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NUMERICAL RANGE AND ORTHOGONALITY IN NORMED SPACES

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Abstract

Introducing the concept of the normalized duality mapping on normed linear space and normed algebra, we extend the usual definitions of the numerical range from one operator to two operators. In this note we study the convexity of these types of numerical ranges in normed algebras and linear spaces. We establish some Birkhoff-James orthogonality results in terms of the algebra numerical range $V(T)_A$ which generalize those given by J.P. William and J.P. Stamplfli. Finally, we give a positive answer of the Mathieu's question.

1 Introduction

Let E be a normed space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), S_E its unit sphere, E^* its dual topologic space. Let D be the normalized duality mapping from E into E^* given by

$$D(x) := \{\varphi \in E^* : \varphi(x) = \|x\|^2,\ \|\varphi\| = \|x\|\}, \text{ for all } x \in E. \tag{1.1}$$

Let $\mathcal{B}(E)$ be the normed space of all bounded linear operators acting on E. For any operator $T \in \mathcal{B}(E)$ and $x \in E$, $W_x(T)$ denotes the convex subset of the complex plane defined by

$$W_x(T) := \{\varphi(Tx) : \varphi \in D(x)\}\cdot \tag{1.2}$$

and

$$W(T) = \cup\{W_x(T) : x \in S_E\}$$

is called the spatial numerical range of T, which may also be defined as

$$W(T) := \{\varphi(Tx) : x \in S_E;\ \varphi \in D(x)\}. \tag{1.3}$$

This definition was extended to arbitrary elements of a normed algebra $\mathcal{A}$ by F.F. Bonsall [4, 5, 6] who defined the numerical range of an element $a \in \mathcal{A}$ as

$$V(a) = W(T_a),$$

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where T_a is the left regular representation of $\mathcal{A}$ in $\mathcal{B}(\mathcal{A})$, that is, $T_a(b) = ab$ for all $b \in \mathcal{A}$. $V(a)$ is known as the algebra numerical range of $a \in \mathcal{A}$, and according to the above definitions, $V(a)$ is defined by

$$V(a) := \{\varphi(ab) : b \in S_{\mathcal{A}};\ \varphi \in D(b)\}. \tag{1.4}$$

If the normed algebra $\mathcal{A}$ has a unit e with $\|e\| = 1$, it was proven that in this case $V(a) = \{\varphi(b) : \varphi \in D(a)\}$ is a convex and weak*-compact subset of $\mathcal{A}^*$. In particular, if $\mathcal{A} = \mathcal{B}(E)$, it's was shown that

$$V(T) = \overline{coW(T)}\cdot$$

For further details on the numerical range as well as various applications of this pioneering tool in operator theory, we refer the reader to [1, 2, 19, 20].

In this paper, we mainly investigate qualitative properties of numerical range and study orthogonality on a normed space.
Our paper is organized as follows: in Section 2, we give some preliminary concepts needed in the sequel. Section 3 mainly concerns the description of the numerical range of two operators and the investigation on the orthogonality in the sense of the Birkhoff-James's definition. In section 4, we give a positive answer to the following question of Mathieu [15]:
Does the inequality $\|M_{a,b} + M_{b,a}\| \geq \|a\| \cdot \|b\|$ holds for any elements a, b in a prime C^*-algebra ?

2 Preliminaries

Throughout this paper, $\mathbb{K}$ is the field of real numbers or complex numbers, E is a normed linear space over $\mathbb{K}$ with the unit sphere S_E and E^* its dual topological space (the norm in E and E^* will be denoted by the same symbol $\|\cdot\|$).
$\mathcal{B}(E)$ is the algebra of all bounded linear operators on E, I denotes the identity operator on E. If M is a non-empty set of E, then $[M]$ denotes the closed linear subspace of E spanned by M, if $M = \{x : x \neq 0\}$, we write in short $[x]$. coS denotes the convex hull of a subset S and $\overline{S}$ denotes the closure of S.
Recall that the support of a functional φ at $x \in E$ is a norm-one linear functional in E^* such that $\varphi(x) = \|x\|$. Recall also that a convex function $f : E \to \mathbb{R}$ is said to be Gâteaux differentiable at $x \in E$ if the following limit

$$f_x(y) = \lim_{t \to 0,\ t \in \mathbb{R}} \frac{f(x + ty) - f(x)}{t}$$

exists for all $y \in E$. It is well known that if f is continuous and Gâteaux differentiable at $x \in E$, then the function $f_x : E \to \mathbb{R},\ y \mapsto f_x(y)$ is a real bounded linear functional [17]. This function is called the Gâteaux differential of f at x.

Definition 2.1. A normed space E is said to be smooth at $x \in E$ if there exists a unique support functional at x. E is said to be smooth if it is smooth at every point (see [2, part 3, ch. 1, § 2]).

Proposition 2.2. *Let E be a normed linear space over $\mathbb{K}$ equipped with a norm $\|.\|$. E is smooth at $x \in E$ if and only if the norm $\|.\|$ is Gâteaux differentiable at x. Furthermore, if E is smooth at x then the unique support functional ψ_x at x, is given by $\psi_x = f_x$ if $\mathbb{K} = \mathbb{R}$, and by $\psi_x = f_x + if_{ix}$ if $\mathbb{K} = \mathbb{C}$, where f_x , if_{ix} are the Gâteaux differentials of the norm $\|.\|$ at x and ix respectively.*

The Hahn-Banach theorem ensures that there always exists at least one support functional at each vector $x \in E$ and therefore $D(x)$ is non-empty for every $x \in E$, where D is the duality mapping on E defined by equation (1.1). Moreover, it is well known that $D(x)$ is convex and weak*-compact subset of E^*.

It is clear, from the Proposition 2.2, that D is a single-valued mapping on E if and only if E is a smooth space.

3 Numerical range

For an operator $A \in \mathcal{B}(E)$, we extend the usual definitions of the numerical range from one operator to two operators in different ways as follows.

The spatial numerical range $W(T)_A$ of $T \in \mathcal{B}(E)$ relative to A:

$$W(T)_A = \{\varphi(Tx) : x \in S_E \,; \varphi \in D(Ax)\}. \tag{3.1}$$

The spatial numerical range $G(T)_A$ of $T \in \mathcal{B}(E)$ relative to A:

$$G(T)_A = \{\varphi(Tx) : x \in E \,; \|Ax\| = 1, \ \varphi \in D(Ax)\}. \tag{3.2}$$

The *Maximal spatial numerical range* of $T \in \mathcal{B}(E)$ relative to A:

$$M(T)_A = \{\varphi(Tx) : x \in S_E \,; \|Ax\| = \|A\|, \ \varphi \in D(Ax)\}. \tag{3.3}$$

It is clear that if $A = I$, we get $W(T)_I = G(T)_I = M(T)_I = W(T)$ (the usual spatial numerical range) and for any operators $A, T \in \mathcal{B}(E)$, $M(T)_A \subseteq W(T)_A$. Moreover, if $\|A\| = 1$, then $M(T)_A \subseteq G(T)_A$.

Note that $M(T)_A$ may be empty, depends on the space E on which the operator A acts. Indeed, let c_0 be the classical space of sequences $(x_n)_n \subset \mathbb{C} : x_n \to 0$, equipped with the norm $\|(x_n)_n\| = \max\limits_n |x_n|$. And let A be an operator defined on c_0 by $A(x)(n) = \frac{n}{n+1}\varphi_n(x)$ for all $x \in c_0$ where $(\varphi_n)_n \subseteq S_{c_0^*}$ such that $\varphi_n \overset{w^*}{\to} 0$, (the existence of such sequence is assured by Josefson-Nissenzweig's theorem [8]). A is a norm-one operator which doesn't attain its norm and therefore $M(T)_A$ is an empty set for any operator T. Let $NA(E)$ be the subset of the norm attaining operators in $\mathcal{B}(E)$. Henceforth, we suppose that $A \in NA(E)$ and hence $M(T)_A$ will be a non-empty set.

Remark 3.1. Let A be an injective operator in $\mathcal{B}(E)$ and ρ be a function defined by $\rho(x) = \|Ax\|$ for all $x \in E$ and where $\|\cdot\|$ is the original norm in E. The function ρ defines a new norm in E and if A is an invertible operator then ρ defines an equivalent norm to the original norm $\|\cdot\|$ and in this last case we have successively $W(T)_A = M(T)_A$, $G(T)_A = W_\rho(T)_A$, $G(TA)_A = W(T)$ and $G(AT)_A = W(ATA^{-1})$, where $W_\rho(T)_A, W(T)$ are the spatial numerical range of T relative to A with respect to the new norm ρ and the usual numerical norm of T respectively. A^{-1} denotes the inverse of A.

In the same manner as in the monograph of F.F. Bonsall [4, 5], we introduce the map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{A})$; $x \mapsto \phi_x$ such that $\forall y \in \mathcal{A} : \phi_x(y) = xy$ (ϕ is the left regular representation of $\mathcal{A}$ in $\mathcal{B}(\mathcal{A})$). Hence the equations (3.1), (3.2) and (3.3) can be extended from operators over normed linear space to arbitrary elements of a normed algebra $\mathcal{A}$. For $a, b \in \mathcal{A}$, we set

$$W_{\mathcal{A}}(b)_a = W(\phi_b)_{\phi_a},\ G_{\mathcal{A}}(b)_a = G(\phi_b)_{\phi_a},\ M_{\mathcal{A}}(b)_a = M(\phi_b)_{\phi_a}.$$

The sets $W_{\mathcal{A}}(b)_a$, $G_{\mathcal{A}}(b)_a$ and $M_{\mathcal{A}}(b)_a$ will be called the *algebra and the maximal algebra numerical ranges respectively* of $b \in \mathcal{A}$ relative to $a \in \mathcal{A}$ and can be rewritten in the following forms.

$$\begin{aligned} W_{\mathcal{A}}(b)_a &= \{\varphi(bc) : c \in S_{\mathcal{A}}\ ; \varphi \in D(ac)\} \\ G_{\mathcal{A}}(b)_a &= \{\varphi(bc) : c \in \mathcal{A}, \|ac\| = 1\ ; \varphi \in D(ac)\} \\ M_{\mathcal{A}}(b)_a &= \{\varphi(bc) : c \in S_{\mathcal{A}}, \|ac\| = \|a\|\ ; \varphi \in D(ac)\}. \end{aligned} \tag{3.4}$$

Note that, if $\mathcal{A}$ is a unital normed algebra with unit e and $\|e\| = 1$, then $M_{\mathcal{A}}(b)_a$ is always a non-empty subset of $\mathbb{K}$.

In the following, we denote the maximal algebra numerical range $M_{\mathcal{A}}(b)_a$ by $V_{\mathcal{A}}(b)_a$ or simply by $V(b)_a$ and we shall give some useful properties of $W_A(T)$, $G(T)_A$ and $M_A(T)$.

Proposition 3.2. *Let $A, B, C \in \mathcal{B}(E)$ and $\alpha, \beta \in \mathbb{K}$, then*

1. *$W(B)_A$ and $G(B)_A$ are non-empty;*
2. *$M(\alpha B + \beta A)_A = \alpha M(B)_A + \beta \|A\|^2$, $G(\alpha B + \beta A)_A = \alpha G(B)_A + \beta$;*
3. *$W(\alpha B + \beta C)_A \subseteq \alpha W(B)_A + \beta W(C)_A$, $G(\alpha B + \beta C)_A \subseteq \alpha G(B)_A + \beta G(C)_A$;*
4. *$\max(|W(B)_A|, |G(B)_A|, |M(B)_A|) \leq \|A\| \|B\|$, where $|\{.\}| = \sup\{|\lambda| : \lambda \in \{.\}\}$.*

Proof. The proof is elementary and will be omitted. □

Recall from [10, 14] the definition of semi-inner product, if there exists a function $[.,.] : E \times E \to \mathbb{K}$ satisfying the following properties.

(i) $[\alpha x + \beta y, z] = \alpha [x, z] + \beta [y, z]$

(ii) $[y, \alpha x] = \overline{\alpha} [y, x]$

(iii) $[x, x] = \|x\|^2$

(iv) $|[y, x]| \leq \|x\| \|y\|$, for all $\alpha, \beta \in \mathbb{K}$ and all $x, y, z \in E$, then the function $[.,.]$ is called a semi-inner product (in short s.i.p.) on the normed space E which generates the given norm in E.

Define the following mapping (a multi-valued mapping) on $E \times E$ by

$$\langle .,. \rangle_D : E \times E \to 2^{\mathbb{K}} \ ; \ \forall x, y \in E : \quad \langle y, x \rangle_D = \langle Dx \ , \ y \rangle \, , \tag{3.5}$$

where D is the normalized duality mapping defined by equation (1.1) and $\langle .,. \rangle$ denotes the pairing duality. So, from the definition of D, it is not difficult to check that $\langle .,. \rangle_D$ has the same properties mentioned above for a s.i.p. $[.,.]$.

We say that d is a determination of D, if d is a map (single-valued mapping) from E to E^* such that $d(x) \in D(x)$ for any $x \in E$. Then we can write, for any $x, y \in E$, that $\langle y, x \rangle_D = \underset{d}{\cup} \{\langle dx \ , \ y \rangle\}$. Set $[y, x]_d = \langle d(x), y \rangle$, for all $x, y \in E$. Hence $[.,.]_d$ is a s.i.p. in E with the norm $\|x\|^2 = [x, x]_d = \langle d(x), x \rangle$. Conversely, for every s.i.p.$[.,.]$ in normed space E which generates its norm, there is one determination d of D of the form $[y, x] = \langle d(x), y \rangle$ for all $x, y \in E$. So the concept of the mapping $\langle .,. \rangle_D$ generalizes the concept of s.i.p. $[.,.]$ in normed linear space.
From above we can rewrite the equations (3.1), (3.2) and (3.3) as follows

$$\begin{aligned} W(T)_A &= \underset{\|x\|=1}{\cup} \langle Ax, Tx \rangle_D = \underset{d}{\cup} \{[Tx, Ax]_d : \|x\| = 1\} \\ G(T)_A &= \underset{\|Ax\|=1}{\cup} \langle Ax, Tx \rangle_D = \underset{d}{\cup} \{[Tx, Ax]_d : \|Ax\| = 1\} \\ M(T)_A &= \underset{\|x\|=1, \|Ax\|=\|A\|}{\cup} \langle Ax, Tx \rangle_D \\ &= \underset{d}{\cup} \{[Tx, Ax]_d : \|x\| = 1, \ \|Ax\| = \|A\|\} \end{aligned} \tag{3.6}$$

We observe, from equation (3.6), that the numerical range $W(T)_A$, which is associated to a multivalued mapping $\langle .,. \rangle_D$, is a natural generalization of the traditional one (i.e., the Toeplitz's numerical range defined on the Euclidian spaces) and the Lumer's numerical range defined on the s.i.p. spaces. If E is equipped with a s.i.p. $[.,.]$, then

$$\begin{aligned} W(T)_A &= \{[Tx, Ax] : \|x\| = 1\} \\ G(T)_A &= \{[Tx, Ax] : x \in E \ ; \ \|Ax\| = 1\} \\ M(T)_A &= \{[Tx, Ax] : \|x\| = 1, \ \|Ax\| = \|A\|\} \end{aligned} \tag{3.7}$$

So, if E is a smooth normed space, we have only one determination of D and therefore the equalities (3.6) and (3.7) coincide. In particular, if $E = H$ is a Hilbert space with the inner product $(.,.)$ then the previous equalities can be rewritten in the following forms.

$$\begin{aligned} W(T)_A &= W(A^*T) = \{(Tx, Ax) : \|x\| = 1 \ \} \\ G(T)_A &= \{(Tx, Ax) : x \in E \ ; \ \|Ax\| = 1\} \\ M(T)_A &= \{(Tx, Ax) : \|x\| = 1, \ \|Ax\| = \|A\|\} \, , \end{aligned} \tag{3.8}$$

where A^* is the adjoint of A.

Remark 3.3. Let A be an operator on H and $A = UP$ be its polar decomposition. If A is injective then U is an isometry and the function $\|.\|_P$ defined by $\|x\|_P = \|Ax\| = \|Px\|$; for all $x \in H$, is a new norm on H and $(.,.)_P$ is a new inner product in H where $(x,y)_P = (Ax, Ay) = (Px, Py)$, for all $x, y \in H$. Furthermore, if A is invertible, then $G(TA)_A = W(T)$; $G(AT)_A = W_P(T) = W\left(PTP^{-1}\right)$ where $W(.)$ $W_P(.)$ are the usual numerical ranges relative to the original norm $\|.\|$ and to the new norm $\|.\|_P$ respectively. It is well known that the usual Hilbert space numerical range $W(T)$ is convex, and so $W(T)_A = W(A^*T)$ is convex for the Hilbert space operators A and T.
In the following proposition we show that $M(T)_A$ is convex for Hilbert space operators A and T.

Proposition 3.4. *Let $A \in NA(E)$, the Hilbert space numerical range $M(T)_A$ of T relative to A defined in the Equation (3.8) is a non-empty convex subset in $\mathbb{K}$.*

Proof. If $M(T)_A$ is a singleton set, then the result is trivial.
Let $\alpha, \beta \in M(T)_A$ and assume that the Hilbert space E is equipped with the inner product $(.,.)$ over the field $\mathbb{K} = \mathbb{C}$. Also, without loss of generality we may assume that $\|A\| = 1$. By equations (3.8), there exist $x, y \in S_E$ such that $(Ax, Tx) = \alpha$, $(Ay, Ty) = \beta$, $\|Ax\| = 1$ and $\|Ay\| = 1$. Define the function F by

$$F : \mathbb{C}^2 \to \mathbb{C};\ F(\lambda, \eta) = (A(\lambda x + \eta y), T(\lambda x + \eta y)), \text{ for all } \lambda, \eta \in \mathbb{C}.$$

To prove our result, it suffices to prove that the function F must attains every value on the line segment joining α and β while $\|\lambda x + \eta y\| = 1$ and $\|A(\lambda x + \eta y)\| = 1$.

$$\begin{aligned} F(\lambda, \eta) - \beta \|\lambda Ax + \eta Ay\|^2 &= \lambda\overline{\lambda}(\alpha - \beta) + \overline{\lambda}\eta\left[(Ty, Ax) - \eta(Ay, Ax)\right] \\ &\quad + \lambda\overline{\eta}\left[(Tx, Ay) - \beta(Ax, Ay)\right] \end{aligned}$$

and

$$\begin{aligned} \frac{F(\lambda, \eta) - \beta \|\lambda Ax + \eta Ay\|^2}{(\alpha - \beta)} &= \lambda\overline{\lambda} + \overline{\lambda}\eta\frac{(Ty, Ax) - \eta(Ay, Ax)}{(\alpha - \beta)} + \\ &\quad + \lambda\overline{\eta}\frac{(Tx, Ay) - \beta(Ax, Ay)}{(\alpha - \beta)}. \end{aligned}$$

Put

$$\begin{aligned} a &= \frac{(Ty, Ax) - \eta(Ay, Ax)}{(\alpha - \beta)}, \quad b = \frac{(Tx, Ay) - \beta(Ax, Ay)}{(\alpha - \beta)} \\ G(\lambda, \eta) &= \frac{F(\lambda, \eta) - \beta \|\lambda Ax + \eta Ay\|^2}{(\alpha - \beta)}, \end{aligned}$$

then

$$\begin{aligned} G(\lambda, \eta) &= \lambda\overline{\lambda} + a\overline{\lambda}\eta + b\lambda\overline{\eta} \\ \|\lambda Ax + \eta Ay\|^2 &= \lambda\overline{\lambda} + \lambda\overline{\eta}(Ax, Ay) + \overline{\lambda}\eta(Ay, Ax) + \eta\overline{\eta} \\ \|\lambda x + \eta y\|^2 &= \lambda\overline{\lambda} + \lambda\overline{\eta}(x, y) + \overline{\lambda}\eta(y, x) + \eta\overline{\eta}. \end{aligned}$$

Since λ, η are arbitrary then we can choose them as follows.

$$\begin{aligned}\lambda &= |\lambda| e^{i\theta} : \theta = \frac{\pi}{2} + k\pi (k \in \mathbb{Z}) : Im\ (Ax, Ay) = Im,\ (x, y)\\ \theta &= \arctan\left(\frac{Re\ (Ax, Ay) - Re\ (x, y)}{Im\ (Ax, Ay) - Im\ (x, y)}\right) : Im\ (Ax, Ay) \neq Im\ (x, y)\end{aligned}$$

$\eta = \gamma z$ with $\gamma \in \mathbb{R}$ and $|z| = 1$, such that

$$ae^{i\theta}z + b\overline{e^{i\theta}z} \in \mathbb{R} \quad \text{and} \quad Re\, e^{i\theta}z\,(Ax, Ay) \geq 0.$$

Set $\tau = Re\, e^{i\theta}z\,(Ax, Ay)$, $\zeta = ae^{i\theta}z + b\overline{e^{i\theta}z}$ and $\rho = |\lambda|$.
Hence

$$\begin{aligned}G(\lambda, \eta(\gamma)) &= \rho^2 + \zeta\rho\gamma\\ \|\lambda Ax + \eta Ay\|^2 &= \|\lambda x + \eta y\|^2\\ &= \rho^2 + 2\rho\gamma\tau + \gamma^2.\end{aligned}$$

Since $\|\lambda Ax + \eta Ay\|^2 = 1$ and $\gamma = -\rho Re\,\tau + \sqrt{1 - \rho^2 + (\rho\tau)^2}$, then γ is real if $\rho \leq 1$. Therefore $G(\rho, \eta(\gamma))$ can be rewritten as a function of ρ as

$$G(\rho, \eta(\gamma)) = \rho\left[\rho(1 - \tau\zeta) + \zeta\sqrt{1 - \rho^2 + (\rho\tau)^2}\right] = H(\rho).$$

It is clear that the function H is continuous on $0 \leq \rho \leq 1$, $H(0) = 0$ and $H(1) = 1$. Therefore the function H takes all its values from 0 to 1. Thus the function F takes all its values on the line segment joining α and β. □

With slight modifications in the proof of the previous proposition, we can also prove that $G(T)_A$ is convex. But in an arbitrary normed space, we known that $W(T)$ is not necessarily convex or closed, so is for $W(T)_A$, $G(T)_A$ and $M(T)_A$.

For a normed algebra $\mathcal{A}$ with unit e and $\|e\| = 1$, it is known that the usual algebra numerical range $V(a)$ is a convex and closed set in $\mathbb{K}$.
In the following, we prove that the maximal algebra numerical range $V(b)_a$ is a convex and closed set in $\mathbb{K}$.

Lemma 3.5. *The set $\{\varphi(b) : \varphi \in D(a)\}$ is a non-empty, convex and closed set.*

Proof. Since $D(a)$ is non-empty and convex, it yields that the set $\{\varphi(b) : \varphi \in D(a)\}$ is non-empty and convex. On the other hand the set $\{\varphi(b) : \varphi \in D(a)\}$ is the image of $D(a)$ which is weak*-compact subset of $\mathcal{A}^*$, under the weak*-continuous map

$$\varphi \to \varphi(b)\,(\mathcal{A}^* \to \mathbb{K}).$$

Hence $\{\varphi(b) : \varphi \in D(a)\}$ is compact in $\mathbb{K}$ and therefore closed. □

Proposition 3.6. *Let $\mathcal{A}$ be a normed algebra with unit e and $\|e\| = 1$. If $a, b \in \mathcal{A}$, then*

$$V(b)_a = \{\varphi(b) : \varphi \in D(a)\}$$

Proof. We have $\{\varphi(b) : \varphi \in D(a)\} \subseteq V(b)_a$ (it suffices to take $c = e$). Conversely, let $\lambda \in V(b)_a$ then there is $c \in S_{\mathcal{A}}$ with $\|ac\| = \|a\|$ and $\varphi \in D(ac)$ such that $\lambda = \varphi(bc)$. Define the functional ψ as $\psi(x) = \varphi(xc)$; for all $x \in \mathcal{A}$, then $|\psi(x)| = |\varphi(xc)| \leq \|a\| \|x\|$ and $\psi(a) = \varphi(ac) = \|a\|^2$. Thus $\|\psi\| = \|a\|$ and $\psi(a) = \|a\|^2$ and so $\psi \in D(a)$. Also, $\lambda = \varphi(bc) = \psi(b)$. This gives $\lambda \in \{\varphi(b) : \varphi \in D(a)\}$. □

As a consequence of the previous proposition $V(b)_a$ can be rewritten, with respect to the multivalued mapping $\langle .,. \rangle_D$ on $\mathcal{A} \times \mathcal{A}$ as

$$V(b)_a = \langle a, b \rangle_D . \tag{3.9}$$

Remark 3.7. With a similar reasoning as in Proposition 3.6, we can prove that $M(T)_A \subseteq V_{\mathcal{B}(E)}(T)_A$.

Furthermore, for A, $T \in \mathcal{B}(E)$, we put

$$S_E(A) = \{(x_n)_n : x_n \in S_E \ , \ \|Ax_n\| \to \|A\|\}$$

and define the following set

$$\mathcal{M}(T)_A = \{\lim \varphi_n(Tx_n) : (x_n)_n \in S_E(A) \ , \varphi_n \in D(Ax_n)\}, \tag{3.10}$$

we will call this set, the generalized maximal numerical range of T relative to A, it is easy to see, from the definition of the supremum $\|A\| = \sup\{\|Ax\| : x \in S_E\}$, that $\mathcal{M}(T)_A$ is a non-empty closed subset of $\mathbb{K}$ and $M(T)_A \subseteq \mathcal{M}(T)_A \subseteq \overline{W(T)_A}$. The definition of $\mathcal{M}(T)_A$ can be rewritten, with respect to the multivalued mapping $\langle .,. \rangle_D$, as

$$\mathcal{M}(T)_A = \bigcup_{(x_n)_n \in S_E(A)} \lim \langle Ax_n, Tx_n \rangle_D , \tag{3.11}$$

with respect to a s.i.p. $[.,.]$ as

$$\mathcal{M}(T)_A = \{\lim [Tx_n, Ax_n] : (x_n)_n \in S_E(A)\}, \tag{3.12}$$

with respect to an inner product $(.,.)$ as

$$\mathcal{M}(T)_A = \{\lim (Tx_n, Ax_n) : (x_n)_n \in S_E(A) \ \}. \tag{3.13}$$

If E is a Hilbert space and $T = I$, then $\mathcal{M}(I)_A = W_0(A)$ is the maximal numerical range of T and it is convex [19]. With a slight modification in [19], we can prove that $\mathcal{M}(T)_A$ is convex for Hilbert space operators A, T. The following proposition extends the result mentioned in Remark 3.2.

Proposition 3.8. *If $\mathcal{A}$=$\mathcal{B}(E)$ where E is a Banach space and $A, T \in \mathcal{B}(E)$, then $\overline{co}\mathcal{M}(T)_A \subseteq V(T)_A$.*

Proof. Let $\lambda \in \mathcal{M}(T)_A$, then there exists $(x_n)_n \in S_E(A)$ and $\varphi_n \in D(Ax_n)$ such that $\lambda = \lim \varphi_n(Tx_n)$. Let $f_n(X) = \varphi_n(Xx_n)$ for all $X \in \mathcal{B}(E)$, then $f_n \in (\mathcal{B}(E))^*$ for all n and $f_n(A) = \varphi_n(Ax_n) = \|\varphi_n\|^2 = \|Ax_n\|^2$ and $\lim f_n(A) = \|A\|^2$.

Since the sphere of radius $\|A\|$ of $(\mathcal{B}(E))^*$ is weak*-compact, there is a subsequence $(f_{n_k})_{n_k}$ weak*-convergent to $f \in (\mathcal{B}(E))^*$. Hence, from $\lim f_{n_k}(A) = \|A\|^2$and $\|f_{n_k}\| = \|\varphi_{n_k}\| = \|Ax_{n_k}\|$, it follows that $f(A) = \|A\|^2$ and $\|f\| = \|A\|$. Hence $f \in D(A)$. Since $\lambda = \lim \varphi_{n_k}(Tx_{n_k}) = \lim f_{n_k}(T)$, we get $\lambda = f(T)$ and therefore by Proposition 3.6, $\lambda \in V(T)_A$. Finally, from the convexity and the closure of $V(T)_A$ we obtain $\overline{co}\mathcal{M}(T)_A \subseteq V(T)_A$. □

Proposition 3.9. *If $\mathcal{A}$=$\mathcal{B}(E)$ where E is a Banach space and $A, T \in \mathcal{B}(E)$, then $G(T)_A \subseteq G_{\mathcal{B}(E)}(T)_A$.*

Proof. Let $\lambda \in G(T)_A$, then there is $(0 \neq) x \in E$ with $\|Ax\| = 1$ and $\varphi \in D(Ax)$ such that $\lambda = \varphi(Tx)$. Let $\psi_x(X) = \varphi(XAx)$ for all $X \in \mathcal{B}(E)$ and $R_x(y) = \varphi(y)x$. Firstly, it is clear that $\psi_x \in \mathcal{B}^*(E)$, $|\psi_x(X)| = |\varphi(XAx)| \leq \|X\|$ and $\psi_x(AR_x) = \varphi(AR_xAx) = \varphi(Ax) = 1$. Then, $\|\psi_x\| = 1$ and $\psi_x(AR_x) = 1$. Secondly, $\|AR_x(y)\| = \|\varphi(y)Ax\| \leq \|y\|$ and $\|AR_x(Ax)\| = \|Ax\| = 1$. Therefore $\|AR_x\| = 1$ and $\psi_x \in D(AR_x)$. Also, we have $\psi_x(TR_x) = \varphi(TR_xAx) = \varphi(Tx) = \lambda$. This gives $\lambda \in G_{\mathcal{B}(E)}(T)_A$ and the proof is complete. □

For a normed algebra $\mathcal{A}$ with unit e; $\|e\| = 1$ and $a, b \in \mathcal{A}$. We have $V(b)_a \subseteq W_{\mathcal{A}}(b)_a$ and $V(b)_a \subseteq \|a\| G_{\mathcal{A}}(b)_{\frac{a}{\|a\|}}$; $(a \neq 0)$, the reverse inclusions are not verified in general. However, the following proposition gives a weaker result.

Proposition 3.10. *Let $\mathcal{A}$ be a normed algebra with unit e and $\|e\| = 1$. If $(0 \neq) a, b \in \mathcal{A}$, then*

$$V(b)_a = \{\varphi(bc) : \|c\| \leq 1, \|ac\| = \|a\| \ ; \varphi \in D(ac)\}.$$

Proof. The proof is similar as that's given in Proposition 3.6. □

The function $\psi_{x,y}$ defined by

$$\psi_{x,y}(t) = \frac{\|x + ty\| - \|x\|}{t} \tag{3.14}$$

is increasing on $(0, \infty)$ for all $x, y \in E$ and the limit $\lim\limits_{t \to 0^+} \psi_{x,y}(t)$ exists for each $x, y \in E$. Also, for all $0 \neq x, y \in E$,

$$\lim_{t \to 0^{+(-)}} \psi_{x,y}(t) = \frac{1}{\|x\|} \lim_{t \to 0^{+(-)}} \frac{\|x + ty\|^2 - \|x\|^2}{2t}.$$

Set

$$[y, x]^+ = \lim_{t \to 0^+} \frac{\|x + ty\|^2 - \|x\|^2}{2t}, \ [y, x]^- = \lim_{t \to 0^-} \frac{\|x + ty\|^2 - \|x\|^2}{2t} \tag{3.15}$$

The mappings $[.,.]^{+(-)}$ were introduced by Dragomir and their properties are given in the real space [9]. We shall give some of them in the complex case.

Lemma 3.11. *For all $x, y \in E$, we have*

i. $[y,x]^- = -[-y,x]^+$; $[x,x]^{+(-)} = \|x\|^2$;

ii. $[\alpha x + y, x]^{+(-)} = (Re\, y, x]^{+(-)}$, $\alpha \in \mathbb{K}$;

iii. $[\alpha y, \beta x]^{+(-)} = |\alpha\beta| \left[e^{i(\theta-\omega)}y, x\right]^{+(-)}$, $\alpha = |\alpha| e^{i\theta}; \beta = |\beta| e^{i\omega}$;

iv. $\left|[y,x]^{+(-)}\right| \leq \|x\| \|y\|$.

Proof. The proof is elementary and will be omitted. □

Remark 3.12. [13] Let $x, y \in E$ such that $\|x\| = 1$ and y be an arbitrary vector. We note here that if the limits $\lim\limits_{t\to 0^+} \psi_{x,y}(t)$ and $\lim\limits_{t\to 0^-} \psi_{x,y}(t)$ are different, then it follows from the Hahn-Banach theorem that for any γ satisfying

$$\lim_{t\to 0^-} \psi_{x,y}(t) \leq \gamma \leq \lim_{t\to 0^+} \psi_{x,y}(t),$$

there is a real linear functional x^* for which $\|x^*\| = x^*(x) = 1$ and $x^*(y) = \gamma$. If E is a complex space there is a unique complex functional φ with $Re\,\varphi = x^*$and defined by $\varphi(z) = x^*(z) - ix^*(iz)$, for all $z \in E$. Furthermore, $\varphi(x) = \|\varphi\| = 1$ and $Re\,\varphi(y) = \gamma$. Indeed, let $\theta \in \mathbb{R}$, then $|\varphi(z)| = \left|Re\, e^{i\theta}\varphi(z)\right| = \left|x^*\left(e^{i\theta}z\right)\right| \leq \|z\|$ and $1 = \|x^*\| \leq \|\varphi\| \leq 1$. So by $\varphi(x) = 1 - ix^*(ix)$ and $\sqrt{1 + (x^*(ix))^2} \leq \|\varphi\|$ it follows that $\varphi(x) = 1$.

Lemma 3.13. *Let E be a normed linear space over $\mathbb{K}$ and $x, y \in E$, then*

$$\begin{aligned} [y,x]^+ &= \sup\{Re\,\varphi(y) : \varphi \in D(x)\} \\ [y,x]^- &= \inf\{Re\,\varphi(y) : \varphi \in D(x)\}. \end{aligned} \tag{3.16}$$

Proof. The case $x = 0$ is obvious. Let $x \neq 0$, $\varphi \in D(x)$ and $z \in \ker\varphi$, then for any $t > 0$,

$$\varphi(x + tz) = \|x\|^2 \leq \|x\| \|x + tz\|.$$

Hence,

$$\psi_{x,z}(t) = \frac{\|x + tz\| - \|x\|}{t} \geq 0.$$

Since the function $\psi_{x,z}$ is increasing on $(0, \infty)$ and by Lemma 3.11, it follows that

$$[z,x]^- \leq 0 \leq [z,x]^+.$$

Let $z = \varphi(y)x - \varphi(x)y$, for an arbitrary $y \in E$. Then, $z \in \ker\varphi$ and by the last inequalities and Lemma 3.11, we get

$$[y,x]^- \leq Re\,\varphi(y) \leq [y,x]^+, \text{ for all } \varphi \in D(x).$$

Hence

$$[y,x]^- \leq \inf_{\varphi\in D(x)} Re\,\varphi(y) \leq \sup_{\varphi\in D(x)} Re\,\varphi(y) \leq [y,x]^+.$$

If E is a smooth space then $[.,.]^- = [.,.]^+$ and $[.,x]^+$ is a real linear functional and the result holds. If E is not a smooth space then the limits $[y,x]^-$, $[y,x]^+$ are different. Without loss of generality we may suppose that $\|x\| = 1$.

If $[y,x]^+ \neq \sup\limits_{\varphi\in D(x)} Re\,\varphi(y)$ then there is $\varepsilon > 0$ such that

$$\forall\varphi \in D(x) : [y,x]^- < Re\,\varphi(y) < [y,x]^+ - \varepsilon < [y,x]^+ ,$$

by Remark 3.12, we get a contradiction. Therefore, $\sup\limits_{\varphi\in D(x)} Re\,\varphi(y) = [y,x]^+$.

By Lemma 3.11 and $\sup(.) = -\inf(-)$, we obtain $\inf\limits_{\varphi\in D(x)} Re\,\varphi(y) = [y,x]^-$. $\square$

Corollary 3.14. *If $\mathcal{A}$ is a normed algebra then for every a, b in $\mathcal{A}$, we have*

$$\inf Re\,V(b)_a = [b,a]^- \leq [b,a]^+ = \sup Re\,V(b)_a . \tag{3.17}$$

Corollary 3.15. *If $\mathcal{A}$=$\mathcal{B}(E)$ where E is a normed linear space and A, T are operators in $\mathcal{A}$, then*

$$\sup Re\,M(T)_A \leq \sup Re\,\mathcal{M}(T)_A \leq [A,T]^+ . \tag{3.18}$$

Proof. Apply Proposition 3.8 and Corollary 3.14 and the proof is complete. $\square$

Remark 3.16. Let $\mathcal{A}$=$\mathcal{B}(E)$, E be a Banach space and $T \in \mathcal{B}(E)$, it is well known that $\overline{co}W(T) = V(T)$. But, for an arbitrary Banach space operator A, $\overline{co}W(T)_A \neq V(T)_A$. It is easy to see that in the trivial case when $T = A$, we have $V(A)_A = \left\{\|A\|^2\right\}$, $G(T)_A = \{1\}$ and $W(A)_A = \left\{\|Ax\|^2 : \|x\| = 1\right\}$. Then, $\overline{co}G(T)_A \neq V(A)_A$, $\overline{co}W_A(A) \neq \overline{co}G(T)_A$ and $\overline{co}W_A(A) \nsubseteq V(A)_A$. In the previous case if A is a non-injective operator with $A \neq 0$, then $0 \in W(A)_A$ but $0 \notin V(A)_A$.

If $\overline{co}M(T)_A = V(T)_A$ holds in $\mathcal{B}(E)$, where E is a Banach space, then $\max Re\,V(T)_A = \sup Re\,M(T)_A$, where $A, T \in \mathcal{B}(E)$. Therefore, we can write $\overline{co}M(T)_A = V(T)_A$ if and only if

$$\lim_{t\to 0^+}\sup_{x\in S_E}\frac{\|Ax + tTx\|^2 - \|A\|^2}{2t} = \sup_{x\in S_E}\lim_{t\to 0^+}\frac{\|Ax + tTx\|^2 - \|A\|^2}{2t}\cdot \tag{3.19}$$

In the next we give a non-trivial example which shows that, when $A \neq I$, the equality $\overline{co}M(T)_A = V(T)_A$ does not hold. Moreover, it also shows that $\sup Re\,W(T)_A < \max Re\,V(T)_A$ and $\sup Re\,G(T)_A < \max Re\,V(T)_A$ which implies that $V(T)_A \nsubseteq \overline{co}W(T)_A$ and $V(T)_A \nsubseteq \overline{co}G(T)_A$. Therefore, the subsets $\overline{co}W(T)_A$, $V(T)_A$, $\overline{co}G(T)_A$ are in general incomparable.

Example 3.17. Let c_0 be the classical space of sequences $(x_n)_n \subset \mathbb{C} : x_n \to 0$, equipped with the norm $\|(x_n)_n\| = \max\limits_n |x_n|$ and L be an infinite-dimensional Banach space. Let E be a Banach space such that $E = L \oplus c_0$ with the norm $\|x\| = \|x_1 + x_2\| = \max\{\|x_1\|, \|Bx_1\| + \|x_2\|\}$ for all $x \in E$ and B is any norm-one operator from L to c_0 which does not attain its norm (by Josefson-Nissenzweig's theorem [8], we can find a sequence $(\varphi_n)_n \subseteq S_{F^*}$ such that $\varphi_n \overset{w^*}{\to} 0$. Therefore we get the desired operator $B : L \to c_0$; $(Bx)_n = \dfrac{n}{n+1}\varphi_n(x))$. Let A, T be operators on E defined as follows.

$$\begin{aligned}\forall x &= (x_1, x_2) \in L \times c_0 : \\ Ax &= A(x_1 + x_2) = x_1 + 0 \ ; \qquad Tx = T(x_1 + x_2) = 0 + Bx_1.\end{aligned}$$

It is easy to check that A, T are linear bounded operators and $\|A\| = 1$.

Since $\|Ax + tTx\| = \|x_1 + tBx_1\| = \max\{\|x_1\|, (1+t)\|Bx_1\|\}$, then

$$\sup_{x\in S_E}\|Ax + tTx\| = \sup_{\|x_1\|\leq 1}\|x_1 + tBx_1\| = 1 + t.$$

Hence

$$\begin{aligned}\lim_{t\to 0^+}\sup_{x\in S_E}\frac{\|Ax + tTx\|^2 - \|A\|^2}{2t} &= \lim_{t\to 0^+}\sup_{\|x_1\|\leq 1}\frac{\|x_1 + tBx_1\|^2 - 1}{2t} \\ &= 1\end{aligned}$$

and

$$\sup_{\|x_1\|\leq 1}\lim_{t\to 0^+}\frac{\|x_1 + tBx_1\|^2 - 1}{2t} = 0.$$

So, it follows from the last two equations and the equation (3.19), that $\overline{co}M(T)_A \neq V(T)_A$.

Let's us compute $\sup Re\, W(T)_A$ and $\sup Re\, G(T)_A$:

$$\begin{aligned}\sup Re\, W(T)_A &= \sup_{x\in S_E}[Tx, Ax]^+ = \sup_{x\in S_E}\lim_{t\to 0^+}\frac{\|Ax + tTx\|^2 - \|Ax\|^2}{2t} \\ &= \sup_{\|x_1\|\leq 1}\lim_{t\to 0^+}\frac{\|x_1 + tBx_1\|^2 - \|x_1\|^2}{2t} \\ &= \sup_{\|x_1\|\leq 1}\lim_{t\to 0^+}\frac{(1+t)^2\|Bx_1\|^2 - \|x_1\|^2}{2t} \\ &< 1.\end{aligned}$$

Hence $\sup Re\, W(T)_A < \max Re\, V(T)_A$.

$$\begin{aligned}\sup Re\, G(T)_A &= \sup_{\|Ax\|=1}[Tx, Ax]^+ = \sup_{\|Ax\|=1}\lim_{t\to 0^+}\frac{\|Ax + tTx\|^2 - \|Ax\|^2}{2t} \\ &= \sup_{\|x_1\|=1}\lim_{t\to 0^+}\frac{\|x_1 + tBx_1\|^2 - \|x_1\|^2}{2t} < 1.\end{aligned}$$

Hence $\sup Re\, G(T)_A < \max Re\, V(T)_A$.

Definition 3.18. Let F be a Banach space. F is said a superspace of the Banach space E, if there exists an inclusion map J ($J : E \to F$) which is linear and an isometry.

Remark 3.19. The definitions of $W(T)_A$, $G(T)_A$, $M(T)_A$ and $V(T)_A$ remain valid if we replace $\mathcal{B}(E)$ by $\mathcal{B}(E,F)$ where E,F are normed linear spaces and $A,T \in \mathcal{B}(E,F)$. Let F be a superspace of E with inclusion operator J and $T \in \mathcal{B}(E,F)$. It is clear that if we consider $A = J$ then

$$\begin{aligned} W(T)_J &= G(T)_J = M(T)_J = \{\varphi(Tx) : x \in S_E \ ; \varphi \in D(Jx)\} \\ \overline{W(T)}_J &= \mathcal{M}(T)_J = \{\lim \varphi_n(Tx_n) : x_n \in S_E \ ; \varphi_n \in D(Jx_n)\} \\ V(T)_J &= \{\varphi(T) : \ \varphi \in D(J)\}. \end{aligned}$$

These numerical ranges were given by L. Harris [11] for continuous functions. Note that $V(T)_A$ is a closed convex subset of $\mathbb{K}$ and $\overline{co}W(T)_J \subseteq V(T)_A$. Indeed, let for all $S \in \mathcal{B}(E,F) : \psi(S) = \varphi(Sx)$, where $x \in S_E$ and $\varphi \in D(Jx)$. It is clear that ψ is a functional in $\mathcal{B}^*(E,F)$ and $\psi \in D(J)$.

The equality $\overline{co}W(T)_J = V(T)_J$ for an arbitrary operator $T \in \mathcal{B}(E,F)$ with E a proper closed subspace of F does not hold in general. In Example 3.17, take $E = L$, $F = L \oplus c_0$ with the same norm and $Jx_1 = x_1 + 0$; $Tx_1 = 0 + Bx_1$ for all $x_1 \in E$ and B be the same operator defined in the Example 3.17. Then,

$$\begin{aligned} \lim_{t\to 0^+} \sup_{x_1 \in S_E} \frac{\|Jx_1 + tTx_1\|^2 - 1}{2t} &= \lim_{t\to 0^+} \sup_{x_1 \in S_E} \frac{\|x_1 + tBx_1\|^2 - 1}{2t} = 1 \\ &\neq \sup_{x_1 \in S_E} \lim_{t\to 0^+} \frac{\|x_1 + tBx_1\|^2 - 1}{2t} = 0. \end{aligned}$$

4 Orthogonality

Let E be a normed linear space and x, y any elements in E.

Definition 4.1. We say that x is orthogonal to y in the sense of Birkhoff-James [3, 12], in short $x \perp_{B-J} y$, if for any $\lambda \in \mathbb{K}$,

$$\|x + \lambda y\| \geq \|x\|. \tag{4.1}$$

Definition 4.2. We say that x is orthogonal to y with respect to the mapping $\langle .,. \rangle_D$ defined by the equation (3.5), in short $x \perp_D y$, if

$$0 \in \langle x, y \rangle_D. \tag{4.2}$$

If we take one determination d of D, we get the orthogonality with respect to the semi inner product $[.,.]_d$, in short $x \perp_d y$, if

$$[y, x]_d = 0. \tag{4.3}$$

If E is a pre-Hilbert space, we get the usual orthogonality relative to its inner product. So the orthogonality in Definition 4.2, is a naturel generalization of the traditional case.

Remark 4.3. 1. Observe, with respect to the s.i.p. $[.,.]_d$, that $\perp_d \subset \perp_D$.

2. $\langle x,y\rangle_D = 0 \Rightarrow x \perp_D y$, for any $x,y \in E$.
The reverse holds if E is a smooth normed linear space.

Note that $\perp_D$ and $\perp_{B-J}$ are asymmetric orthogonalities in non-inner product spaces, i.e. characterize the non-inner product spaces.

Lemma 4.4. *Let $x,y \in E$.*
$x \perp_{B-J} y \Leftrightarrow \psi_{x,e^{i\theta}y}(t) \geq 0,$ *for all* $t > 0, \theta \in \mathbb{R}$.

Proof. $x \perp_{B-J} y \Leftrightarrow \forall\lambda \in \mathbb{C} : \|x+\lambda y\| \geq \|x\|$. Set $\lambda = te^{i\theta}$ where $t = |\lambda|$ and $\theta \in \mathbb{R}$, then $\left\|x+te^{i\theta}y\right\| - \|x\| \geq 0$, for all $t > 0, \theta \in \mathbb{R}$. Hence
$\psi_{x,e^{i\theta}y}(t) = \frac{\left\|x+te^{i\theta}y\right\|-\|x\|}{t} \geq 0$, for all $t > 0$, $\theta \in \mathbb{R}$. The reverse is obvious. □

The following theorem gives a characterization of Birkhoff-James's orthogonality in arbitrary normed linear space with respect to the set-valued mapping $\langle.,.\rangle_D$.

Theorem 4.5. *Let x,y be any elements in a normed linear space E, then*

$$x \perp_{B-J} y \Leftrightarrow x \perp_D y$$

Proof. Let $x,y \in E$, if $x \perp_D y$, by Definition 4.2 there exists $\varphi \in D(x)$ such that $\langle\varphi,y\rangle = 0$. Then, for all $\lambda \in \mathbb{K}$, $\varphi(x+\lambda y) = \varphi(x) = \|x\|^2$, and
$\|x\|^2 \leq \|\varphi\|\,\|x+\lambda y\| \leq \|x\|\,\|x+\lambda y\|$.
Hence
$\|x\| \leq \|x+\lambda y\|$, for all $\lambda \in \mathbb{K}$. This gives $x \perp_{B-J} y$.
Conversely. Using Lemma 3.13 and properties of $\psi_{x,y}$ we get

$$x \perp_{B-J} y \Leftrightarrow \left[e^{i\theta}y,x\right]^- \leq 0 \leq \left[e^{i\theta}y,x\right]^+, \text{ for all } \theta \in \mathbb{R}.$$

By Hahn-Banach theorem, there exists $\varphi \in E^*$ such that $\varphi \in D(x)$ and $Re\,\varphi\left(e^{i\theta}y\right) = 0$, for all $\theta \in \mathbb{R}$. Consequently $Re\,\varphi(-iy) = 0 = Re\,\varphi(y)$, and so $\langle\varphi,y\rangle = 0$, i.e., $x \perp_D y$. □

As an immediate consequence of the previous theorem and Proposition 3.6, the following corollary generalizes the result given by J.G.Stampfli and J.P. Williams [20].

Corollary 4.6. *Let A,T be operators in $\mathcal{B}(E)$, where E is a normed linear space, then*
. $A \perp_{B-J} T \Leftrightarrow 0 \in V(T)_A$;
. $V(T)_A = \bigcap\limits_{\eta\in\mathbb{C}} \{\lambda : \|T-\eta A\| \geq |\lambda-\eta|\}$; *where* $\|A\| = 1$.

It is easy to check that for all $\alpha,\beta \in \mathbb{K}$:

$$\begin{aligned} V(\alpha T+\beta A)_A &= \alpha V(T)_A + \beta\|A\|^2 \\ W(\alpha T+\beta A)_A &\neq \alpha W(T)_A + \beta\|A\|^2 . \end{aligned} \tag{4.4}$$

The following proposition gives a weaker result than the general case, i.e., $V(T)_A \subseteq \overline{co}W(T)_A$.

Proposition 4.7. $0 \in V(T)_A \Rightarrow 0 \in \overline{co}W(T)_A$.

Proof. Let us suppose that $0 \in V(T)_A$ and $0 \notin \overline{co}W(T)_A$. By rotation, we can suppose that $\overline{co}W(T)_A$ is contained in the right half-plane, and therefore there is a line which separates 0 from $\overline{co}W(T)_A$. So $Re\,\overline{co}W(T)_A \geq 0$ and $[Tx, Ax]^- > 0$, for all $x \in S_E$. Applying the properties of the function ψ and $[Tx, Ax]^{-,+}$ defined by the equations (3.14) and (3.15) respectively, it follows that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\psi_{Ax,-Tx}(t) < \frac{[-Tx, Ax]^+}{\|Ax\|} + \varepsilon \text{ for } 0 < t < \delta.$$

Hence

$$\frac{\|Ax - tTx\| - \|Ax\|}{t} < \frac{-[Tx, Ax]^-}{\|Ax\|} + \varepsilon.$$

Thus,

$$\frac{\|Ax\| - \|Ax - tTx\|}{t} > \frac{[Tx, Ax]^-}{\|Ax\|} - \varepsilon \geq \frac{[Tx, Ax]^-}{\|A\|} - \varepsilon.$$

Choose a real number ε_0 satisfying $\frac{[Tx,Ax]^-}{\|A\|} - \varepsilon > 0$, and then there is a δ_{ε_0} such that $\|Ax - \mu Tx\| < \|Ax\|$, for all $0 < \mu < \delta_{\varepsilon_0}$ and all $x \in S_E$.

Hence, $\|A - \mu T\| < \|A\|$; contraction by Corollary 4.6. □

As an application of the Corollary 4.6, we obtain a better result than the previous one as follows.

Proposition 4.8. *If E is a smooth normed linear space and $A, T \in \mathcal{B}(E)$ then $V(T)_A = \overline{co}\mathcal{M}(T)_A$.*

Proof. Evidently,

$$\begin{aligned} \overline{co}\mathcal{M}(\alpha T + \beta A)_A &= \alpha\overline{co}\mathcal{M}(T)_A + \beta\|A\|^2 \\ V(\alpha T + \beta A)_A &= \alpha V(T)_A + \beta\|A\|^2, \text{ for all } \alpha, \beta \in \mathbb{K}\cdot \end{aligned}$$

Then, by Proposition 3.8, it suffices to prove that

$$0 \in V(T)_A \Rightarrow 0 \in \overline{co}\mathcal{M}(T)_A.$$

Suppose that $0 \in V(T)_A$ and $0 \notin \overline{co}\mathcal{M}(T)_A$. By rotation, we may suppose that $\overline{co}\mathcal{M}(T)_A$ is contained in the right half-plane, and therefore there is a line which separates 0 from $\overline{co}\mathcal{M}(T)_A$. So there is $\tau > 0$ such that $Re\,\overline{co}\mathcal{M}(T)_A \geq \tau$.

Since E is a smooth space, then $[.,.]^- = [.,.]^+ = [.,.]$ and $[.,.]$ is a s.i.p.

Let

$$K = \left\{x : x \in S_E \;;\; [Tx, Ax] \leq \frac{\tau}{2}\right\}, \quad \eta = \sup_{x \in K}\|Ax\|$$

Then, for all $x \in K$ and $0 < \mu \leq \frac{\|A\|-\eta}{2\|T\|}$, we get

$$\|(A+\mu T)\, x\| \leq \|Ax\| + \mu \|Tx\| < \eta + \mu \|T\| \leq \|A\| .$$

If $x \in S_E \backslash K$ then, $[Tx, Ax] > \frac{\tau}{2}$. The function $\psi_{_{Ax,Tx}}$ defined by equation (3.14) is increasing and continuous on $\mathbb{R} \backslash \{0\}$ with

$$\lim_{t \to 0^-} \psi_{_{Ax,Tx}}(t) = \frac{[Tx, Ax]^-}{\|Ax\|} = \frac{[Tx, Ax]}{\|Ax\|}.$$

Set $\mu = -t$, we get

$$\lim_{t \to 0^-} \frac{\|Ax + tTx\| - \|Ax\|}{t} = \lim_{\mu \to 0^+} \frac{\|Ax\| - \|Ax - \mu Tx\|}{\mu} = \frac{[Tx, Ax]}{\|Ax\|} > \frac{\tau}{2}.$$

Hence for any $\varepsilon > 0$ there is $\delta > 0$ such that, for $0 < \mu < \delta$,

$$\frac{\|Ax\| - \|Ax - \mu Tx\|}{\mu} > \frac{[Tx, Ax]}{\|Ax\|} - \varepsilon > \frac{\tau}{2\|A\|} - \varepsilon$$

Choose $\varepsilon = \frac{\tau}{2\|A\|}$ then there is δ_τ such that, for $0 < \mu < \delta_\tau$,

$$\|Ax - \mu Tx\| < \|Ax\| \leq \|A\| , \text{ for all } \ x \in S_E \backslash K.$$

Hence for $0 < \mu \leq \min\left\{ \frac{\|A\|-\eta}{2\|T\|}, \frac{\delta_\tau}{2} \right\}$, we have $\|Ax - \mu Tx\| < \|Ax\| \leq \|A\|$, for all $x \in S_E$. Finally, there is μ such that $\|A - \mu T\| < \|A\|$, this implies that A is not orthogonal to T with respect to $\perp_{_{B-J}}$ and by Corollary 4.6, we get $0 \notin V(T)_A$. This is a contraction. $\square$

The following corollary gives a characterization of Birkhoff-James's orthogonality in $\mathcal{B}(H)$ where H is a Hilbert space equipped with an inner product $(.,.)$ over the field $\mathbb{K}$.

Corollary 4.9. *If A, T are operators in $\mathcal{B}(H)$ then the following conditions are equivalents.*

(i) $A \perp_{B-J} T$;

(ii) $\exists\, (x_n)_n \subseteq S_H$, $\|Ax_n\| \to \|A\|$ and $(Tx_n, Ax_n) \to 0$;

(iii) $\left[e^{i\theta}T, A\right]^- \leq 0 \leq \left[e^{i\theta}T, A\right]^+$, for all $\theta \in \mathbb{R}$.

Proof. The Hilbert generalized maximal numerical range $\mathcal{M}(T)_A$ is convex and closed, so it suffices to apply the Corollary 4.6 and Proposition 4.8. $\square$

5 Application

Let a, b be any two elements in an algebra $\mathcal{A}$. A 2-sided multiplication operator $a \bullet b$ on $\mathcal{A}$ is defined by the equation $(a \bullet b)(x) = axb$. An elementary operator E on $\mathcal{A}$ is an operator of the form $E=\sum\limits_{i=1}^{n} a_i \bullet b_i$ with fixed $a_i, b_i \in \mathcal{A}$ and finite integer n, the length l of E is defined to be the smallest number of multiplication terms required for any representation $\sum a_j \bullet b_j$ for E, the representation $E=0$ is defined to have length zero. In particular, we have for some elements a, b in $\mathcal{A}$ the following elementary operators in $\mathcal{B}(\mathcal{A})$: $L_a(x)=ax$, $R_b(x)=xb$, $\delta_{a,b}=L_a-R_b$, $M_{a,b}=L_aR_b$ the left multiplication operator, the right multiplication operator, the generalized derivation, and the elementary multiplication operator respectively.

Proposition 5.1. *Let A, B be operators in $\mathcal{B}(E)$, where E is a normed linear space, then*

$$\begin{aligned}
\|A\|\,\|M_{A,B}+M_{B,A}\| &\geq \sup_{\alpha\in\overline{co}\mathcal{M}(B)_A}\left\|\alpha A+\|A\|^2 B\right\| \\
\|B\|\,\|M_{A,B}+M_{B,A}\| &\geq \sup_{\alpha\in\overline{co}\mathcal{M}(A)_B}\left\|\alpha B+\|B\|^2 A\right\|.
\end{aligned}$$

Proof. If $A=0$ or $B=0$, the inequalities are obvious. Suppose that $A\neq 0$.

We have seen before that $\mathcal{M}(B)_A$, see the equations (3.10), is a non-empty set, so let $\alpha \in co\mathcal{M}(B)_A$, then $\alpha=\sum_{k=1}^{k=m}\zeta_k\alpha_k$ where$\sum_{k=1}^{k=m}\zeta_k=1$, $\alpha_k\in\mathcal{M}(B)_A$; $(1\leq k\leq m)$ and there is $\left(x_n^k\right)_n\in S_E(A)$ and $\varphi_n^k\in D\left(Ax_n^k\right)$ such that $\alpha_k=\lim_n\varphi_n^k\left(Bx_n^k\right)$. Define the operators $X_n^k\in\mathcal{B}(E)$ as follows, $X_n^k\left(x_n^k\right)=\varphi_n^k\left(x_n^k\right)y$, for any n, k and y any vector in S_E. Hence $\left\|X_n^k\right\|\leq\|A\|$; for all n, k.

$$\begin{aligned}
\left\|(M_{A,B}+M_{B,A})X_n^k\left(x_n^k\right)\right\| &= \left\|\left(AX_n^kB+BX_n^kA\right)x_n^k\right\| \\
&= \left\|AX_n^k\left(Bx_n^k\right)+BX_n^k\left(Ax_n^k\right)\right\| \\
&= \left\|A\varphi_n^k\left(Bx_n^k\right)y+B\varphi_n^k\left(Ax_n^k\right)y\right\|.
\end{aligned}$$

Then

$$\begin{aligned}
\|M_{A,B}+M_{B,A}\| &\geq \frac{\left\|(M_{A,B}+M_{B,A})\sum_{k=1}^{k=m}\zeta_kX_n^k\left(x_n^k\right)\right\|}{\left\|\sum_{k=1}^{k=m}\zeta_kX_n^k\right\|}, \text{ for all } n \\
&\geq \frac{1}{\|A\|}\left\|\left[\sum_{k=1}^{k=m}\zeta_k\left(\varphi_n^k\left(Bx_n^k\right)A+\left\|Ax_n^k\right\|^2B\right)\right]y\right\| \\
&= \frac{1}{\|A\|}\left\|\left[A\sum_{k=1}^{k=m}\zeta_k\varphi_n^k\left(Bx_n^k\right)+B\sum_{k=1}^{k=m}\zeta_k\left\|Ax_n^k\right\|^2\right]y\right\|.
\end{aligned}$$

Letting $n\to\infty$,

$$\|M_{A,B}+M_{B,A}\|\geq\frac{1}{\|A\|}\left\|\left(\alpha A+\|A\|^2B\right)y\right\|.$$

So for $\alpha\in co\mathcal{M}(B)_A$ and any $y\in S_E$, we get

$$\|M_{A,B}+M_{B,A}\|\geq\frac{1}{\|A\|}\left\|\left(\alpha A+\|A\|^2B\right)y\right\|.$$

Hence,

$$\|M_{A,B}+M_{B,A}\| \geq \frac{1}{\|A\|} \sup_{\alpha \in \overline{co}\mathcal{M}(B)_A} \left[\sup_{y \in S_E} \left\| \left(\alpha A + \|A\|^2 B \right) y \right\| \right].$$

Therefore,

$$\|M_{A,B}+M_{B,A}\| \geq \frac{1}{\|A\|} \sup_{\alpha \in \overline{co}\mathcal{M}(B)_A} \left\| \alpha A + \|A\|^2 B \right\|.$$

We proceed similarly as the case $\alpha \in co\mathcal{M}(A)_B$ with $B \neq 0$, we obtain

$$\|M_{A,B}+M_{B,A}\| \geq \frac{1}{\|B\|} \sup_{\alpha \in \overline{co}\mathcal{M}(A)_B} \left\| \alpha B + \|B\|^2 A \right\|.$$

$\square$

As an immediate consequence of the previous proposition, Proposition 4.8 and Corollary 4.6, we obtain

Corollary 5.2. *Let E be a normed linear space and A, B be operators in $\mathcal{B}(E)$, then*
i) $0 \in \overline{co}\mathcal{M}(B)_A \cup \overline{co}\mathcal{M}(A)_B \Rightarrow \|A\| \|B\| \leq \|M_{A,B}+M_{B,A}\| \leq 2\|A\| \|B\|.$
ii) If E is smooth space, then
$A \perp_{B-J} B$ *or* $B \perp_{B-J} A \Rightarrow \|A\| \|B\| \leq \|M_{A,B}+M_{B,A}\| \leq 2\|A\| \|B\|.$

Remark 5.3. The left estimate in the previous corollary is, in general, the best possible.

The following proposition gives a condition for which the equality holds in the previous corollary.

Proposition 5.4. *If $\|A\| \|B\| \in \overline{co}\mathcal{M}(B)_A \cap \overline{co}\mathcal{M}(A)_B$, then*

$$\|M_{A,B}+M_{B,A}\| = 2\|A\| \|B\|.$$

Proof. If $\|A\| \|B\| \in co\mathcal{M}(B)_A \cap co\mathcal{M}(A)_B$ then

$$\begin{aligned} \|A\| \|B\| &= \sum_{k=1}^{k=m} \zeta_k \alpha_k = \sum_{j=1}^{j=l} \mu_j \beta_j, \text{ where } \sum_{k=1}^{k=m} \zeta_k = \sum_{j=1}^{j=l} \mu_j = 1 \\ \alpha_k, \beta_j &\in \mathcal{M}(B)_A ; (1 \leq k \leq m, 1 \leq j \leq l). \end{aligned}$$

Also, there exist $\left(x_n^k\right)_n \in S_E(A)$, $\varphi_n^k \in D\left(Ax_n^k\right)$ and $\left(y_n^j\right)_n \in S_E(B)$, $\psi_n^j \in D\left(Ay_n^j\right)$ such that $\alpha_k = \lim_n \varphi_n^k\left(Bx_n^k\right)$, $\beta_j = \lim_n \psi_n^j\left(Ay_n^j\right)$. Define the operators $X_n^{k,j} \in \mathcal{B}(E)$ as follows, $X_n^{k,j}\left(x_n^k\right) = \varphi_n^k\left(x_n^k\right) y_n^j$, for any n, k, j.

Hence $\left\|X_n^{k,j}\right\| \leq \|A\|,$ for all $n,k,j.$ and

$$\begin{aligned}
\left\|\left(M_{A,B}+M_{B,A}\right)X_n^{k,j}\left(x_n^k\right)\right\| &= \left\|\left(AX_n^{k,j}B+BX_n^{k,j}A\right)x_n^k\right\| \\
&= \left\|AX_n^{k,j}\left(Bx_n^k\right)+BX_n^{k,j}\left(Ax_n^k\right)\right\| \\
&= \frac{\left\|\psi_n^j\right\|}{\left\|\psi_n^j\right\|}\left\|A\varphi_n^k\left(Bx_n^k\right)y_n^j+B\varphi_n^k\left(Ax_n^k\right)y_n^j\right\| \\
&\geq \frac{1}{\left\|\psi_n^j\right\|}\left\|\psi_n^j\left(A\varphi_n^k\left(Bx_n^k\right)y_n^j+B\varphi_n^k\left(Ax_n^k\right)y_n^j\right)\right\| \\
&= \frac{1}{\left\|\psi_n^j\right\|}\left\|\varphi_n^k\left(Bx_n^k\right)\psi_n^j\left(Ay_n^j\right)+\varphi_n^k\left(Ax_n^k\right)\psi_n^j\left(By_n^j\right)\right\| \\
&= \frac{1}{\left\|By_n^j\right\|}\left\|\varphi_n^k\left(Bx_n^k\right)\psi_n^j\left(Ay_n^j\right)+\left\|Ax_n^k\right\|^2\left\|By_n^j\right\|^2\right\|.
\end{aligned}$$

Also

$$\|M_{A,B}+M_{B,A}\| \geq \frac{\left\|\left(M_{A,B}+M_{B,A}\right)\left(\sum_{k=1}^{k=m}\sum_{j=1}^{j=l}\zeta_k\mu_j X_n^{k,j}\left(x_n^k\right)\right)\right\|}{\left\|\sum_{k=1}^{k=m}\sum_{j=1}^{j=l}\zeta_k\mu_j X_n^{k,j}\right\|}, \text{ for all } n,$$

and

$$\|M_{A,B}+M_{B,A}\| \geq \frac{\sum_{k=1}^{k=m}\zeta_k\varphi_n^k\left(Bx_n^k\right)\sum_{j=1}^{j=l}\mu_j\psi_n^j\left(Ay_n^j\right)+\sum_{k=1}^{k=m}\zeta_k\left\|Ax_n^k\right\|^2\sum_{j=1}^{j=l}\mu_j\left\|By_n^j\right\|^2}{\|A\|\|B\|}$$

Letting $n\to\infty$,

$$\begin{aligned}
\|M_{A,B}+M_{B,A}\| &\geq \frac{1}{\|A\|\,\|B\|}\left\|\sum_{k=1}^{k=m}\zeta_k\alpha_k\sum_{j=1}^{j=l}\mu_j\beta_j+\|A\|^2\,\|B\|^2\sum_{k=1}^{k=m}\zeta_k\sum_{j=1}^{j=l}\mu_j\right\| \\
&= 2\,\|A\|\,\|B\|.
\end{aligned}$$

Since $\|M_{A,B}+M_{B,A}\| \leq 2\,\|A\|\,\|B\|.$ Therefore, $\|M_{A,B}+M_{B,A}\| = 2\,\|A\|\,\|B\|.$ □

Corollary 5.5. *Let H be a Hilbert space and $A,B\in\mathcal{B}(H)$.*

1. *If $A\perp_{B-J}B$ or $B\perp_{B-J}A$, then $\|A\|\,\|B\| \leq \|M_{A,B}+M_{B,A}\| \leq 2\,\|A\|\,\|B\|.$*
2. *If there exist two sequences $(x_n)_n$, $(y_n)_n \subseteq S_H$ with $\|Ax_n\|\to\|A\|$, $\|By_n\|\to\|B\|$ and $\lim(Bx_n,Ax_n)=\lim(Ay_n,By_n)=\|A\|\,\|B\|$, then*

$$\|M_{A,B}+M_{B,A}\| = 2\,\|A\|\,\|B\|.$$

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