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Reproducing Kernel Method for Singular Fourth Order Four-Point Boundary Value Problems

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Abstract. This paper investigates the analytical approximate solutions of singular fourth order four-point boundary value problems using reproducing kernel method (RKM). The solution obtained by using the method takes the form of a convergent series with easily computable components. However, the RKM can not be used directly to solve singular fourth order four-point boundary value problems (BVPs), since there is no method of obtaining reproducing kernel (RK) satisfying four-point boundary conditions. The aim of this paper is to fill this gap. A method for obtaining RK satisfying four-point boundary conditions is proposed so that RKM can be used to solve singular fourth order four-point BVPs. Results of numerical examples demonstrate that the method is quite accurate and efficient for singular fourth order four-point BVPs.

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1. Introduction

In this paper, we consider the following fourth order four-point boundary value problems:

$$\begin{cases} Lu(x) \equiv a_0(x)u''''(x) + a_1(x)u'''(x) + a_2(x)u''(x) \\ \quad +a_3(x)u'(x) + a_4(x)u(x) = f(x),\ 0 < x < 1, \\ u(0) = 0,\ u(\alpha) = 0,\ u(\beta) = 0,\ u(1) = 0, \end{cases} \tag{1.1}$$

where $\alpha\ \beta \in (0,1)$, $a_i(x)$, $f(x) \in C[0,1]$, $i = 0, 1, 2, 3, 4$ and maybe $a_0(0) = 0$ or $a_0(1) = 0$. That is, the equation may be singular at $x = 0, 1$. We consider $u(0) = 0$, $u(\alpha) = 0$, $u(\beta) = 0$, $u(1) = 0$ since the boundary conditions $u(0) = \gamma_0$, $u(\alpha) = \gamma_1$, $u(\beta) = \gamma_2$, $u(1) = \gamma_3$ can be reduced to $u(0) = 0$, $u(\alpha) = 0$, $u(\beta) = 0$, $u(1) = 0$.

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Multi-point boundary value problems arise in a variety of applied mathematics and physics. Two-point boundary value problems have been extensively studied in the literature. Also, the existence and multiplicity of solutions of fourth order four-point boundary value problems have been studied by many authors, see [1–3, 13, 15–18] and the references therein. However, research for methods for solving singular fourth order four-point boundary value problems has proceeded very slowly. Geng [7] proposed a method for a class of second order three-point BVPs by convert the original problem into an equivalent integro-differential equation. In this paper, we will apply reproducing kernel method (RKM) presented by Cui, Geng, *et. al* [4–6, 8–12, 14] to solve singular fourth order four-point BVP (1.1).

The rest of the paper is organized as follows. In the next section, the RK satisfying four-point boundary conditions is constructed. The RKM is applied to (1.1) in Section 3. The numerical examples are presented in Section 4. Section 5 ends this paper with a brief conclusion.

2. Construction of RK satisfying four-point boundary conditions

By using the methods in [3–5, 7–12, 14], it is impossible to obtain reproducing kernel (RK) satisfying four-point boundary conditions of (1.1). In this section, we will make great efforts to fill this gap.

First, we construct a RK space $W^5[0,1]$ in which every function satisfies $u(0) = u(1) = 0$.

$W^5[0,1]$ is defined as $W^5[0,1] = \{u(x) \mid u(x), u'(x), u''(x), u'''(x), u''''(x)$ are absolutely continuous real value functions, $u^{(5)}(x) \in L^2[0,1], u(0) = 0, u(1) = 0\}$. The inner product and norm in $W^5[0,1]$ are given, respectively, by

$$(u(y), v(y))_{W^5} = u(0)v(0) + u(1)v(1) + u'(0)v'(0) + u'(1)v'(1) + u''(0)v''(0) + \int_0^1 u^{(5)}v^{(5)}dy$$

and

$$\| u \|_{W^5} = \sqrt{(u,u)_{W^5}},\ u, v \in W^5[0,1].$$

By [3, 5, 7–12], it is easy to obtain the following theorem.

Theorem 2.1. *The space $W^5[0,1]$ is a RK Hilbert space. That is, there exists $k(x,y) \in W^5[0,1]$, for any $u(y) \in W^5[0,1]$ and each fixed $x \in [0,1]$, $y \in [0,1]$, such that $(u(y), k(x,y))_{W^5} = u(x)$. The RK $k(x,y)$ can be denoted by*

$$k(x,y) = \begin{cases} h(x,y), & y \le x, \\ h(y,x), & y > x, \end{cases} \tag{2.1}$$

where $h(x,y) = \frac{1}{362880}[y((3x^4 - 4x^3 + 1)y^8 - 9(x-1)^2x(2x+1)y^7 + 36(x-1)^2x^2y^6 + 3x(x^8 - 6x^7 + 12x^6 - 42x^4 + 635100x^3 - 907225x^2 + 30240x + 241920)y^3 - x(4x^8 - 27x^7 + 72x^6 - 84x^5 + 2721675x^3 - 3991720x^2 + 181440x + 1088640)y^2 + 90720(x-1)^2x^2y + 362880(x-1)^2x2x + 1))]$.

Next, we construct a RK space $W^5_{\alpha\beta}[0,1]$ in which every function satisfies $u(0) = u(\alpha) = u(\beta) = u(1) = 0$.

$W^5_{\alpha\beta}[0,1]$ is defined as $W^5_{\alpha\beta}[0,1] = \{u(x) \mid u(x) \in W^5[0,1], u(\alpha) = 0, u(\beta) = 0\}$.

Clearly, $W^5_{\alpha\beta}[0,1]$ is a closed subspace of $W^5[0,1]$.

The following theorem give the RK of $W^5_{\alpha\beta}[0,1]$.

Theorem 2.2. *If $k(\alpha,\alpha)k(\beta,\beta) - k^2(\alpha,\beta) \neq 0$, then the RK $k_{\alpha\beta}$ of $W^5_{\alpha\beta}[0,1]$ is given by*

$$\begin{aligned} k_{\alpha\beta}(x,y) = k(x,y) &+ \frac{k(x,\beta)k(\alpha,y)k(\alpha,\beta) + k(x,\alpha)k(\beta,y)k(\alpha,\beta)}{k(\alpha,\alpha)k(\beta,\beta) - k^2(\alpha,\beta)} \\ &- \frac{k(x,\beta)k(\beta,y)k(\alpha,\alpha) + k(x,\alpha)k(\alpha,y)k(\beta,\beta)}{k(\alpha,\alpha)k(\beta,\beta) - k^2(\alpha,\beta)}. \end{aligned} \tag{2.2}$$

Proof. It is easy to see that $k_{\alpha\beta}(x,\alpha) = k_{\alpha\beta}(x,\beta) = 0$, and therefore $k_{\alpha\beta}(x,y) \in W^5_{\alpha\beta}[0,1]$.

For $\forall\ u(y) \in W^5_{\alpha\beta}[0,1]$, obviously, $u(\alpha) = 0$, $u(\beta) = 0$, it follows that

$$(u(y), k_{\alpha\beta}(x,y))_{W^5} = (u(y), k(x,y))_{W^5} = u(x).$$

That is, $k_{\alpha\beta}(x,y)$ is of "reproducing property". Thus, $k_{\alpha\beta}(x,y)$ is the RK of $W^5_{\alpha\beta}[0,1]$ and the proof is complete. ∎

3. Application of RKM to (1.1)

In (1.1), it is clear that $L : W^5_{\alpha\beta}[0,1] \to W^1_2[0,1]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{k}(x_i,x)$ and $\psi_i(x) = L^*\varphi_i(x)$ where $\overline{k}(x_i,x)$ is the RK of $W^1_2[0,1]$, L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W^5_{\alpha\beta}[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), (\beta_{ii} > 0, i = 1,2,\ldots). \tag{3.1}$$

By RKM presented in [3–5, 7–12], we have the following theorem.

Theorem 3.1. *For (1.1), if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W^5_{\alpha\beta}[0,1]$ and $\psi_i(x) = L_y k_{\alpha\beta}(x,y)|_{y=x_i}$.*

Theorem 3.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and the solution of (1.1) is unique, then the solution of (1.1) is*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x). \tag{3.2}$$

Now, the approximate solution u_n can be obtained by taking finitely many terms in the series representation of $u(x)$ and

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x).$$

4. Numerical examples

In this section, two numerical examples are studied to demonstrate the accuracy of the present method. The examples are computed using Mathematica 5.0. Results obtained by the method are compared with the analytical solution of each example and are found to be in good agreement with each other.

Example 4.1. Consider the following singular fourth order four-point BVP

$$\begin{cases} x^4(1-x)u''''(x) + \frac{e^{\frac{x}{2}}}{2}u'''(x) + 2e^x \sin\sqrt{x}u''(x) + 2u'(x) + xu(x) = f(x), \\ \quad 0 \le x \le 1, \\ u(0) = 0,\ u(\frac{1}{2}) = \sinh\frac{1}{2},\ u(\frac{1}{4}) = \sinh\frac{1}{4},\ u(1) = \sinh 1, \end{cases}$$

where $f(x) = (1-x)\sinh xx^4 + \sinh xx + \frac{1}{2}e^{x/2}\cosh x + 2\cosh x + 2e^x \sin\sqrt{x}\sinh x$. The exact solution is given by $u(x) = \sinh x$. Using our method, taking $x_i = \frac{i-1}{n-1}, i = 1, 2, \cdots, n, n = 6,\ 11,\ 51$, the absolute errors $\mid u_n(x) - u(x) \mid$ between the approximate solution and exact solution are given in Figure 1.

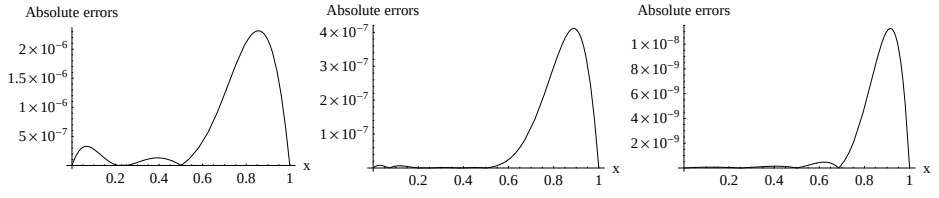

Figure 1. Absolute errors for Example 4.1 ($n = 6,\ 11,\ 51$) (The left: $n = 6$; the middle: $n = 11$; the right: $n = 51$).

Example 4.2. Consider the following singular fourth order four-point BVP

$$\begin{cases} \sin x(e^x - 1)^2 u''''(x) + e^{\frac{x}{2}}u'''(x) + 2\sin\sqrt{x}u''(x) + \sinh xu'(x) + xu(x) = f(x), \\ \quad 0 \le x \le 1, \\ u(0) = 0,\ u(\frac{1}{2}) = \sinh\frac{1}{2},,\ u(\frac{1}{4}) = \sinh\frac{1}{4},\ u(1) = \sinh 1, \end{cases}$$

where $f(x) = (e^x - 1)^2 \sin^2 x + x\sin x - 2\sin\sqrt{x}\sin x - e^{x/2}\cos x + \cos x\sinh x$. The exact solution is given by $u(x) = \sin x$. Using our method, taking $x_i = \frac{i-1}{n-1}, i = 1, 2, \cdots, n, n = 6,\ 11,\ 51$, the absolute errors $\mid u_n(x) - u(x) \mid$ between the approximate solution and exact solution are given in Figure 2.

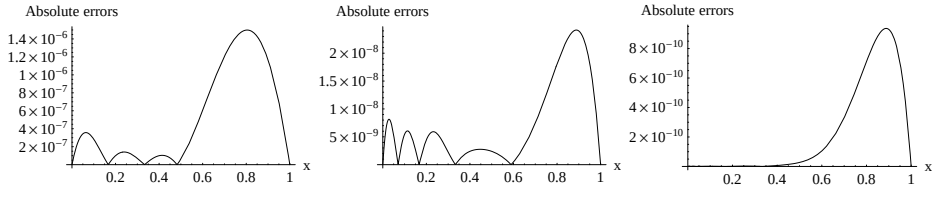

Figure 2. Absolute errors for Example 4.2 ($n = 6,\ 11,\ 51$) (The left: $n = 6$; the middle: $n = 11$; the right: $n = 51$).

5. Conclusion

In this paper, we apply RKM to singular fourth order four-point boundary value problems and obtain approximate solutions with a high degree of accuracy. Therefore, RKM is an accurate and reliable analytical technique for singular fourth order four-point boundary value problems.

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