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LPV Analysis and Control Using Fast Iterative Solutions to Rationally Parametric Lyapunov and Riccati Equations

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Abstract— **We consider the problem of analysis and control of Linear Parameter Varying(LPV) systems. With regard to such problems, solving rationally parametric Lyapunov and Riccati equations for parametric matrices often arises. In this paper, we develop computationally efficient iterative methods for finding rational approximations to solutions to such problems to arbitrary accuracy.**

I. Introduction

In recent years, system models in many important applications have been shown to have a Linear Parameter Varying(LPV) structure, e.g. wind turbines [1], automated lane keeping systems [2], steam generators for nuclear power plants [3], biomedical applications [4], web servers [5], and bicycles [6]. Correspondingly, there has been a burgeoning field of LPV analysis and synthesis techniques, see e.g. [7][8][9][10][11] and the references therein.

An important problem in LPV analysis and control is to find parameter dependent solutions to parameter dependent Lyapunov and Riccati equations and inequalities of both the algebraic and differential type, valid over some set of admissible parameters and parameter rates. Such solutions can be used to guarantee exponential LPV stability or performance [12] or to construct parameter dependent controllers with guaranteed stability and performance(see e.g. [13] and references therein). Unfortunately, finding such parameter dependent solutions (or even verifying that a given solution satisfies a parameter dependent equation or inequality over the entire parameter set) turns out to be quite a difficult problem; even though the inequalities are linear in the unknowns, we must search or optimize over the (often infinite) space of admissible parameters. Such problems can often be cast as verifying or solving a parametric matrix equation (Lyapunov, Riccati) or Linear Matrix Inequality(LMI) for which various techniques have been developed: e.g. [14][15](parametric equations) [16][17] [18][19][20][21][22][23](parametric LMI's). However, these techniques either require some restrictive *a priori* assumption on the form of the solution's parameter dependence (e.g affine, polynomial or rational of order n, etc), the gridding of the parameter space, conservative relaxations, or relaxations which are asymptotically nonconservative but lead to very large (and thus computationally expensive) LMI's.

However, under certain system assumptions, we can develop computationally efficient techniques based on transfer function arithmetic and the matrix sign function which avoids these problems, and allows us to solve certain parameter dependent algebraic Lyapunov and Riccati equations arbitrarily accurately, thus providing a different approach to such LPV problems. Our idea is to construct a sequence of rationally parametric matrices which approach the exact(and perhaps irrational) solution of the Lyapunov or Riccati equation quadratically fast. Instead of using the computationally expensive operations on transfer matrices, we work only in stable realizations (with a mixed-causal LTI interpretation) thereof, for which we build a structure preserving arithmetic.

In section II we will overview the type of LPV systems we will consider, and show how to convert them into a form that is convenient for our computational methods. In section III, we will build our computationally efficient, structure preserving arithmetic of rationally dependent matrix realizations, and in section IV we will show how this arithmetic can be used with the matrix sign function to efficiently solve rationally parameter dependent Lyapunov and Riccati equations to arbitrary accuracy. In section V we then show how these tools can be used for some practical LPV analysis and synthesis problems, with an example in section VI.

II. Background, notation, and $\mathcal{S}_c$ realizations

We will consider systems of the sort:

$$\Sigma: \quad \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{1}$$

where $\rho \in \hat{\mathbb{R}}_+$ (the extended positive real line; $\hat{\mathbb{R}}_+ = \mathbb{R}_+ \bigcup +\infty$) and $\alpha < \dot{\rho} < \beta$ are bounds on the rate of variation of the measurable parameter $\rho(t)$, and the system matrices are assumed to have a proper rational parameter dependence, e.g. $A(\rho) = H + G(\rho I - E)^{-1}F$. At first glance this formulation seems ungainly; parameters of most physical systems won't vary over $\rho \in \hat{\mathbb{R}}_+$, however, some common situations may be transformed into this one:

Example 1 (Affine Parameter Dependence on an Interval): Say we have an affine LPV operator $K(\mu) = \varsigma\mu + \kappa$ valid over $\mu \in [\alpha, \beta]$. We can perform a transformation to parameterize $\mu = \frac{\alpha-\beta}{\rho+1} + \beta$ over $\rho \in \hat{\mathbb{R}}_+$ and equivalently write: $K(\rho) = \varsigma\frac{\alpha-\beta}{\rho+1} + \varsigma\beta + \kappa$ or as a state space realization of an LFT:

$$K(\rho) = \underbrace{\varsigma}_{G}(\rho I - \underbrace{(-1)}_{E})^{-1}\underbrace{(\alpha-\beta)}_{F} + \underbrace{\varsigma\beta+\kappa}_{H}$$

over $\rho \in \hat{\mathbb{R}}_+$. Note that this construction is well posed and easily generalizable to matrices. Also note that the above

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parametrization can be extended to *any polynomial* in μ, a practical example of which we will exhibit in section VI. ★

For our computational methods, the system in (1) is not yet suitable. Given some $K(\rho) = G(\rho I - E)^{-1}F + H$, $\rho \in \hat{\mathbb{R}}_+$, addition, multiplication, and inversion of such operators is trivial (and identical in form to that of LTI transfer functions over the imaginary axis or unit circle), but calculating the induced norm $\|K(\rho)\|$, or finding order reduced approximations $\tilde{K}(\rho) \approx K(\rho) \forall \rho \in \hat{\mathbb{R}}_+$, which will be integral to our iterative procedures, is not. Hence we will further reparametrize such $K(\rho) \quad \forall \rho \in \hat{\mathbb{R}}_+$ into mixed-causal transfer functions over the extended imaginary axis: $K(s), \quad \forall s \in \Im$.

For any well posed $K(\rho)$, this can trivially be accomplished via a change of variables

$$K(\rho) = H + G(\rho I - E)^{-1}F = H - G(s^2 I + E)^{-1}F$$

where $\rho = -s^2$. This parametrization can in turn be split up into stable and antistable parts, for a stable 'mixed causal' realization:

Lemma 1: Assuming that $K(-s^2) = H - G(s^2 I + E)^{-1}F$ is well posed for all $s \in \Im$, then equivalently,

$$K(-s^2) = H - G(sI - (-Z))^{-1}XF - GX(s^*I - (-Z))^{-1}F$$

where $Z := (-E)^{1/2}$ with $\Re(\lambda(Z)) > 0$ and X is the unique solution to the Sylvester equation $ZX + XZ + I = 0$.

Proof: see appendix. ■

We have thus successfully rewritten the system in

$$K(s) = D + P(sI - R)^{-1}Q + U(s^*I - W)^{-1}V \tag{2}$$

form, where R and W are both strictly stable.

But such stable mixed causal realizations of $K(s)$ are highly non unique, and given some $K(s)$ in transfer function form with high rational orders, finding a minimal realization, while trivial, is extremely computationally inefficient. Hence we will avoid this step in the following by only dealing with stable state space representations of $K(s)$ (instead of the elementwise transfer function version), which we will represent using the following notation:

$$\overline{K} = \mathcal{S}_c(P, R, Q, D, U, W, V) \tag{3}$$

and we will use the notation $\bar{A} \in \mathcal{S}_c$ to indicate that $\bar{A}$ is a mixed causal realization of $A(s)$ with stable causal and anticausal parts. We will hereafter develop an arithmetic of such realizations, and this notation will serve to show how we avoid ever dealing explicitly with the awkward elementwise rational transfer function form.

This notation will also show how part of this work parallels work on spatially invariant systems in [24], except on a continuous domain instead of discrete, and we will note the analogous results when they occur.

III. ARITHMETIC, ORDER REDUCTION

Transfer matrix arithmetic is slow (especially inversion) and may lead to unnecessarily high orders, so instead we will work in $\mathcal{S}_c$ realizations, for which we will now develop a structure and stability preserving arithmetic. A $\bullet$ will indicate a term that can be trivially derived from the surrounding information, but is omitted due to space constraints.

A. Addition and Multiplication

Lemma 2: Given

$$\bar{X} = \mathcal{S}_c\{P_X, R_X, Q_X, D_X, U_X, W_X, V_X\}$$
$$\bar{Y} = \mathcal{S}_c\{P_Y, R_Y, Q_Y, D_Y, U_Y, W_Y, V_Y\}$$

Then a realization of the sum: $\bar{Z} = \bar{X} + \bar{Y}$ is:

$$\bar{Z} = \mathcal{S}_c\{\begin{bmatrix} P_X^T \\ P_Y^T \end{bmatrix}^T, \begin{bmatrix} R_X & 0 \\ 0 & R_Y \end{bmatrix}, \begin{bmatrix} Q_X \\ Q_Y \end{bmatrix}, (D_X + D_Y), \bullet, \bullet, \bullet\}$$

Proof: This is verifiable by inspection. ■

Lemma 3: $\bar{X}\bar{Y} = \bar{Z} = \mathcal{S}_c\{P_Z, R_Z, Q_Z, D_Z, U_Z, W_Z, V_Z\}$ with

$$D_Z = D_X D_Y, \quad P_Z = \begin{bmatrix} P_X & D_X P_B + U_X S \end{bmatrix}$$
$$R_Z = \begin{bmatrix} R_X & Q_X P_Y \\ 0 & R_Y \end{bmatrix}, \quad Q_Z = \begin{bmatrix} Q_X D_Y + T V_Y \\ Q_Y \end{bmatrix}$$
$$U_Z = \bullet, W_Z = \bullet, V_Z = \bullet$$

where S and T are the unique solutions to the Sylvester equations:

$$W_X S + S R_Y + V_X P_Y = 0, \quad R_X T + T W_Y + Q_X U_Y = 0$$

Proof: This is easy to verify using LTI continuous time systems theory, where we consider $\overline{X}$ and $\overline{Y}$ to be mixed-causal system realizations that we put in series. ■

Note that the proper rational transfer function structure, and also the stability of the realizations, is preserved under these addition and multiplication algorithms: $\bar{X}, \bar{Y} \in \mathcal{S}_c \Rightarrow \bar{Z} \in \mathcal{S}_c$. This result is the continuous domain analog of [24].

B. Order Reduction

The alert reader will have noticed that through the above two operations, $Z(s)$ will be a rational transfer matrix with order larger than $X(s)$ or $Y(s)$. Fortunately, since we represent $\bar{Z}$ as the sum of $Z(s) = L(s) + U(s)$ with LTI causal and anticausal interpretations, respectively, we can efficiently perform order reduction on Z by performing standard LTI state space model order reduction on its causal and anticausal parts separately. Since each is stable, using, e.g. balanced truncation, we also obtain an H_∞ bound on the error of the transfer function(see e.g. [25]). If we perform balanced truncations such that $\|L(s) - \tilde{L}(s)\|_\infty < e_L$, $\|U(s) - \tilde{U}(s)\|_\infty < e_U$, then the reduced order realization $\tilde{\bar{Z}} = \tilde{\bar{L}} + \tilde{\bar{U}}$ has error bound $\|\bar{Z} - \tilde{\bar{Z}}\| < e_L + e_U$. This result is the continuous domain analog of [24].

C. Inversion

We have shown the closure of $\mathcal{S}_c$ under addition and multiplication, but for controller synthesis, we will also need inversion. First we need some preliminaries:

Lemma 4 (Positive Real Lemma): Let $G(s) = D + C(sI - A)^{-1}B$ be a stable rational transfer matrix with $R := (D + D^*)^{-1} \succ 0$. The following two statements are equivalent:

- $G(s) + G^*(s) \succ 0, \quad \forall s \in \Im$
- $\exists P \succeq 0$, unique, such that:

$$A^*P + PA + (PB - C^*)R(PB - C^*)^* = 0 \qquad (4)$$

and $\lambda(A + BR(B^*P - C)) \in \mathbb{C}_-$

Proof: ([25], Lemma 13.27) ∎

We will first consider the inverse of Hermitian operators:

Lemma 5: Given a Hermitian $\bar{X} = \mathcal{S}_c\{P, R, Q, Y, Q^*, R^*, P^*\}$ with $Y \succ 0$, if the Riccati and Lyapunov equations:

$$RG + GR^* + \Phi Y \Phi^* = 0 \qquad (5)$$

$$\Pi^* H + H\Pi + P^* Y^{-1} P = 0 \qquad (6)$$

where $\Phi = (Q - GP^*)Y^{-1}$ and $\Pi = R - \Phi P$, have solutions, G and H, then there exists a $\bar{X}^{-1} = \bar{Z} = \mathcal{S}_c\{P_Z, R_Z, Q_Z, Y_Z, \bullet, \bullet, \bullet\}$, which may be calculated as:

$$P_Z = \Phi^* H - Y^{-1}P, \quad R_Z = \Pi, \quad Q_Z = \Phi, \quad Y_Z = Y^{-1}$$

Proof: The derivation is omitted for brevity, but basically follows by assuming that $\bar{X}$ has an outer factorization $\bar{X} = \bar{L}\bar{L}^*$ and then calculating $\bar{X}^{-1} = \bar{L}^{-*}\bar{L}^{-1}$. Note that the adjoint is characterized as: $\mathcal{S}_c\{P, R, Q, D, U, W, V\}^* = \mathcal{S}_c\{V^*, W^*, U^*, D^*, Q^*, R^*, P^*\}$: ∎

Lemma 6: The Riccati equation (5) will have a (positive semidefinite)stabilizing solution if and only if $\bar{X} \succ 0$. When this is the case, $\Pi = R - \Phi P$ will be strictly stable, the Lyapunov equation (6) has a unique solution, and $\bar{Z} = \bar{X}^{-1} \in \mathcal{S}_c$.

Proof: Set $G(s) = \frac{1}{2}Y + P(sI - R)^{-1}Q$, and this follows directly from Lemma 4. Note that $X(s) \succ 0 \forall s \in \Im$ implies $Y \succ 0$, and that $\lambda(R - \Phi P) \in \mathbb{C}_-$ is sufficient for (6) to have a unique solution and for $\bar{Z}$ to be stable. ∎

We can now extend to the nonsymmetric case:

Lemma 7: Assume $\bar{A} \in \mathcal{S}_c$. Then $\exists \bar{A}^{-1} \in \mathcal{S}_c \Leftrightarrow 0 \notin \lambda(\bar{A})$. Furthermore, we can calculate it using the formulas in Lemma 5.

Proof:
$\Leftarrow$ Then clearly $0 \prec \bar{A}\bar{A}^* \in \mathcal{S}_c$, and we can use Lemmas 5 and 6 to calculate $\bar{A}^{-1} = \bar{A}^*(\bar{A}\bar{A}^*)^{-1} \in \mathcal{S}_c$
$\Rightarrow$ A bounded $\bar{A}^{-1}$ always implies $0 \notin \lambda(\bar{A})$. ∎

We will call such $\bar{A} \in \mathcal{S}_c$ with $\bar{A}^{-1} \in \mathcal{S}_c$ 'regular'. Note that we have been assuming $\bar{A}$ square, but these results could easily be extended to nonsquare $\bar{A}$ using left and right inverses.

We also note that the rational order of $\bar{A}^{-1}$, as calculated above, will be generally 3 times the rational order of $\bar{A}$. However, often this can be avoided by inverting $\bar{A}$ directly, without making it symmetric, using nonsymmetric Riccati and Lyapunov equations:

Lemma 8: Given $\overline{A} = \mathcal{S}_c\{P, R, Q, D, U, W, V\}$, if the nonsymmetric CARE:

$$\begin{aligned}(R - QD^{-1}P)T + T(W - VD^{-1}U) + \ldots \\ TVD^{-1}PT + QD^{-1}U &= 0 \qquad (7)\end{aligned}$$

has a *stabilizing* solution, e.g., a T for which both

$$\begin{aligned} W^F &= W - VD^{-1}(U - PT) \\ R^F &= R - (Q - TV)D^{-1}P \end{aligned}$$

are stable, then $\overline{A}$ has an inverse $\bar{A}^{-1} = \bar{F} = \mathcal{S}_c\{P^F, R^F, Q^F, D^F, U^F, W^F, V^F\}$ where

$$D^F = D^{-1}, \quad U^F = D^F(U - PT), \quad Q^F = (Q - TV)D^F$$
$$V^F = SQ^F - VD^F, \quad P^F = U^F S - D^F P$$

where S is the (unique) solution to the Sylvester equation

$$W^F S + SR^F + VD^F P = 0 \qquad (8)$$

Proof: The proof proceeds by first assuming that $\bar{A}$ has an outer factorization $\bar{A} = \bar{L}\bar{U}$ with $\bar{L}, \bar{U} \in \mathcal{S}_c$ and with $\bar{L}$ ($\bar{U}$) causal and anti-causal, with causal and anticausal inverses $\in \mathcal{S}_c$ respectively, this requirement gives us the Riccati equation (7). If the stabilizing solution exists, it then provides an appropriate $\bar{L}, \bar{U}$ pair. We then calculate $\bar{A}^{-1} = \bar{U}^{-1}\bar{L}^{-1}$ using Lemma 3, giving the Sylvester equation, (8) which has a unique solution because W^F and R^F are both stable. See [26] for the details. ∎

The Riccati equation (7) *will not* have a stabilizing solution for every invertible $\bar{A}$, like that in the inversion method in Lemma 7. However, when (7) does have a stabilizing solution (which can always be calculated using the sign iteration [27]), notice that the resulting $\bar{F}$ does not increase in order from $\bar{A}$. This is often the case in practice, allowing us to greatly speed up our iterative computations. This result is the continuous domain analog of [24].

D. How to calculate $\|A(s)\|_\infty$

In the following, we will perform iterative calculations on $\mathcal{S}_c$ realizations, and thus must have some measure of convergence. Given a realization, $\bar{X} \in \mathcal{S}_c$, however, $\|\bar{X}\|$ is not trivial to calculate. First we need an outer factorization:

Lemma 9 (Outer Factorization): Assume we have a Hermitian $\bar{X} = \mathcal{S}_c\{P, R, Q, Y, Q^*, R^*, P^*\}$, $\bar{X} \succ 0$. Then $\exists$

$$\begin{aligned} \bar{L} &= \mathcal{S}_c\{P_L, R_L, Q_L, D_L, 0, 0, 0\} \\ \bar{L}^{-1} &= \mathcal{S}_c\{(D_L^{-1}P_L), (R_L - Q_L D_L^{-1}P_L), (-Q_L D_L^{-1}), \\ &\qquad D_L^{-1}, 0, 0, 0\} \end{aligned}$$

with D_L invertible such that $\bar{L}\bar{L}^* = \bar{X}$. Furthermore, such an $\bar{L}$ can be calculated as: $P_L = P, \quad R_L = R, \quad D_L D_L^* = Y, \quad Q_L = (Q - GP^*)Y^{-1}D_L$, where $G \succeq 0$ is the stabilizing solution to the Riccati equation(5). D_L can be calculated via Cholesky factorization.

Proof: The derivation is part of that of the inversion. Since $\bar{X} \succ 0$, a stabilizing G exists by the positive real lemma. $\bar{L} \in \mathcal{S}_c$ since $R_L = R$ is stable, and $\bar{L}^{-1} \in \mathcal{S}_c$ since G is stabilizing and thus $(R_L - Q_L D_L^{-1}P_L)$ is stable. This is the continuous time analog of [24]. ∎

If we have some non-Hermitian $\bar{A} \in \mathcal{S}_c$, (maybe singular), and we want to find the norm $\|\bar{A}\|$, then we can equivalently calculate: $\|\bar{A}\| = \sqrt{\|\bar{L}\bar{L}^*\| - 1}$, where $\bar{L}\bar{L}^* = \bar{A}^*\bar{A} + I \succ 0$ is an outer factorization, and hence $\bar{L}$ has a stable causal representation, $L(s)$. Since $\|\bar{L}\bar{L}^*\| = \|\bar{L}\|^2 = \|L(s)\|_\infty^2$ we can then use the Bounded Real Lemma for stable continuous LTI systems [25] to find the infinity norm of $L(s)$ and thus $A(s)$.

IV. SIGN FUNCTION FOR TRANSFER MATRICES

The matrix sign function [28] has been shown to be a very powerful tool for finite dimensional linear systems analysis and control synthesis (see [29] for an overview). The Newton's method of calculation (called the 'sign iteration', see below) also converges extremely fast (locally quadratically [29]), making it one of the most efficient computational techniques for solving Riccati equations and other common control problems. In the following we will extend the sign function definition, some convergence bounds, and numerical robustness calculations from the finite dimensional case to the rational transfer matrix case.

A. Definition

We can define the sign iteration and sign function:

Algorithm 1 Sign Iteration [28]

$$\begin{aligned} & Z_0 = X \\ Z_{k+1} &= \frac{1}{2}(Z_k + Z_k^{-1}), \quad k = 0, 1, 2, ... \\ sign(X) &= \lim_{k\to\infty} Z_k \end{aligned}$$

Extending Algorithm IV-A to stable rational transfer matrices is straightforward; for some $\bar{X} \in \mathcal{S}_c$ we can consider the above iteration to calculate $sign(\bar{X})$ just as the finite matrix sign computation at each complex matrix $X(s_0)$, $\forall s_0 \in \Im$, although we actually perform the calculations using the $\mathcal{S}_c$ realization arithmetic developed in section III. Just as in the matrix case [28], if $\bar{X}$ is regular then every $\bar{Z}_k$ is regular.

Lemma 10: For some $\bar{X} \in \mathcal{S}_c$, assume that $\lambda(\bar{X})$ does not touch the imaginary axis. Then $Z_k(s)$ converges uniformly on the extended imaginary axis, and $sign(X(s))$ is continuous and bounded

Proof: $X(s)$ is continuous on the extended imaginary axis, which is compact, and hence the proof is formally identical to that for the sign iteration on transfer functions on the unit circle in [30]. ■

We note that $Z_\infty(s) = sign(X(s))$ will **not** always be rational, since the space of rational functions is not complete. However, $\bar{Z}_k \in \mathcal{S}_c, \quad \forall k < \infty$ and thus we can approximate $sign(\bar{X})$ arbitrarily close in $\mathcal{S}_c$, and the approximation generated by the halted sign iteration converges locally quadratically fast to $sign(\bar{X})$ in operator norm, so in practice this is not a problem.

B. Applications

As discussed in [31] and references therein, in finite dimensions, the matrix sign function is useful for many things in control analysis and design, such as checking matrix stability($sign(X) = -I \Leftrightarrow \lambda(X) \in \mathbb{C}_-$) and solving Lyapunov and Riccati equations. Most of these results extend directly to the sign function on rational transfer matrices in a trivial way, e.g. $sign(X(s)) = -I, \forall s \in \Im \Leftrightarrow \lambda(X(s)) \in \mathbb{C}_-, \forall s \in \Im$) Hence we can extend many finite dimensional results with the sign iteration, such as stability, H_2 and H_∞ performance analysis, and controller synthesis to $\mathcal{S}_c$ realizations. Please note, as pointed out in the introduction, that rationally parametric Riccati equations often have exact solutions that are *irrational*, but which can be approximated to arbitrary accuracy by rational transfer matrices; it is these high-order rational transfer matrices that our techniques will produce.

C. Numerical Difficulties

The alert reader will also have noticed that the rational order of the $\mathcal{S}_c$ realizations during the sign iteration approximately double during each step, and thus repeated order-reducing approximations, as mentioned in subsection III-B must be used to prevent the complexity from blowing up (the size of the finite dimensional Riccati equation (5) needed to invert $\bar{Z}_k$ is proportional to its order). Such iterative approximations, if too aggressive, can cause numerical instability in the sign iteration [31].

However, in linear systems analysis and control applications, *a posteriori* closed loop stability and performance(e.g. H_2, H_∞), can be relaxed to ϵ-sub-optimal problems involving Lyapunov and Riccati inequalities. The verification of solutions to such problems can thus be reduced to checking the positive definiteness of a Hermitian matrix [31], which can be simply checked using the Positive Real Lemma (Lemma 4).

V. USE FOR LPV ANALYSIS AND SYNTHESIS

As we have outlined it above, we can check the positive definiteness of operators (and hence check whether some $X(\rho)$ satisfies a useful LMI or Riccati inequality), and solve Lyapunov and Riccati equations of the sort:

$$\begin{aligned} A(\rho)T(\rho) + T(\rho)A(\rho)^T + Q(\rho) &= 0 \\ A(\rho)S(\rho) + S(\rho)A(\rho)^T + Q(\rho) + S(\rho)R(\rho)S(\rho) &= 0 \end{aligned}$$

to arbitrary accuracy over all $\rho \in \hat{\mathbb{R}}_+$.

However, as discussed in section II, LPV analysis and control for dynamically varying parameters requires more than this: there is a derivative term in the inequalities. For *verifying* solutions to such LMI's, this provides no added difficulty, for example:

Lemma 11:

$$A(\rho)S(\rho) + S(\rho)A(\rho)^T + Q(\rho) + \dot{\rho}\frac{\partial S(\rho)}{\partial \rho} \prec 0 \quad (9)$$

over all $\rho \in \hat{\mathbb{R}}_+$, $\dot{\rho} \in [\alpha, \beta]$ if and only if

$$A(\rho)S(\rho) + S(\rho)A(\rho)^T + Q(\rho) + \gamma\frac{\partial S(\rho)}{\partial \rho} \prec 0$$

$\forall \rho \in \hat{\mathbb{R}}_+$ for both $\gamma \in \{\alpha, \beta\}$.

Proof: (9) is convex in $\dot{\rho}$. ■

where the derivative with respect to ρ can be calculated as:

Lemma 12: Given $X(\rho) = P(\rho I - R)^{-1}Q$, bounded on $\rho \in \hat{\mathbb{R}}_+$,

$$\frac{\partial X(\rho)}{\partial \rho} = \begin{bmatrix} P & 0 \end{bmatrix} \left(\rho I - \begin{bmatrix} R & -I \\ 0 & R \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ Q \end{bmatrix} \quad (10)$$

Proof: Chain rule. ■

note that (10) is in the usual first order LFT form in ρ, which we can use the results of lemma 1 to change back into a transfer matrix on s for further computations. Note also that $\frac{\partial X(\rho)}{\partial \rho}$ is bounded on $\rho \in \hat{\mathbb{R}}_+$ if $X(\rho)$ is bounded on $\rho \in \hat{\mathbb{R}}_+$.

So verifying (9) is as easy as checking positive definiteness of two parametric matrices over $\rho \in \hat{\mathbb{R}}_+$, which can be efficiently done in the $s \in \Im$ domain using the sign based techniques in section IV-B.

However, actually finding such an $S(\rho)$ is much more difficult, and we give only a suggestion for solving such problems. We propose the following strategy (for Lyapunov inequalities): pick some $\wp \in \mathbb{R}$, and solve sequentially for S_0, S_1, ... in:

$$\begin{aligned} A(\rho)S_0(\rho) + S_0(\rho)A(\rho)^T + Q(\rho) &= -\epsilon I \\ A(\rho)S_1(\rho) + S_1(\rho)A(\rho)^T + Q(\rho) + \wp\frac{\partial S_0(\rho)}{\partial \rho}) &= -\epsilon I \\ A(\rho)S_2(\rho) + S_2(\rho)A(\rho)^T + Q(\rho) + \wp\frac{\partial S_1(\rho)}{\partial \rho}) &= -\epsilon I \\ \cdots & \end{aligned} \quad (11)$$

Since $A(\rho)$ is assumed stable, $S_{i+1}(\rho)$ is continuous in $\wp\frac{\partial S_i(\rho)}{\partial \rho}$, and we showed in lemma 12 that $\frac{\partial S_i(\rho)}{\partial \rho}$ is continuous in $S_i(\rho)$, assuming that $S_i(\rho)$ is bounded on ρ, there thus exists a $\wp$ small enough such that $S_i(\rho)$ will converge to a unique $S_\infty(\rho)$ by the Banach Fixed Point Theorem [32], satisfying

$$A(\rho)S_\infty(\rho) + S_\infty(\rho)A(\rho)^T + Q(\rho) + \wp\frac{\partial S_\infty(\rho)}{\partial \rho}) = -\epsilon I$$

It then remains to check for what bounds $\wp_+ < \dot{\rho} < \wp_-$ our solution S_∞ satisfies the inequality (9), a problem like that in Lemma 11, which we can easily solve. We note that this same idea works for the Riccati inequalities(where the convergence follows using the analyticity results from [33]).

Of course, this will only provide *a posteriori* rate bounds $\wp_+ < \dot{\rho} < \wp_-$ in which our closed loop system has stability or performance, and it is not yet clear how to pick $\wp$ and ϵ(or some other offset matrix) in (11) in order to achieve *a priori* desired bounds. Future work will be devoted to exploring this point, and testing the above described method on dynamically parameter varying examples.

VI. Airfoil Flutter Example

For now we will apply our method to a simple static parameter model of a fluttering airfoil. Our model comes from section 4.9 of [34], wherein a 2-D quasi-static flutter model is derived for a 'smart' airfoil; i.e. one with trailing edge flap actuators. The model has four states, one controlled input (flap angle), and two measured outputs (angle of attack and vertical displacement), and is polynomially dependent on the freestream velocity, v, in the following form:

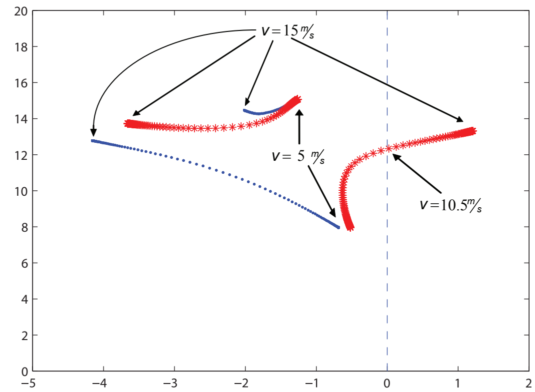

Fig. 1. Open loop eigenvalues(*) vs closed loop eigenvalues(·)

$$\dot{x} = (A_0 + A_1 v + A_2 v^2)x + (B_2 v^2)u, \qquad y = C_0 x$$

We would like to design a controller, $u = -K(v)x$ that stabilizes the system for all static values of $v \in [5, 15]\frac{m}{s}$. We do this via LQR; by solving the parameter dependent Riccati equation:

$$\begin{aligned} A(v)^T X(v) + X(v)A(v) + C^T C \ldots & \\ + X(v)B(v)B^T(v)X(v) &= 0 \end{aligned}$$

for $X(v)$, and then using the feedback gain $K(v) = B^T(v)X(v)$. Using our methods outlined above, we found an approximate $\tilde{X}(v)$ that satisfied the Riccati equation with a residual error norm of only $\max_{v \in [5,15]} \|A(v)^T \tilde{X}(v) + \tilde{X}(v)A(v) + C^T C + \tilde{X}(v)B(v)B^T(v)\tilde{X}(v)\| \approx 3.79 \times 10^{-7}$. In figure 1 we see a comparison of the open loop vs closed loop spectrum: for open loop, the system goes unstable around $v = 10.5\frac{m}{s}$, but for closed loop the system is stable for the entire range $v \in [5, 15]\frac{m}{s}$.

This particular example is actually very ill-suited for our technique. As discussed in [31], while the ultimate convergence of the sign iteration for calculating $sign(H)$ is quadratic, the initial steps may be very slow, largely dependent on the location of the spectrum of H. If the elements of the spectrum are close to $+1$ or -1, are mostly real, and stay away from the imaginary axis, convergence will be fast, otherwise it will be initially slow and non-monotonic, requiring high rational orders. In our case of the flutter system above, as we see in figure 1, the closed loop spectrum is badly damped, with a relatively very large imaginary component, requiring ~ 17 sign iterations for convergence (usually 5-10 suffices).

VII. Conclusion

We have demonstrated a new computational approach for efficiently and accurately finding parametric solutions to rationally parametric Lyapunov and Riccati equations, and shown how this might be used in a number of LPV analysis and synthesis problems. We note that this might also be useful in certain robust control applications, and that while we have only developed the framework for a single parameter in this paper, all of the techniques should be extendable to multiple parameters in the same way that the 1-D distributed system techniques in [24] can be extended to n-D in [30].

Future work will be devoted to case study comparisons of this technique with other control methods for parameter dependent systems in terms of performance, and investigations of under what conditions(on the closed loop spectrum) this method provides significant gains in computational complexity.

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APPENDIX

Proof of Lemma 1: Now since we assume that $K(-s^2)$ is well posed for all $s \in \Im$, then E must have no purely real eigenvalues in $\mathbb{C}_+$. Hence $-E$ has no purely real eigenvalues in $\mathbb{C}_-$, and a unique real $(-E)^{1/2} := Z$ exists with $\Re(\lambda(Z)) > 0$ [35]. We can hence rewrite:

$$K(-s^2) = H - G\left[(sI + Z)(sI - Z)\right]^{-1} F$$

or $K(-s^2) = H - \begin{bmatrix} G & 0 \end{bmatrix} (sI - \begin{bmatrix} -Z & I \\ 0 & Z \end{bmatrix})^{-1} \begin{bmatrix} 0 \\ F \end{bmatrix}$. If we then solve the Sylvester equation $ZX + XZ + I = 0$ for the unique X, and apply the similarity transformation $\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$, then we get:

$$K(-s^2) = H - \begin{bmatrix} G & GX \end{bmatrix} (sI - \begin{bmatrix} -Z & 0 \\ 0 & Z \end{bmatrix})^{-1} \begin{bmatrix} XF \\ -F \end{bmatrix}$$

which we cut into stable 'causal' and 'anticausal' parts:

$$K(\rho) = H - G(sI - (-Z))^{-1} XF - GX(s^* I - (-Z))^{-1} F$$

(note that for s on the imaginary axis, $s^* = -s$). Where we remember that $\Re(\lambda(-Z)) < 0$ and thus both parts are stable.

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