# On a new type of Double Sequences of Fuzzy Numbers.

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## ABSTRACT

In this article we introduce the notion of $\Delta$-statistically pre-Cauchy double sequence of fuzzy numbers and establish a criterion for arbitrary double sequence of fuzzy numbers to be $\Delta$-statistically pre-Cauchy.

(Keywords: double sequence of fuzzy numbers, difference sequence, statistically pre-Cauchy double sequence, Orlicz function)

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## INTRODUCTION

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability, and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [35] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [20] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [15], Nanda [21], Esi [13], Dutta [7, 8, 9], and many others.

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u: R \to [0,1]$ which satisfies the following four conditions:

(*i*) $u$ is normal, i.e., there exist an $x_0 \in R$ such that $u(x_0)=1$.
(*ii*) $u$ is fuzzy convex, i.e. $u[\lambda x+(1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in R$ and for all $\lambda \in [0,1]$.
(*iii*) $u$ is upper semi-continuous.
(*iv*) The set $[u]_0 = \overline{\{x \in R : u(x) > 0\}}$ is compact, where $\overline{\{x \in R : u(x) > 0\}}$ denotes the closure of the set $\{x \in R : u(x) > 0\}$ in the usual topology of $R$.

We denote the set of all fuzzy numbers on $R$ by $E^1$ and called it as the space of fuzzy numbers. $\lambda$-level set $[u]_\lambda$ of $u \in E^1$ is defined by:

$$[u]_\lambda = \{t \in R : u(t) \geq \lambda\} \text{ if } (0<\lambda\leq 1),$$
$$= \overline{\{t \in R : u(t) > \lambda\}} \text{ if } (\lambda = 0)$$

The set $[u]_\lambda$ is a closed, bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. $R$ can be embedded in $E^1$, since each $r \in R$ can be regarded as a fuzzy number,

$$\overline{r(t)} = 1 \text{ if } t = r$$
$$= 0 \text{ if } t \neq r$$

Let $W$ be the set of all closed bounded intervals $A$ of real numbers such that $A = [A_1, A_2]$. Define the relation $d$ on $W$ as follows:

$$d(A,B) = \max\{|A_1 - B_1|, |A_2 - B_2|\}.$$

Then $(W, d)$ is a complete metric space (See Diamond and Kloeden [14], Nanda [21]).

A fuzzy double sequence is a double infinite array of fuzzy real numbers. We denote a fuzzy double sequence by $(a_{mn})$, where $a_{mn}$ are fuzzy real numbers for each $m, n \in N$. The initial works on double sequences of real or complex terms is found in Bromwich [1]. Hardy [18] introduced the notion of regular convergence for double sequences of real or complex terms. The works on double sequence was further investigated by Basarir and Solancan [2], Moricz [23], Tripathy and Dutta [31], Tripathy and Sarma [32] and many others.

The concept of statistical convergence was first introduced by Fast [16] and also independently by Buck [3] and Schoenberg [28] for real and complex sequences. Further this concept was studied by Salat [27], Fridy [17], Cannor [4, 5] and many others.

A double sequence of fuzzy number $(x_{kl})$ is called $\Delta$-statistically convergent to *L* if

$$\lim_{m,n\to\infty} \frac{1}{mn}\left|\{(j,k): d(\Delta x_{jk}, L) \geq \varepsilon,\ j \leq m, k \leq n\}\right| = 0 .$$

where the vertical bars indicate the number of elements in the set.

**Definition 1:** A double sequence of fuzzy number $(x_{kl})$ is called $\Delta$-statistically pre-Cauchy if for every $\varepsilon > 0$ there exist $p = p(\varepsilon)$ and $q(\varepsilon)$ such that

$$\lim_{m,n\to\infty} \frac{1}{m^2 n^2}\left|\{(j,k): d(\Delta x_{jk}, \Delta x_{pq}) \geq \varepsilon,\ j \leq m, k \leq n\}\right| = 0$$

In fact, the first order difference operator $\Delta$ can be viewed as an infinite triangular matrix as follows:

$$\Delta = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

The fuzzy double sequence by $(a_{mn})$ can be expressed as an infinite matrix of fuzzy numbers as follows:

$$(a_{mn}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & \dots \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & \dots \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & & \dots & \dots \end{pmatrix}$$

Now for any fuzzy double sequence by
*a* = $(a_{mn})$, we have

$$\Delta a = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & \dots \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & \dots \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & & \dots & \dots \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}-a_{21} & a_{12}-a_{22} & \dots & a_{1n}-a_{2n} & \dots \\ a_{21}-a_{31} & a_{22}-a_{32} & \dots & a_{2n}-a_{3n} & \dots \\ a_{31}-a_{41} & a_{32}-a_{42} & \dots & a_{3n}-a_{4n} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

This approach of construction of difference sequences is useful to study some properties of the spaces of such sequences.

An Orlicz Function is a function $M:[0,\infty) \to [0,\infty)$ which is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

If convexity of *M* is replaced by $M(x+y) \leq M(x) + M(y)$, then it is called a Modulus function (see Maddox[22]). An Orlicz function may be bounded or unbounded.
For example, $M(x) = x^p$ $(0 < p \leq 1)$ is unbounded and $M(x) = \dfrac{x}{x+1}$ is bounded.

Lindesstrauss and Tzafriri [19] used the idea of Orlicz sequence space and introduced the sequence space $\ell_M$ as follows:

$$l_M = \{x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,$$

for some $\rho > 0$

They proved that $\ell_M$ is a Banach space normed by:

$$\| x \|_M = \| (x_k) \| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

The space $l_M$ is closely related to the space $l_p$, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$. An Orlicz function M satisfies the $\Delta_2$-condition $(M \in \Delta_2$ for short) if there exist constant $k \geq 2$ and $u_0 > 0$ such that $M(2u) \leq KM(u)$ whenever $|u| \leq u_0$. Note that an Orlicz function satisfies the inequality.

$M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 < \lambda < 1$.

The study of Orlicz sequence spaces have been made recently by various authors (cf [10], [11], [12], [25], [33], [34]).

In [6] Connor, Fridy and Kline proved that a bounded sequence $x = (x_i)$ is statistically pre-Cauchy if and only if $\lim_n \sum_{j,i \leq n} |x_i - x_j| = 0$.

We establish the following criterion for arbitrary double sequence of fuzzy numbers to be statistically pre-Cauchy.

## MAIN RESULTS

**Theorem 1:** Let $x = (x_{jk})$ be the double sequence of fuzzy number and let *M* be a bounded Orlicz function. Then *x* is $\Delta$-statistically pre-Cauchy if and only if

$$\lim_{m,n} \frac{1}{m^2 n^2} \sum_{j,p \leq m} \sum_{k,q \leq n} M\left(\frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho}\right) = 0,$$

for some $\rho > 0$.

**Proof :** Suppose

$$\lim_{m,n} \frac{1}{m^2 n^2} \sum_{j,p \leq m} \sum_{k,q \leq n} M\left(\frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho}\right) = 0,$$

for some $\rho > 0$. For each $\varepsilon > 0$, $\rho > 0$ and $m, n \in N$, we have

$$\frac{1}{m^2 n^2} \sum_{j,p \leq m} \sum_{k,q \leq n} M\left(\frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho}\right)$$

$$= \frac{1}{m^2 n^2} \sum_{j,p \leq m, d(\Delta x_{jk}, \Delta x_{pq}) < \varepsilon} \sum_{k,q \leq n} M\left(\frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho}\right)$$

$$+ \frac{1}{m^2 n^2} \sum_{j,p \leq m, d(\Delta x_{jk}, \Delta x_{pq}) \geq \varepsilon} \sum_{k,q \leq n} M\left(\frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho}\right)$$

$$\geq \frac{1}{m^2 n^2} \sum_{j,p \leq m, d(\Delta x_{jk}, \Delta x_{pq}) \geq \varepsilon} \sum_{k,q \leq n} M\left(\frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho}\right)$$

$$\geq M(\varepsilon)\left(\frac{1}{m^2 n^2} |\{(j,k) : d(\Delta x_{jk}, \Delta x_{pq}) \geq \varepsilon, j \leq m, k \leq n\}|\right)$$

$\geq 0$.

Now suppose that *x* is $\Delta$-statistically pre-Cauchy and that $\varepsilon$ has been given. Let $\varepsilon > 0$ be such that $M(\delta) < \frac{\varepsilon}{2}$. Since *M* is bounded, there exist an integer *B* such that $M(x) < \frac{B}{2}$ for all $x \geq 0$. Note that, for each $n \in N$.

$$\frac{1}{m^2 n^2} \sum_{j,p \leq m} \sum_{k,q \leq n} M\left(\frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho}\right)$$

$$= \frac{1}{m^2 n^2} \sum_{j,p \leq m, d(\Delta x_{jk}, \Delta x_{pq}) < \delta} \sum_{k,q \leq n} M\left(\frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho}\right)$$

$$+ \frac{1}{m^2 n^2} \sum_{j,p \leq m, d(\Delta x_{jk}, \Delta x_{pq}) \geq \delta} \sum_{k,q \leq n} M\left(\frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho}\right)$$

$$\leq M(\delta) + \frac{1}{m^2 n^2} \sum_{j,p \leq m, d(\Delta x_{jk}, \Delta x_{pq}) \geq \delta} \sum_{k,q \leq n} M\left(\frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho}\right)$$

$$\leq \frac{\varepsilon}{2} + \frac{B}{2}\left( \frac{1}{m^2n^2} |\{(j,k): d(\Delta x_{jk}, \Delta x_{pq}) \geq \delta, j \leq m, k \leq n\}| \right)$$

$$\leq \varepsilon + B\left( \frac{1}{m^2n^2} |\{(j,k): d(\Delta x_{jk}, \Delta x_{pq}) \geq \delta, j \leq m, k \leq n\}| \right) \quad \textbf{(1)}$$

Since $x$ is $\Delta$-statistically pre-Cauchy, there is $N$ such that R.H.S of equation (1) is less than $\varepsilon$ for all $n \in N$. Hence

$$\lim_{m,n} \frac{1}{m^2n^2} \sum_{j,p\leq m} \sum_{k,q\leq n} M\left( \frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho} \right) = 0 .$$

**Theorem 2:** Let $x = (x_{jk})$ be the double sequence of fuzzy number and let $M$ be a bounded Orlicz function. Then $x$ is $\Delta$-statistically convergent to $L$ if and only if

$$\lim_{m,n} \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} M\left( \frac{d(\Delta x_{jk}, L)}{\rho} \right) = 0$$

**Proof:** Suppose that

$$\lim_{m,n} \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} M\left( \frac{d(\Delta x_{jk}, L)}{\rho} \right) = 0$$

with an Orlicz function $M$, then $x$ is $\Delta$-statistically convergent to $L$ (see[22]). Conversely suppose that $x$ is $\Delta$-statistically convergent to $L$. We can prove in the similar manner to Theorem 1 that

$$\lim_{m,n} \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} M\left( \frac{d(\Delta x_{jk}, L)}{\rho} \right) = 0$$

using that $M$ is an Orlicz function.

**Corollarly 1:** Let $x = (x_{jk})$ be the double sequence of fuzzy number. Then $x$ is $\Delta$-statistically pre-Cauchy if and only if

$$\lim_{m,n} \frac{1}{m^2n^2} \sum_{j=1}^{m} \sum_{k=1}^{n} d(\Delta x_{jk}, \Delta x_{pq}) = 0$$

**Proof:** Let $A = \sup_{j,k} d\left(\Delta x_{jk}, \bar{0}\right)$ and define

$$M(x) = \frac{(1+2A)x}{1+x}$$

Then

$$M\left( \frac{d(\Delta x_{jk,} \Delta x_{pq})}{\rho} \right) \leq (1+2A) d(\Delta x_{jk}, \Delta x_{pq})$$

and

$$M\left( \frac{d(\Delta x_{jk}, \Delta x_{pq})}{\rho} \right) = (1+2A) \frac{d(\Delta x_{jk}, \Delta x_{pq})}{1+d(\Delta x_{jk}, \Delta x_{pq})}$$

$$\geq \frac{(1+2A) d(\Delta x_{jk}, \Delta x_{pq})}{1+d(\Delta x_{jk}, \Delta x_{pq})}$$

$$\geq \frac{(1+2A) d(\Delta x_{jk}, \Delta x_{pq})}{1+2A}$$

$$= d(\Delta x_{jk}, \Delta x_{pq})$$

Hence

$$\lim_{m,n} \frac{1}{m^2n^2} \sum_{j=1}^{m} \sum_{k=1}^{n} d(x_{jk}, x_{pq}) = 0$$

if and only if

$$\lim_{m,n} \frac{1}{m^2n^2} \sum_{j=1}^{m} \sum_{k=1}^{n} M\left( \frac{d(x_{jk}, x_{pq})}{\rho} \right) = 0$$

and an immediate application of Theorem 1 completes the proof.

**Corollary 2:** Let $x = (x_{jk})$ be the double sequence of fuzzy number. Then $x$ is $\Delta$-statistically convergent to $L$ if and only if

$$\lim_{m,n} \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} d(\Delta x_{jk}, L) = 0$$

**Proof:** Let $A = \sup_{j,k} d\left(\Delta x_{jk}, \bar{0}\right)$ and define

$$M(x) = \frac{(1+A+L)x}{1+x} .$$

We can prove in the similar manner as in the proof of Corollary 1.

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