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On the stability analysis of a class of multiple models

Mohammed CHADLI, Didier MAQUIN, José RAGOT

Centre de Recherche en Automatique de Nancy, CNRS, UMR 7039
2, Avenue de la forêt de Haye, 54516 Vandœuvre les Nancy – France.
Phone: (33) 3 83 59 57 02, Fax: (33) 3 83 59 56 44
Email: {Mohamed.Chadli, Didier.Maquin, José.Ragot}@ensem.inpl-nancy.fr

Abstract
This paper proposes a method to discuss the stability analysis of multiple models. The proposed stability analysis is based on the use of scalar Lyapunov functions and the properties of M-matrices. An example is included to illustrate the proposed method.
Keywords: multiple models, stability analysis, Lyapunov methods, Linear Matrix Inequalities (LMIs).

1 Introduction

The multiple model representation, which includes the Polytopic Linear Differential Inclusion [7] and the Takagi-Sugeno model [4], consists to construct nonlinear dynamics by means of interpolating the behaviour of several LTI subsystems

$$\dot{x}(t) = \sum_{i=1}^{n} \mu_i(x(t)) A_i x(t) \tag{1}$$

where $x(t) \in \mathbb{R}^p$ is the state vector, n is the number of LTI subsystems and the matrices $A_i \in \mathbb{R}^{p \times p}$ define the i^{th} LTI model. The nonnegative functions $\mu_i(.)$ called *activation functions,* satisfy the following properties

$$\begin{cases} \sum_{i=1}^{n} \mu_i(x(t)) = 1 \\ \mu_i(x(t)) \geq 0 \quad \forall i \in \{1, \dots, n\} \end{cases} \tag{2}$$

Many researches have addressed the issue of stability of such multiple models. The stability depends on the existence of a common positive definite matrix satisfying a set of LMIs [3][7]. To reduce the conservativness of some previous constraints, a nonquadratic Lyapunov functions have also been used [1][6].

Some results use the properties of M-matrices to study subclass of systems which admit a Common Quadratic Lyapunov Function (CQLF), see [5], [8] and references therein. In this paper, with the aid of a Scalar Lyapunov Function (SLF) and the properties of M-matrices, sufficient conditions for global exponential stability are established for the class of multiple models accepting a CQLF.
Next, the following useful notation is introduced. $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote respectively the minimum and the maximum eigenvalues of the matrix X, $\|x\|_F^2 = x^T x$ is the Frobenius norm, $|x_i|$ is the absolute value of the real variable x_i and $y > 0$ is a positive vector, i.e. all of its entries are positive.

2 Main result

In general, there is not any systematic method allowing to obtain a quadratic Lyapunov function. However, in the particular case where matrices A_i are Hurwitz and commute pairwise, it is possible to build, in a systematic way, a CQLF [2]. Thereafter, an approach stressing another class of multiple models admitting CQLF will be presented. This study is based on the properties of M-matrices and the SLF candidate of the form

$$V(x(t)) = x(t)^T P(\alpha) x(t) \tag{3}$$

where: $P(\alpha) = \sum_{i=1}^{n} \alpha_i P_i$, , $P_i = P_i^T > 0$ and $\alpha_i \in \mathbb{R}^+$.

In the following, we will suggest a new sufficient global exponential stability condition of multiple model (1).

Theorem 1 : Suppose that there exists symmetric positive definite matrices $P_i > 0$ and symmetric matrices $S_{ii} > 0$ and S_{ij} such that

i) $A_i^T P_i + P_i A_i \leq -S_{ii}, \quad i \in \{1, \dots, n\}$ (4a)

ii) $A_j^T P_i + P_i A_j \leq S_{ji} \quad i \neq j, (i,j) \in \{1, \dots, n\}^2$ (4b)

iii) Φ is a M-matrix. (4c)

where

$$\Phi = \begin{pmatrix} \lambda_{\min}(S_{11}) & -|\lambda_{\max}(S_{12})| & . & -|\lambda_{\max}(S_{1n})| \\ -|\lambda_{\max}(S_{21})| & \lambda_{\min}(S_{22}) & . & . \\ . & . & . & -|\lambda_{\max}(S_{(n-1)n})| \\ -|\lambda_{\max}(S_{n1})| & . & -|\lambda_{\max}(S_{n(n-1)})| & \lambda_{\min}(S_{nn}) \end{pmatrix} \tag{5}$$

Then the multiple model (1) is globally exponentially stable.

Proof *:* Consider the SLF candidate (3). This is a radialy unbounded Lyapunov function since that :

$$\lambda_{\min}(P(\alpha)) \|x\|_F^2 \leq V(x) \leq \lambda_{\max}(P(\alpha)) \|x\|_F^2, \; x \in \mathbb{R}^p \tag{6}$$

The derivative of the SLF (3) along the trajectory of the continuous multiple model (1) gives with conditions (4a) and (4b)

$$\dot{V}(x(t)) \leq -x^T(t)\sum_{i=1}^{n}\alpha_i\mu_i(x(t))S_{ii}x(t) + x^T(t)\sum_{i=1}^{n}\sum_{j\neq i:1}^{n}\alpha_i\mu_j(x(t))S_{ji}x(t) \quad (7)$$

Then taking into account the following properties :

$$x^T S_{ji} x \leq \left|\lambda_{\max}(S_{ji})\right| \, \|x\|_F^2 \quad (8a)$$

$$\lambda_{\min}(S_{ii}) \, \|x\|_F^2 \leq x^T S_{ii} x \leq \lambda_{\max}(S_{ii}) \, \|x\|_F^2 \quad (8b)$$

the inequality (7) becomes

$$\dot{V}(x(t)) \leq -\nu\big(x(t)\big)\Phi\alpha \, \|x(t)\|_F^2 \quad (9)$$

where $\nu\big(x(t)\big) = \big(\mu_1(x(t)),\ldots,\mu_n(x(t))\big)$ and $\alpha = \big(\alpha_1,\ldots,\alpha_n\big)^T$ are positive vectors and the matrix Φ is defined in (5). Since Φ is an M-matrix there exists a positive vector α such that $\Phi\alpha > 0$. The exponential part of the proof is obtained by considering definitions (6) and (9). ■

It is important to note that, in some cases, the obtaining of an M-matrix is not direct. Such a matrix may be obtained by replacing the matrices S_{ii} by $S_{ii} + \tau I$ where τ is a real parameter and the matrix $I \in \mathbb{R}^{p \cdot p}$ is the identity matrix. An M-matrix that depends on $\big(\lambda_{\min}(S_{ii}) + \tau\big)$ is then computed by an iterative method modifying the τ parameter. This case is illustrate by the following example.

3 Numerical example

Consider the multiple model (1) with

$$n = 2, \; A_1 = \begin{pmatrix} 0 & 1 \\ -0.06 & -1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 2.04 \\ -0.5 & -1.5 \end{pmatrix} \quad (10)$$

Conditions of theorem 1 fail to derive directly an M-matrix and then to prove the stability of this example. By modifying these conditions as stated above, the theorem 1 leads to solve the following LMIs constraints:

$$P_1 > 0, P_2 > 0, S_{11} + \tau I > 0, S_{22} + \tau I > 0 \quad (11a)$$

$$A_1^T P_1 + P_1 A_1 \leq -\big(S_{11} + \tau I\big), A_2^T P_2 + P_2 A_2 \leq -\big(S_{22} + \tau I\big) \quad (11b)$$

$$A_1^T P_2 + P_2 A_1 \leq S_{12}, A_2^T P_1 + P_1 A_2 \leq S_{21} \quad (11c)$$

Solving the LMI system (11), we obtain

$$P_1 = \begin{pmatrix} 1.094 & 1.011 \\ 1.011 & 1.649 \end{pmatrix}, \; P_2 = \begin{pmatrix} 0.8462 & 0.6593 \\ 0.6593 & 1.3542 \end{pmatrix}$$

$$\tau = 0.3, S_{11} = \begin{pmatrix} 60.537 & 7.998 \\ 7.998 & 637.672 \end{pmatrix}, S_{22} = \begin{pmatrix} 329.503 & -30.139 \\ -30.139 & 686.174 \end{pmatrix} \quad (12a)$$

$$S_{12} = \begin{pmatrix} 666.874 & 105.686 \\ 105.686 & -643.808 \end{pmatrix}, S_{21} = \begin{pmatrix} -265.461 & -109.216 \\ -109.216 & -75.108 \end{pmatrix} \quad (12b)$$

The expression (5) with the result in (12) allow the matrix Φ to be deduced:

$$\Phi = \begin{pmatrix} 60.726 & -675.341 \\ -25.417 & 327.274 \end{pmatrix} \quad (13)$$

As Φ is clearly an M-matrix, the considered multiple model (10) is exponentially stable

4 Conclusion

Using the properties of M-matrices, sufficient conditions for the global exponential stability for multiple models are obtained. The method studies classes of multiple models admitting CQLF. An heuristic method, based on LMIs formulation, to compute such an M-matrix is proposed when this latter is not directly obtained. The result can be extended to discrete case.

5 References

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