x^* to (P), the range of (PSA) using x^* is calculated using the following (Yang, 1990):

$$Y_{c_k}(x^*) = \left\{ \theta \left| \begin{array}{l} \begin{bmatrix} A_\sigma^T \\ A_{\underline{\sigma}}^T \\ A_{\overline{\sigma}}^T \end{bmatrix} v - \begin{bmatrix} w_\sigma \\ w_{\underline{\sigma}} \\ w_{\overline{\sigma}} \end{bmatrix} + \begin{bmatrix} w_{s_\sigma} \\ w_{s_{\underline{\sigma}}} \\ w_{s_{\overline{\sigma}}} \end{bmatrix} = \begin{bmatrix} c_\sigma + (\theta e_k)_\sigma \\ c_{\underline{\sigma}} + (\theta e_k)_{\underline{\sigma}} \\ c_{\overline{\sigma}} + (\theta e_k)_{\overline{\sigma}} \end{bmatrix}, \\ w_{s_\sigma} = w_\sigma = w_{\underline{\sigma}} = w_{s_{\overline{\sigma}}} = 0, w_{s_{\underline{\sigma}}}, w_{\overline{\sigma}} \geq 0 \end{array} \right. \right\} \tag{3.3}$$

$$Y_{b_h}(x^*) = \left\{ \gamma \mid A_\sigma x_\sigma + A_{\overline{\sigma}} x_{\overline{\sigma}} = b + \gamma e_h, 0 \leq x_\sigma \leq u_\sigma, x_{\underline{\sigma}} = 0, x_{\overline{\sigma}} = u_{\overline{\sigma}} \right\} \tag{3.4}$$

where $\sigma = \eta(x^*), \underline{\sigma} = \underline{\eta}(x^*), \overline{\sigma} = \overline{\eta}(x^*)$.

As proved in Section 2, there exists at least one pair of optimal solution (x^*, v^*, w^*, w_s^*) to the (P) and (D) which is strictly complementary, i.e.,

$$x_j^* + w_{s_j}^* > 0 \ \ and \ \ w_j^* + (u_j - x_j^*) > 0 \ \forall \ j = 1,2,\cdots,n$$

Let $B^* = \eta(x^*), N^* = \eta(w_s^*), M^* = \eta(w^*)$. The partition $\pi^* = (B^*, N^*, M^*)$ of the indices of variables is called the optimal partition of (P) and (D). (Throughout this paper $\pi^* = (B^*, N^*, M^*)$ denotes the optimal partition of (P) and (D)). The definition of sensitivity analysis using the optimal partition is as follows:

Definition 3.3. (Optimal Partition Sensitivity Analysis, OSA) Let $\pi^* = (B^*, N^*, M^*)$ be the optimal partition of (P) and (D). The sensitivity analysis using the optimal partition on c_k is to find the range of θ within which optimal partition of (P_θ) and (D_θ) is equal to π^*. Similarly, the sensitivity analysis using the optimal partition on b_h is to find the range of γ within which the optimal partition of (P_γ) and (D_γ) is equal to π^*. The range of OSA is calculated using the following (Roos et al., 1997):

$$O_{c_k}(B^*, N^*, M^*) = \left\{ \theta \left| \begin{array}{l} \begin{bmatrix} A_{B^*} \\ A_{N^*} \\ A_{M^*} \end{bmatrix} v - \begin{bmatrix} w_{B^*} \\ w_{N^*} \\ w_{M^*} \end{bmatrix} + \begin{bmatrix} w_{s_{B^*}} \\ w_{s_{N^*}} \\ w_{s_{M^*}} \end{bmatrix} = \begin{bmatrix} c_{B^*} + (\theta e_k)_{B^*} \\ c_{N^*} + (\theta e_k)_{N^*} \\ c_{M^*} + (\theta e_k)_{M^*} \end{bmatrix}, \\ w_{s_{B^*}} = w_{B^*} = w_{N^*} = w_{s_{M^*}} = 0, w_{s_{N^*}}, w_{M^*} \geq 0 \end{array} \right. \right\} \tag{3.5}$$

$$O_{b_h}(B^*, N^*, M^*) = \left\{ \gamma \left| \begin{array}{l} A_{B^*} x_{B^*} + A_{M^*} x_{M^*} = b + \gamma e_h, 0 \leq x_{B^*} \leq u_{B^*}, \\ x_{N^*} = 0, x_{M^*} = u_{M^*} \end{array} \right. \right\} \tag{3.6}$$

Note that $O_{c_k}(B^*, N^*, M^*)$ and $O_{b_h}(B^*, N^*, M^*)$ include the boundary values where the optimal partition of the perturbed problem differs from $\pi^* = (B^*, N^*, M^*)$.

So far, we have defined three kinds of sensitivity analysis for linear programming. It is trivial that if x^* is a non-degenerate optimal basic solution to (P), then the range of PSA using x^* is equal to that of BSA. However, if x^* is a degenerate optimal basic solution, the range of PSA may differ from that of BSA. The case will be discussed in Section 4. In addition, we know easily by definition that the range of PSA using a strictly complementary optimal solution is equal to that of OSA. Since the range of PSA using an optimal solution x^* is equal to the range of perturbations within which the partition $(\eta(x^*), \underline{\eta}(x^*), \overline{\eta}(x^*))$ of the indices of variables remains invariant, OSA can be regarded as a special case of PSA.

4. The range of PSA using different optimal solutions

Let $z(\theta)$ denote the optimal value of the objective function of (P_θ) and (D_θ). Also, for any optimal solution x^* to (P), let $L_k(x^*)$ denote the range of θ such that $z(\theta) = z(0) + \theta x_k^*$, i.e., $L_k(x^*) = \{\theta \mid z(\theta) = z(0) + \theta x_k^*\}$.

Jansen et al. (1992) proved that $Y_{c_k}(x^*) = L_k(x^*)$ for an optimal basic feasible solution x^* to linear programming problem with non-negative variables. The result can be easily proved for problem (P).

Theorem 6. *Let x^* be an optimal basic solution to (P). Then $Y_{c_k}(x^*) = L_k(x^*)$.*

Proof. Let $\theta \in L_k(x^*)$. Then x^* is an optimal solution of (P_θ), hence $\theta \in Y_{c_k}(x^*)$. Conversely, let $\theta \in Y_{c_k}(x^*)$. So, there exists an optimal solution $x^{'}$ of (P_θ) such that $\pi(x^{'}) = \pi(x^*)$. We prove that $c^T x^{'} + \theta x_k^{'} = c^T x^* + \theta x_k^*$.

Let B, N_1, N_2 be the index set of basic variables and non-basic variables at their lower and upper bounds respectively associated with x^*. As $\pi(x^*) = \pi(x^{'})$, so $x_j^{'} = x_j^* = 0 \ \forall j \in N_1$, $x_j^{'} = x_j^* = u_j \ \forall j \in N_2$, and $0 \le x_j^*, x_j^{'} \le u_j \ \forall j \in B$.

Also, $x_B^* = A_B^{-1}(b - A_{N_2} u_{N_2})$. Let $\sigma = \eta(x^{'}), \underline{\sigma} = \underline{\eta}(x^{'}), \overline{\sigma} = \overline{\eta}(x^{'})$. Clearly, $N_1 \subset \underline{\sigma}, N_2 \subset \overline{\sigma}$ and $\sigma \subset B$. By the suitable permutation of the components of $x^{'}$, we can write $x^{'}$ as a basic feasible solution. Define $x_j^{'}$ as basic variable if x_j^* is basic and non-basic at lower and upper bounds if corresponding x_j^* is non-basic at lower and upper bounds respectively. So we have

$$A_B x_B^{'} + A_{N_2} x_{N_2}^{'} = b,$$

$$A_B x_B^{'} + A_{N_2} x_{N_2}^{*} = b,$$

$$A_B x_B^{'} = b - A_{N_2} x_{N_2}^{*},$$

$$A_B x_B^{'} = A_B x_B^* \implies x_B^{'} = x_B^*,$$

$$(c^T + \theta e_k) x^* = (c^T + \theta e_k) x^{'}.$$

Hence the result.

Any optimal feasible solution of (P) satisfies the following results which All can be easily proved on the same lines as discussed in Park et.al. (2004):

Lemma 3. *For an arbitrary optimal solution* x^* *to* (P), $Y_{c_k}(x^*) = L_k(x^*)$.

Theorem 7. *Let* $\bar{x}$ *and* $\tilde{x}$ *be two different optimal solutions to* (P). *If* $Y_{c_k}(\bar{x}) \cap Y_{c_k}(\tilde{x}) \neq \{0\}$, *then* $Y_{c_k}(\bar{x}) = Y_{c_k}(\tilde{x})$.

Theorem 8. *Let* x^*, x^1 *and* x^2 *be distinct optimal solutions to* (P) *such that* $x^* = \lambda x^1 + (1-\lambda)x^2$ *for some* λ *with* $0 < \lambda < 1$. *Then* $Y_{c_k}(x^*) = Y_{c_k}(x^1) \cap Y_{c_k}(x^2)$.

Corollary 2. *Let* $x^*, x^1, x^2, \cdots, x^r$ *be optimal solutions to* (P) *such that for some* $\lambda_i \ (i = 1,2,\cdots,r)$

$$x^* = \lambda_1 x^1 + \lambda_2 x^2 + \cdots + \lambda_r x^r, \sum_{i=1}^{r} \lambda_i = 1, \lambda_i > 0 \ \forall\, i.$$

Then, $Y_{c_k}(x^*) = \cap_{1 \le i \le r} Y_{c_k}(x^i)$. Moreover, if $Y_{c_k}(x^*) \neq \{0\}$, then $Y_{c_k}(x^1) = \cdots = Y_{c_k}(x^r)$.

If x^* in Corollary 2 is a strictly complementary solution, then we find that $O_{c_k}(B^*, N^*, M^*) = \cap_{1 \le i \le r} Y_{c_k}(x^i)$ because $Y_{c_k}(x^*) = O_{c_k}(B^*, N^*, M^*)$. That is, the range of OSA is the intersection of the ranges of PSA using optimal solutions whose convex combination leads to a strictly complementary solution.

Next , consider the case when b_h is perturbed. For an arbitrary matrix $E \in R^{m \times r}$ with r being a positive integer, let $Pos(E)$ denote a set of vectors as follows:

$$Pos(E) = \left\{ x \in R^m \mid x = \sum_{1 \le j \le r} \lambda_j E_{.j}, \lambda_j \ge 0 \right\},$$

where $E_{.j}$ is the j^{th} column vector of E. In the next theorem, the relationship between ranges of PSA using different optimal solutions is presented when b_h is changed.

Theorem 9. *Let* x^*, x^1, x^2 *be optimal solutions to* (P) *such that* $x^* = \lambda x^1 + (1-\lambda)x^2$ *for some* λ *with* $0 < \lambda < 1$. *Then*

$$Y_{b_h}(x^i) \subset Y_{b_h}(x^*) \text{ for } i = 1,2.$$

Proof. Let $\sigma = \eta(x^*), \bar{\sigma} = \bar{\eta}(x^*)$ and $\sigma^i = \eta(x^i), \bar{\sigma}^i = \bar{\eta}(x^i)$ for $i = 1,2$. By assumption, $\sigma^i \cup \bar{\sigma}^i \subset \sigma \bigcup \bar{\sigma}$, so $Pos(A_{\sigma^i} \cup A_{\bar{\sigma}^i}) \subset Pos(A_\sigma \cup A_{\bar{\sigma}})$. This together with equation (3.4), implies that $Y_{b_h}(x^i) \subset Y_{b_h}(x^*)$ for each $i = 1,2$.

From the above theorem, we may conjecture that $Y_{b_h}(x^*) = \cup_{1 \le i \le r} Y_{b_h}(x^i)$ where x^* and x^i are defined in the same way as in Corollary 2. However. from the following example we find that, in general, $Y_{b_h}(x^*)$ is not equal to $\cup_{1 \le i \le r} Y_{b_h}(x^i)$:

$$(P_1): \ \min \ -x_1 - 2x_2$$

subject to

$$-x_1 + x_2 + x_3 = 2, \ x_1 + 2x_2 + x_4 = 13, \ 4x_1 + x_2 + x_5 = 25,$$
$$0 \le x_1 \le 5, 0 \le x_2 \le 5, 0 \le x_3 \le 10, 0 \le x_4 \le 15, 0 \le x_5 \le 25.$$
$$(D_1): \ \max \ 2v_1 + 13v_2 + 25v_3 - 5w_1 - 5w_2 - 10w_3 - 15w_4 - 25w_5$$
$$-v_1 + v_2 + 4v_3 - w_1 + w_{s_1} = -1.$$
$$v_1 + 2v_2 + v_3 - w_2 + w_{s_2} = -2,$$
$$v_1 - w_3 + w_{s_3} = 0,$$
$$v_2 - w_4 + w_{s_4} = 0,$$
$$v_3 - w_5 + w_{s_5} = 0,$$
$$w_j, w_{s_j} \ge 0 \ \forall \ j = 1,2,...,5.$$

The problem (P_1) has two optimal basic solutions, x^1 and x^2:

$$x^1 = (3,5,0,0,8), x^2 = (5,4,3,0,1).$$

When b_1 is changed, the ranges of PSA using x^1 and x^2 are $Y_{b_1}(x^1) = \{0\}$ and $Y_{b_1}(x^2) = [-3,7]$. However, the range of PSA using an optimal non-basic solution $x^* = \frac{1}{2}(x^1 + x^2) = (4,9/2,3/2,0,9/2)$ is $[-3,10]$.

In addition, if x^* is a strictly complementary solution in Theorem 9, then we find that $O_{b_h}(B^*, N^*, M^*) \supset \cup_{1 \le i \le r} Y_{b_h}(x^i)$ because $Y_{b_h}(x^*) = O_{b_h}(B^*, N^*, M^*)$.

In the rest of this section, we present a necessary and sufficient condition that c_k can be perturbed while an optimal solution to (P) remains optimal to the perturbed problem. Let P^* denote the set of all optimal solutions to (P).

Lemma 4. *let* x^* *be an optimal solution to* (P) *and* $\sigma = \eta(x^*), \underline{\sigma} = \underline{\eta}(x^*)$ *and* $\overline{\sigma} = \overline{\eta}(x^*)$. *Then*

(i) $\sigma \subset B^*$,

(ii) $\underline{\sigma} \subset (B^* - \sigma) \cup N^*$,

(iii) $N^* \subset \underline{\sigma}$ and $M^* \subset \overline{\sigma}$,

(iv) $\overline{\sigma} = [B^* - (\sigma \cup \underline{\sigma})] \cup M^*$.

Proof. **(i)** Let $j \in \sigma$, this implies that $0 < x_j^* < u_j$. Let $x^{'}$ be the strictly complementary solution to (P) and $w_s^{'}$ be the dual slack vector corresponding to $x^{'}$. Then, by Corollary 1, we have

$$x_j^{'} w_{s_j}^{'} = 0, w_j^{'}(u_j - x_j^{'}) = 0, x_j^{'} + w_{s_j}^{'} > 0, w_j^{'} + (u_j - x_j^{'}) > 0, \quad (4.1)$$
$$x_j^* w_{s_j}^* = 0, w_j^*(u_j - x_j^*) = 0, \quad (4.2)$$

$$x_j' w_{s_j}^* = 0, w_j'(u_j - x_j^*) = 0, \quad (4.3)$$

$$x_j^* w_{s_j}' = 0, w_j^*(u_j - x_j') = 0. \quad (4.4)$$

As $0 < x_j^* < u_j$, this implies that $w_{s_j}' = 0$ and $w_j' = 0$. Then from (4.1), it follows that $0 < x_j' < u_j$. Thus, $j \in B^*$.

(ii) Let $j \in \underline{\sigma}$, this implies that $x_j^* = 0$. From (4.3), we have $w_j' = 0$. Using relation (4.1), $0 \le x_j' < u_j$, i.e., $j \in (B^* - \sigma) \cup N^*$. So, $\underline{\sigma} \subset (B^* - \sigma) \cup N^*$.

(iii) Let $j \in N^*$, this implies that $x_j' = 0$. Then, $w_{s_j}' > 0$, using (4.1), which along with relation (4.4) implies that $x_j^* = 0$ i.e. $j \in \underline{\sigma}$. So, $N^* \subset \underline{\sigma}$.

Let $j \in M^*$, i.e. $x_j' = u_j$. Then $w_j' > 0$, using relation (4.1), which along with relation (4.3) implies that $x_j^* = u_j$, i.e. $j \in \overline{\sigma}$. So, $M^* \subset \overline{\sigma}$.

(iv) Let $j \in \overline{\sigma}$, this implies that $x_j^* = u_j$. So, using relation (4.4), $w_{s_j}' = 0$. This together with (4.1) implies that $x_j' > 0$.

Also, $w_j^* \ge 0$, so $x_j' \le u_j$ (using (4.4)). This implies that $0 < x_j' \le u_j$, i.e.

$$j \in [(B^* - \sigma) \cup N^* - \underline{\sigma}] \cup M^*, \text{ because } j \notin \sigma \cup \underline{\sigma} .$$

Conversely, let $j \in [(B^* - \sigma) \cup N^* - \underline{\sigma}] \cup M^*$. Let $j \in M^*$, then $x_j' = u_j$. This implies that $w_j' > 0$. But $w_j'(u_j - x_j^*) = 0$, which implies that $x_j^* = u_j$. So, $j \in \overline{\sigma}$.

Let $j \in [(B^* - \sigma) \cup N^*] - \underline{\sigma}$, i.e., $j \notin \underline{\sigma}$ but $j \in (B^* - \sigma) \cup N^*$.

$$\Rightarrow j \notin \underline{\sigma} \text{ but } j \in (B^* - \sigma) \text{ using } N^* \subset \underline{\sigma},$$
$$\Rightarrow j \notin \underline{\sigma} \text{ and } j \notin \sigma,$$
$$\Rightarrow j \in \overline{\sigma} \text{ and } x_j^* = u_j.$$

Hence the result.

Theorem 10. *Let* x^* *be an optimal solution to* (P). *Then ,* $\theta \in Y_{c_k}(x^*)$ *for some* $\theta > 0$ *iff* $x_k^* \le x_k$ *for all* $x \in P^*$.

Proof. First we will show that the "only if" part holds. Suppose that $\theta \in Y_{c_k}(x^*)$ for some $\theta > 0$. In addition, suppose that $x_k < x_k^*$ for some $x \in P^*$. Then,

$$[c + \theta e_k]^T x = c^T x + \theta x_k < c^T x^* + \theta x_k^* = [c + \theta e_k]^T x^*.$$

This contradicts the assertion that x^* is an optimal solution to (P_θ). Therefore, $x_k^* \le x_k \ \forall \ x \in P^*$.

Next, we will show that ``if'' part holds. Let $\sigma = \eta(x^*)$ and $\overline{\sigma} = \overline{\eta}(x^*), \underline{\sigma} = \underline{\eta}(x^*)$. Also, let $\pi^* = (B^*, N^*, M^*)$ be an optimal partition of (P) and (D). By Lemma 4, we have $\sigma \subset B^*, \underline{\sigma} \subset (B^* - \sigma) \cup N^*$ and $\overline{\sigma} = [B^* - (\sigma \cup \underline{\sigma})] \cup M^*$.

(i)In case $k \in N^*$: As $N^* \subset \underline{\sigma}$, this implies that $k \in \underline{\sigma}$ i.e. $x_k^* = 0$. If (v^*, w^*, w_s^*) is an optimal solution to (D), then $(v^*, w^*, w_s^* + \theta e_k)$ with $\theta > 0$ is a feasible solution to the following linear equation system:

$$\left\{\begin{array}{ll} A_\sigma^T v = c_\sigma + (\theta e_k)_\sigma, & w_{s_\sigma} = w_\sigma = 0, \\ A_{M^*}^T v - w_{M^*} + w_{s_{M^*}} = c_{M^*} + (\theta e_k)_{M^*}, & w_{s_{N^*}} \geq 0, w_{N^*} = 0, \\ A_{M^*}^T v - w_{M^*} + w_{s_{M^*}} = c_{M^*} + (\theta e_k)_{M^*}, & w_{s_{M^*}} = 0, w_{M^*} \geq 0 \\ A_{(B^*-\sigma)}^T v - w_{(B^*-\sigma)} + w_{s_{(B^*-\sigma)}} = c_{(B^*-\sigma)} + (\theta e_k)_{(B^*-\sigma)}, & w_{s_{(B^*-\sigma)}} w_{(B^*-\sigma)} \geq 0, \end{array}\right\}$$

(4.5)

As (x^*, v^*, w^*, w_s^*) is an optimal solution pair to (P) and (D). All the equation except the second one of the system (4.5) are automatically satisfied. Also,

$$A_{N^*}^T v - w_{N^*} + w_{s_{N^*}} = c_{N^*},\ w_{s_{N^*}} \geq 0, w_{N^*} = 0,$$

because $N^* \subset \underline{\sigma}$ and $w_{s_{\underline{\sigma}}} \geq 0$ and $w_{\underline{\sigma}} = 0$.

Adding both sides $(\theta e_k)_{N^*}$ of the above relation, we have

$$A_{N^*}^T v - w_{N^*} + w_{s_{N^*}} + (\theta e_k)_{N^*} = c_{N^*} + (\theta e_k)_{N^*}.$$

Since $(x^*, v^*, w^*, w_s^* + \theta e_k)$ is an optimal solution pair to (P_θ) and (D_θ). We get $[0, \theta] \subset Y_{c_k}(x^*)$, where $\theta > 0$.

(ii) In the case $k \in B^*$. Consider the following linear programming:

$$\begin{array}{ll} (P') \quad \min\, (e_k)^T x_{B^*} & (D') \quad \max\, (b - A_{M^*} u_{M^*})^T v - u_{B^*}^T w_{B^*} \\ A_{B^*} x_{B^*} = b - A_{M^*} u_{M^*} & A_\sigma^T v - w_\sigma + w_{s_\sigma} = (e_k)_\sigma \\ 0 \leq x_{B^*} \leq u_{B^*}, x_{N^*} = 0 & A_{(B^*-\sigma)}^T v - w_{(B^*-\sigma)} + w_{s_{(B^*-\sigma)}} = (e_k)_{(B^*-\sigma)} \\ & w_{s_\sigma}, w_{s_{(B^*-\sigma)}}, w_{B^*} \geq 0 \end{array} \qquad (4.6)$$

The optimal solutions of (P) are going to be the feasible solutions of (P') (proved in Lemma 5). By assumption , x^* is an optimal solution to (P'), and hence the dual (D') also has at least one optimal solution. Let $(\Delta v, \Delta w_{B^*}, \Delta w_{s_{B^*}})$ be an optimal solution to (D') which satisfies the following:

$$\begin{array}{c} A_\sigma^T \Delta v - \Delta w_\sigma + \Delta w_{s_\sigma} = (e_k)_\sigma \\ A_{(B^*-\sigma)}^T \Delta v - \Delta w_{(B^*-\sigma)} + \Delta w_{s_{(B^*-\sigma)}} = (e_k)_{(B^*-\sigma)} \\ \Delta w_{s_\sigma} = \Delta w_\sigma = 0, \Delta w_{s_{(B^*-\sigma)}}, \Delta w_{(B^*-\sigma)} \geq 0 \end{array} \qquad (4.7)$$

In addition, let

$$\Delta w_{s_{N^*}} = (e_k)_{N^*} - A_{N^*}^T \Delta v,\ \Delta w_{M^*} = (e_k)_{M^*} - A_{M^*}^T \Delta v,\ \Delta w_{s_{M^*}} = 0 = \Delta w_{N^*},$$

and (v^*, w^*, w_s^*) be a strictly complementary solution to (D) i.e.

$$x_j^{'} + w_{s_j}^{*} > 0, \quad w_j^* + (u_j - x_j^{'}) > 0.$$

We set $\hat{\theta}$ as the following:

$$\hat{\theta} = \min\left\{\left(\frac{-w_{s_j}^{*}}{\Delta w_{s_j}} \mid \Delta w_{s_j} < 0,\ j \in N^*\right), \left(\frac{-w_j^*}{\Delta w_j} \mid \Delta w_j < 0,\ j \in M^*\right)\right\}$$

Note that $\hat{\theta}$ is positive. Let $\overline{\theta}$ be a real number such that $0 < \overline{\theta} \le \hat{\theta}$. Then we get a solution $(v^{'}, w^{'}, w_s^{'})$ that satisfies the linear system (4.5) where

$$v^{'} = v^* + \overline{\theta}\Delta v,$$
$$w^{'} = w^* + \overline{\theta}(\Delta w_{B^*}^T, 0, \Delta w_{M^*}^T)^T,$$
$$w_s^{'} = w_s^* + \overline{\theta}(\Delta w_{s_{B^*}}^T, \Delta w_{s_{N^*}}^T, 0)^T.$$

Since $(x^{'}, v^{'}, w^{'}, w_s^{'})$ is an optimal solution pair to $(P_{\overline{\theta}})$ and $(D_{\overline{\theta}})$, we find that $\overline{\theta} \in Y_{c_k}(x^*)$.

(iii) In the case $k \in M^*$: Let $k \in M^* \subset \overline{\sigma}$, this implies that $x_k^* = u_k$. Let (v^*, w^*, w_s^*) be a strictly complementary solution to (D). Then $(w_k^* + (u_k - x_k^{'}) > 0$. But $x_k^* = u_k$ and $x_k^* \le x_k$ implies that $x_k^{'} = u_k$ and hence $w_k^* > 0$. Then, $(v^*, w^* - \overline{\theta} e_k, w_s^*)$ with $0 < \overline{\sigma} \le w_k^* = \hat{\theta}$ is a feasible solution to the linear system (4.5). Since $(x^{'}, v^*, w^* - \overline{\theta} e_k, w_s^*)$ is an optimal solution pair to $(P_{\overline{\theta}})$ and $(D_{\overline{\theta}})$, we get $[0, \overline{\theta}] \subset Y_{c_k}(x^*)$ where $\overline{\theta} > 0$.

In the next lemma we prove that the set of feasible solutions of problem $(P^{'})$ contains only optimal solutions of (P).

Lemma 5. *The set of feasible solutions of problem* $(P^{'})$ *is* (P^*).

Proof. Consider the problem $(P^{'})$

$$\min\, (e_k)_{B^*}^T x_{B^*}$$

subject to constraints

$$A_{B^*} x_{B^*} + A_{M^*} x_{M^*} = b,$$
$$0 \le x_{B^*} \le u_{B^*},\ x_{N^*} = 0,\ x_{M^*} = u_{M^*}.$$

We will show that for a feasible solution, $x^{'}$ but not an optimal solution of (P) either $x_j^{'} \ne u_j$ for some $j \in M^*$ or $x_j^{'} \ne 0$ for some $j \in N^*$.

Let (x, x_s) and (v, w, w_s) be feasible solutions to (P) and (D) where $x_s = u - x$, vector of primal slack variables associated with bound constraints. Then, we have

$$x^T w_s + w^T x_s = c^T x - (b^T v - u^T w). \tag{4.8}$$

Using, weak duality theorem,

$$x^T w_s + w^T x_s = x^T w_s + w^T (u - x) \geq 0. \qquad (4.9)$$

Let (v^*, w^*, w_s^*) be an optimal solution of dual and $x^{'}$ is a feasible solution of (P) but not an optimal solution. Then from (4.9),

$$x^{'T} w_s^* + w^{*T} (u - x^{'}) > 0. \qquad (4.10)$$

Above relation implies that either there exists some j such that $x_j^{'} w_{s_j}^* > 0$ or $w_j^* (u_j - x_j^{'}) > 0$ or both.

Let there exists some j such that $x_j^{'} w_{s_j}^* > 0$, which implies that $0 < x_j^{'} \leq u_j$, and $w_{s_j}^* \neq 0$. Let $\hat{x}$ be a strictly complementary solution to (P). Then, using complementary slackness theorem, $\hat{x}_j w_{s_j}^* = 0$. This implies that $\hat{x}_j = 0$ and hence $j \in N^*$. But $x_j^{'} \neq 0$, so $x_{N^*}^{'} \neq 0$, which shows that $x^{'}$ is not a feasible solution to $(P^{'})$.

Let there exists some j such that $w_j^* (u_j - x_j^{'}) > 0$, which implies that $w_j^* > 0$ and $0 \leq x_j^{'} < u_j$. But $w_j^* (u_j - \hat{x}_j) = 0$, which implies that $\hat{x}_j = u_j$, i.e., $j \in M^*$. So, we have $x_{M^*}^{'} \neq u_{M^*}$. This shows that $x^{'}$ is not a feasible solution to $(P^{'})$.

Thus for any feasible solution to (P) but not optimal either $x_{M^*} \neq u_{M^*}$ or $x_{N^*} \neq 0$ or both. So any non-optimal solution to (P) cannot be a feasible solution to $(P^{'})$.

In other words, every feasible solution of $(P^{'})$ is an element of (P^*). Conversely by Lemma 5, every optimal solution of (P) is a feasible solution of $(P^{'})$.

Theorem 11. *Let x^* be an optimal solution to (P). Then $\theta \in Y_{c_k}(x^*)$ for some $\theta < 0$ iff $x_k^* \geq x_k$ for all $x \in P^*$.*

Proof. First, we will show that the "Only if" part holds. Suppose that $\theta \in Y_{c_k}(x^*)$ for some $\theta < 0$. In addition suppose that $x_k > x_k^*$ for some $x \in P^*$. Then,

$$(c + \theta e_k)^T x = c^T x + \theta x_k < c^T x^* + \theta x_k^* = (c + \theta e_k)^T x^*.$$

This contradicts the assertion that x^* is an optimal solution to (P_θ). Therefore $x_k^* \geq x_k$ for all $x \in P^*$.

Next we will show that "if" part holds. Let $\sigma = \eta(x^*), \underline{\sigma} = \underline{\eta}(x^*)$ and $\overline{\sigma} = \overline{\eta}(x^*)$. Also, let $\pi^* = (B^*, N^*, M^*)$ be the optimal partition to (P) and (D).

(i) In the case $k \in N^*$: As $k \in N^*$ and since $N^* \subset \underline{\sigma} \Rightarrow k \in \underline{\sigma}$. So, $x_k^* = 0$.. let (v^*, w^*, w_s^*) be a strictly complementary solution to (D). Then $w_{s_k}^* > 0$ and $(v^*, w^*, w_s^* + \underline{\theta} e_k)$ satisfies the linear system (4.5) where $-w_{s_k}^* \leq \underline{\theta} < 0$.

(ii) In the case $k \in B^*$: It can be easily shown that $\underline{\theta} \in Y_{c_k}(x^*)$ for some $\underline{\theta}$, by replacing vector e_k in (4.6) and (4.7) with $(-e_k)$ and applying the same technique as in Theorem 10.

(iii) In the case $k \in M^*$**:** As $k \in M^* \subset \overline{\sigma}$, $x_k^* = u_k$. Let (v^*, w^*, w_s^*) be a strictly complementary solution to (D). (Note that $w_k^* > 0$)). Then $(v^*, w^* + \underline{\theta}, w_s^*)$ satisfies the linear system (4.5) where $-w_k^* \le \underline{\theta} < 0$. Since $(x^*, v^*, w^* + \underline{\theta}, w_s^*)$ is an optimal solution pair to $(P_{\underline{\theta}})$ and $D_{\underline{\theta}})$, we get $[\underline{\theta}, 0] \subset Y_{c_k}(x^*)$.

By Theorem 10 and 11, we know that the range of PSA using x^* on c_k includes both a positive and a negative value iff for any optimal solution x the k^{th} element x_k has the same value. In addition, we arrive at another interesting result about the range of OSA as follows:

Corollary 3. *Let* $\pi^* = (B^*, N^*, M^*)$ *be the optimal partition of* (P) *and* (D). *Then,* $O_{c_k}(B^*, N^*, M^*) \ne [0,0]$ *iff* $x_k^* = \alpha$ *for all* $x^* \in P^*$, *where* $\alpha \in [0, u_k]$.

Proof. First, suppose that $O_{c_k}(B^*, N^*, M^*) \ne [0,0]$. Let x' be a strictly complementary optimal solution. Since $Y_{c_k}(x') = O_{c_k}(B^*, N^*, M^*), Y_{c_k}(x') \ne [0,0]$. If x' is a unique optimal solution to (P), then corollary trivially holds. Otherwise, let x^1 be an optimal solution to (P) such that $x^1 \ne x'$, then, there exists an optimal solution x^2 such that

$$x' = \lambda x^1 + (1-\lambda) x^2, \quad \text{for some } \lambda > 0. \tag{4.11}$$

By Theorem 10 and 11, $Y_{c_k}(X') \ne [0,0]$ implies that $x_k' \le x_k^i$ or $x_k' \ge x_k^i$ for $i = 1,2$. This, together with equation (4.11), implies that $x_k' = x_k^1 = x_k^2$. Since x^1 is chosen arbitrary, $x_k^* = \alpha$ for all $x^* \in P^*$ where α is a non-negative real number and $\alpha \in [0, u_k]$. Next, we will show that the reverse holds. Suppose that $x_k^* = \alpha$ for all $x^* \in P^*$. hen by Theorem 10 and 11, there exists $\underline{\theta}$ and $\overline{\theta}$ such that $[\underline{\theta}, \overline{\theta}] \subset O_{c_k}(B^*, N^*, M^*), \underline{\theta} < 0$ and $\overline{\theta} > 0$.

5. **The relationship between PSA and BSA under degeneracy**

In this section, we discuss the relationship between PSA and BSA by comparing PSA with BSA under degeneracy. Let x^* be an optimal basic solution to (P). If x^* is degenerate, there can be more than one optimal basis associated with BSA, using each optimal basis may produce a different range of perturbation θ. For example, consider the following linear programming problem (LP_2):

$$(P_2): \quad \min \ -x_1 - 2x_2$$
$$-x_1 + 2x_2 + x_3 = 6, \ 3x_1 + x_2 + x_4 = 17, \ x_1 + x_2 + x_5 = 9,$$
$$0 \le x_1 \le 5, 0 \le x_2 \le 5, 0 \le x_3 \le 6, 0 \le x_4 \le 17, 0 \le x_5 \le 6.$$
$$(D_2): \quad \max \ 6v_1 + 17v_2 + 9v_3 - 5w_1 - 5w_2 - 6w_3 - 17w_4 - 6w_5$$

$$-v_1+3v_2+v_3-w_1+w_{s_1}=-1,$$
$$2v_1+v_2+v_3-w_2+w_{s_2}=-2,$$
$$v_1-w_3+w_{s_3}=0,$$
$$v_2-w_4+w_{s_4}=0,$$
$$v_3-w_5+w_{s_5}=0,\quad w_j,w_{s_j}\ge 0.$$

The unique optimal solution x^* to (P_2) is $(4,5,0,0,0)^T$, that is a degenerate basic solution. There are three primal-optimal bases, $(A_{B^1}, A_{N_1^1}, A_{N_2^1}), (A_{B^2}, A_{N_1^2}, A_{N_2^2})$ and $(A_{B^3}, A_{N_1^3}, A_{N_2^3})$ where

$B^1=\{1,4,5\}$, $N_1^1=\{3\}$, $N_2^1=\{2\}$; $B^2=\{1,3,5\}$, $N_1^2=\{4\}$, $N_2^2=\{2\}$; $B^3=\{1,3,4\}$, $N_1^3=\{5\}$, $N_2^3=\{2\}$. Both $(A_{B^i}, A_{N_1^i}, A_{N_2^i})$, for $i=2,3$ are optimal bases and $(A_{B^1}, A_{N_1^1}, A_{N_2^1})$ is a primal optimal basis, but not an optimal basis. When c_1 is changed, the range of BSA using $(A_{B^2}, A_{N_1^2}, A_{N_2^2})$ and $(A_{B^3}, A_{N_1^3}, A_{N_2^3})$ are $[-5,1]$ and $[-1,1]$ respectively and using $(A_{B^1}, A_{N_1^1}, A_{N_2^1})$ is $[1,2]$. On the other hand, the range of PSA using x^* is $[-5,2]$.

Ward et al. (1990) showed that when a cost vector is changed, the range of θ within which an optimal basic solution x^* remains an optimal solution to (P_θ) is the union of the ranges of sensitivity analysis using all primal-optimal bases associated with x^*. Park et al.(2004) proved the similar result for $Y_{c_k}(x^*)$. Since $Y_{c_k}(x^*)$ is the range of θ within which x^* remains optimal to (P_θ), we obtain the following theorem:

Theorem 12. *Let x^* be an optimal degenerate basic solution. Let $(B^1, N_1^1, N_2^1),\cdots,$ (B^r, N_1^r, N_2^r) be the index set of basic variables, non-basic variables at lower and upper bounds respectively of all the primal-optimal bases associated with x^*. Then*

$$Y_{c_k}(x^*)=\bigcup_{1\le i\le r} T_{c_k}(A_{B^i}, A_{N_1^i}, A_{N_2^i}).$$

Proof. Let $\theta\in T_{c_k}(A_{B^i}, A_{N_1^i}, A_{N_2^i})$, where

$$T_{c_k}(A_{B^i}, A_{N_1^i}, A_{N_2^i})=\left\{\theta\left|\begin{array}{l}\begin{bmatrix}A_{B^i}^T\\ A_{N_1^i}^T\\ A_{N_2^i}^T\end{bmatrix}v-\begin{bmatrix}w_{B^i}\\ w_{N_1^i}\\ w_{N_2^i}\end{bmatrix}+\begin{bmatrix}w_{s_B^i}\\ w_{s_{N_1^i}}\\ w_{s_{N_2^i}}\end{bmatrix}=\begin{bmatrix}c_{B^i}+(\theta e_k)_{B^i}\\ c_{N_1^i}+(\theta e_k)_{N_1^i}\\ c_{N_2^i}+(\theta e_k)_{N_2^i}\end{bmatrix},\\ w_{s_B^i}=w_{B^i}=w_{N_1^i}=w_{s_{N_2^i}}=0,\ w_{s_{N_1^i}},w_{N_2^i}\ge 0\end{array}\right.\right\}$$

We will show that $\theta\in Y_{c_k}(x^*)$ i.e.

$$\begin{bmatrix} A_{\sigma}^{T} \\ A_{\underline{\sigma}}^{T} \\ A_{\overline{\sigma}}^{T} \end{bmatrix} v - \begin{bmatrix} w_{\sigma} \\ w_{\underline{\sigma}} \\ w_{\overline{\sigma}} \end{bmatrix} + \begin{bmatrix} w_{s_{\sigma}} \\ w_{s_{\underline{\sigma}}} \\ w_{s_{\overline{\sigma}}} \end{bmatrix} = \begin{bmatrix} c_{\sigma} + (\theta e_k)_{\sigma} \\ c_{\underline{\sigma}} + (\theta e_k)_{\underline{\sigma}} \\ c_{\overline{\sigma}} + (\theta e_k)_{\overline{\sigma}} \end{bmatrix},$$
$$w_{s_{\sigma}} = w_{\sigma} = w_{\underline{\sigma}} = w_{s_{\overline{\sigma}}} = 0, w_{s_{\underline{\sigma}}}, w_{\overline{\sigma}} \geq 0$$

We know that

(i) $\sigma \subset B^i$

(ii) $\underline{\sigma} \subset (B^i - \sigma) \cup N_1^{'}$, $N_1^i \subset \underline{\sigma}$, and $\underline{\sigma} = (B^i - \overline{\sigma \cup \overline{\sigma}}) \cup N_1^i$,

(iii) $\overline{\sigma} = (B^i - \overline{\sigma \cup \underline{\sigma}}) \cup N_2^i$, $\quad N_2^i \subset \overline{\sigma}$.

As $w_{s_{B^i}} = w_{B^i} = 0$ and $\sigma \subset B^i$, this implies that $w_{\sigma} = w_{s_{\sigma}} = 0$. Also, $(B^i - \sigma) \subset B^i$, so $w_{(B^i-\sigma)} = w_{s_{(B^i-\sigma)}} = 0$.

As $w_{N_1^i} = 0, w_{s_{N_1^i}} \geq 0$ and $w_{N_2^i} \geq 0, w_{s_{N_2^i}} = 0$; this implies that $w_{\underline{\sigma}} = 0$ and $w_{s_{\underline{\sigma}}} \geq 0$. Also, for $j \in (B^i - \overline{\sigma \cup \underline{\sigma}}), w_j = 0$, $w_{s_j} = 0$ and $w_{N_2^i} \geq 0, w_{s_{N_2^i}} = 0.$

$$\Rightarrow w_{\overline{\sigma}} \geq 0 \text{ and } w_{s_{\overline{\sigma}}} = 0,$$
$$\Rightarrow \theta \in Y_{c_k}(x^*).$$

Conversely, let $\theta \in Y_{c_k}(x^*)$ and x^* is an optimal degenerate basic solution and out of (B^i, N_1^i, N_2^i) there exists at least one basis s.t.

$$Y_{c_k}(x^*) = T_{c_k}(B^i, N_1^i, N_2^i),$$
$$\text{i.e. } \theta \in \bigcup_{1 \leq i \leq r} T_{c_k}(B^i, N_1^i, N_2^i)$$

Hence the result.

Theorem 13. *Let x^* be an optimal degenerate basic solution to (P). Let $(B^1, N_1^1, N_2^1), \cdots, (B^r, N_1^r, N_2^r)$ be all the optimal bases associated with x^*. Then*

$$Y_{b_h}(x^*) \subset \bigcap_{1 \leq i \leq r} T_{b_h}(A_{B^i}, A_{N_1^i}, A_{N_2^i}).$$

Moreover, if $Y_{b_h}(x^*) \neq 0$, then $Y_{b_h}(x^*) = T_{b_h}(A_{B^i}, A_{N_1^i}, A_{N_2^i})$ for $1 \leq i \leq r$.

Proof. By definition,

$$T_{b_h}(A_{B^i}, A_{N_1^i}, A_{N_2^i}) = \left\{\gamma \mid A_B x_B + A_{N_2} x_{N_2} = b + \gamma e_h, x_{N_1} = 0, x_{N_2} = u_{N_2}, 0 \leq x_B \leq u_B \right\} \quad (5.12)$$

$$Y_{b_h}(x^*) = \left\{\gamma \mid A_{\sigma} x_{\sigma} + A_{\overline{\sigma}} x_{\overline{\sigma}} = b + \gamma e_h, 0 \leq x_{\sigma} \leq u_{\sigma}, x_{\underline{\sigma}} = 0, x_{\overline{\sigma}} = u_{\overline{\sigma}} \right\} \quad (5.13)$$

where $\sigma = \eta(x^*), \underline{\sigma} = \underline{\eta}(x^*)$ and $\overline{\sigma} = \overline{\eta}(x^*)$. For each B^i and N_2^i, $\sigma \subset B^i$. First we will prove that $\overline{\sigma} \subset (B^i \cup N_2^i)$. Let $j \in \overline{\sigma}$. Then $x_j^* = u_j$, so there are two possibilities, either x_j^* is basic and at its upper bound such that solution is degenerate or x_j^* is non-basic and at its upper bound i.e. either $j \in B^i$ or $j \in N_2^i$.

$$\Rightarrow\ j \in (B^i \cup N_2^i), \text{i.}e.\, (\sigma \cup \overline{\sigma}) \subset (B^i \cup N_2^i).$$

Then $Pos(\sigma \cup \overline{\sigma}) \subset Pos(B^i \cup N_2^i)$. This together with equation (5.12) and (5.13) implies that $Y_{b_h}(x^*) \subset T_{b_h}(A_{B^i}, A_{N_1^i}, A_{N_2^i})$ for each i. Therefore,

$$Y_{b_h}(x^*) \subset \bigcap_{1\le i\le r} T_{b_h}(A_{B^i}, A_{N_1^i}, A_{N_2^i}).$$

Suppose that $Y_{b_h}(x^*)$ includes any non-zero value. For any arbitrary optimal basis B^i, $\sigma \subset B^i$ for each i and each column in A_σ is linearly independent with all the remaining columns in A_{B^i}. Let $\gamma \in T_{b_h}(A_{B^i}, A_{N_1^i}, A_{N_2^i})$. Then

$$A_{B^i}x_{B^i} + A_{N_2^i}x_{N_2^i} = b + \gamma e_h, 0 \le x_{B^i} \le u_{B^i}, x_{N_1^i} = 0, x_{N_2^i} = u_{N_2^i} \qquad (5.14)$$

Note that $\sigma \subset B^i$. If relation (5.14) holds, then

$$A_\sigma x_\sigma + A_{\overline{\sigma}} x_{\overline{\sigma}} = b + \gamma e_h \qquad (5.15)$$

holds trivially.

In order to prove that $\gamma \in Y_{b_h}(x^*)$, it is sufficient to show that $x_{\underline{\sigma}} = 0, x_{\overline{\sigma}} = u_{\overline{\sigma}}, 0 \le x_\sigma \le u_\sigma$.

Since $B^i \cup N_2^i = (B^i - \overline{(\sigma \cup \overline{\sigma})}) \cup \sigma \cup \overline{\sigma}$ (5.14) can be written as

$$A_\sigma x_\sigma + A_{\overline{\sigma}} x_{\overline{\sigma}} + A_{(B^i-(\sigma\cup\overline{\sigma}))}\, x_{(B^i-(\sigma\cup\overline{\sigma}))} = b + \gamma e_h \qquad (5.16)$$

Equation (5.15) and (5.16) imply that

$$A_{(B^i-(\sigma\cup\overline{\sigma}))}\, x_{(B^i-(\sigma\cup\overline{\sigma}))} = 0.$$

This implies that $x_{(B^i-(\sigma\cup\overline{\sigma}))} = 0$, since columns of $A_{(B^i-(\sigma\cup\overline{\sigma}))}$ are linearly independent.

$\Rightarrow$ $(B^i - \overline{(\sigma \cup \overline{\sigma})}) \cup N_1^i \subset \underline{\sigma}$ as $N_1^i \subset \underline{\sigma}$. We will show that $\underline{\sigma} = (B^i - \overline{(\sigma \cup \overline{\sigma})}) \cup N_1^i$. It can be shown that if $j \notin (B^i - \overline{(\sigma \cup \overline{\sigma})})$, then $j \notin \underline{\sigma}$.

Let $j \notin (B^i - \overline{(\sigma \cup \overline{\sigma})}) = (B^i - \sigma) \cap (B^i - \overline{\sigma})$. There are following possibilities:

(i) $j \notin (B^i - \sigma)$ and $j \notin (B^i - \overline{\sigma})$. If x_j is basic, then $0 < x_j < u_j$ and $x_j = u_j$, which is not feasible.

(ii) $j \notin (B^i - \sigma)$ and $j \in (B^i - \overline{\sigma})$. It implies that x_j is basic and strictly between its bounds. So, $j \notin \underline{\sigma}$.

(iii) $j \in (B^i - \sigma)$ and $j \notin (B^i - \overline{\sigma})$. It implies that x_j is basic and is at its upper bound, so $j \notin \underline{\sigma}$.

Thus $\underline{\sigma} = (B^i - \overline{(\sigma \cup \overline{\sigma})}) \cup N_1^i$. This proves that $x_{\underline{\sigma}} = 0$.

Similarly, $\overline{\sigma} = (B^i - \overline{(\sigma \cup \underline{\sigma})}$ which implies that $x_{\overline{\sigma}} = u_{\overline{\sigma}}$.

$$x_{\underline{\sigma}} = 0, x_{\overline{\sigma}} = u_{\overline{\sigma}}, 0 \le x_\sigma \le u_\sigma.$$

$$\Rightarrow\ \gamma \in Y_{b_h}(x^*) \text{ and hence } T_{b_h}(A_{B^i}, A_{N_1^i}, A_{N_2^i}) = Y_{b_h}(x^*).$$

That is, if $Y_{b_h}(x^*)$ includes any non-zero value, we know that

$\bigcap_{1\le i\le r} T_{b_h}(A_{B^i}, A_{N_1^i}, A_{N_2^i}) = Y_{b_h}(x^*)$, which is similar to Theorem 12. However, when $Y_{b_h}(x^*) = \{0\}$, $Y_{b_h}(x^*)$ may not be equal to $\bigcap_{1\le i\le r} T_{b_h}(A_{B^i}, A_{N_1^i}, A_{N_2^i})$, which is illustrated by the following linear programming (LP_3):

$$(P_3):\ \min x_1$$
$$x_1 + x_2 + x_3 = 5,\ x_1 + 2x_2 + 2x_3 = 5,$$
$$0 \le x_1 \le 5, 0 \le x_2 \le 4, 0 \le x_3 \le 5.$$
$$(D_3):\ \max 5v_1 + 5v_2 - 5w_1 - 4w_2 - 5w_3$$
$$v_1 + v_2 - w_1 + w_{s_1} = 1,$$
$$v_1 + 2v_2 - w_2 + w_{s_2} = 0,$$
$$v_1 + 2v_2 - w_3 + w_{s_3} = 0,$$
$$w_j, w_{s_j} \ge 0\ \forall\ j = 1,2,3.$$

The unique optimal solution to (P_3) is $x^* = (5,0,0)$, and there are two optimal bases associated with x^*:

$$A_{B^1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, A_{N_1^1} = \begin{bmatrix} 1 \\ 2 \\ \ \end{bmatrix}, A_{N_2^1} = \phi,$$

$$A_{B^2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} A_{N_1^2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, A_{N_2^2} = \phi,$$

$$\sigma = \phi, \underline{\sigma} = \{2,3\}, \overline{\sigma} = \{1\},$$

where $B^1 = \{1,2\}, N_1^1 = \{3\}, N_2^1 = \phi; B^2 = \{1,3\}, N_1^2 = \{2\}, N_2^2 = \phi.$ When b_1 is perturbed, the ranges of BSA using $(A_{B^1}, A_{N_1^1}, A_{N_2^1})$ and $(A_{B^2}, A_{N_1^2}, A_{N_2^2})$ are $T_{b_1}(A_{B^1}, A_{N_1^1}, A_{N_2^1}) = [-5,0], T_{b_1}(A_{B^2}, A_{N_1^2}, A_{N_2^2}) = [-5,0]$. However, the range of PSA using x^* is $Y_{b_1}(x^*) = [0,0]$, by using the following equation:

$$Y_{b_1}(x^*) = \left\{ \gamma \,|\, \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 5+\gamma \\ 5 \end{bmatrix}, x_2 = x_3 = 0, x_1 = u_1 = 5 \right\}.$$

Consequently, we find that $Y_{b_1}(x^*) \ne T_{b_1}(A_{B^1}, A_{N_1^1}, A_{N_2^1}) \cap T_{b_1}(A_{B^2}, A_{N_1^2}, A_{N_2^2})$.

6. Concluding Remarks

In this paper, we have established the existence of strictly complementary solution for linear programming problem with bounds on the variables, leading to the study of optimal partition sensitivity analysis for such problems. The properties of PSA and its relationship with other sensitivity analysis methods, BSA and OSA have been discussed. The main advantage of PSA is that it can be performed with any optimal solution which is a non-basic or basic solution. PSA finds the range within which there exists an optimal solution to the perturbed problem whose induced partition is equal to the induced partition of a given optimal solution. PSA focuses only on the induced partition of primal-optimal solutions. Hence the properties of PSA on a cost coefficient differ from those of PSA on a right-hand side.

We presented some properties of PSA that are useful for comparing PSA with the other two sensitivity analysis methods. When a cost coefficient is perturbed, the range of PSA is equal to the interval where a given optimal solution remains optimal to the perturbed problem. On the other hand, when right-hand side is changed, the range of PSA finds the interval where the induced partition of a given optimal solution remains the induced partition of some optimal solution to the perturbed problem. Another important property of PSA on a cost coefficient is the ranges of PSA using an optimal non-basic solution is the intersection of the ranges of PSA using optimal basic solutions, whose convex combination leads to the optimal non-basic solution.

Finally, further studies will be needed, which will deal with the computational performance and numerical experience of sensitivity analysis methods. Given an optimal basis, BSA is obviously the most efficient where the computational time is concerned. However, most codes using interior-point methods often produce an optimal non-basic solution and in this case PSA is expected to be a good alternate because PSA can be applied without obtaining an optimal basis or optimal partition which may require more computational time if a problem is ill-conditioned.

Park et.al. (2004) discussed the properties and relationships of PSA, OSA and BSA by partitioning the decision variables into two categories and this motivated us to extend this study to the case where decision variables are bounded and hence they have to be partitioned into three categories depending upon BSA, OSA and BSA. Although the bounds on the decision variables can be treated as additional constraints but this leads to considerably increase in the size of the problem under consideration, thus it is not preferable to consider the bounds on decision variables as additional constraints.

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