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Research Article

Synthesis of Adaptive Gain Robust Controllers for Polytopic Uncertain Systems

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We present a new adaptive gain robust controller for polytopic uncertain systems. The proposed adaptive gain robust controller consists of a state feedback law with a fixed gain and a compensation input with adaptive gains which are tuned by updating laws. In this paper, we show that sufficient conditions for the existence of the proposed adaptive gain robust controller are given in terms of LMIs. Finally, illustrative examples are presented to show the effectiveness of the proposed adaptive gain robust controller.

1. Introduction

In order to design control systems, it is necessary to derive a mathematical model of the controlled system. It is well known that if the mathematical model describes the controlled system completely, one can design a satisfactory control system by using some controller design methods. However, there are some gaps between the controlled system and its mathematical model and the gaps are referred to as uncertainties. The uncertainties included in the mathematical model may occur deterioration of control performance and, at the worst, control systems become unstable. Therefore, robust stability analysis and robust stabilization for uncertain dynamical systems have received much attention for a long time (e.g., see [1] and references therein). In particular, there are lots of existing results for state feedback robust control such as quadratic stabilizing controllers, $\mathscr{H}^{\infty}$ control systems (e.g., see [2–4]), and so on. In addition for a class of uncertain linear systems a connection between $\mathscr{H}^{\infty}$ control and quadratic stabilization has been established [5].

By the way in most practical situations, it is desirable to design robust control systems which achieve not only robust stability but also an adequate level of performance. Thus, robust controllers with achievable performance level have also been well studied. For instance, guaranteed cost control is introduced by Chang and Peng [6], and Riccati equation approach [7] and LMI one [8] have been suggested. Additionally, several researches have dealt with the problem of designing a robust controller that satisfies additional constraints on the closed-loop poles' location (e.g., see [9, 10]). Besides, some design methods of variable gain controllers for uncertain linear systems have also been shown (e.g., see [11–13]). Maki and Hagino [11] have presented a robust controller with adaptation mechanism for linear systems with time-varying parameter uncertainties. In their work, the control gain is tuned online based on the information about parameter uncertainties and a target model with adjustable parameters is also introduced. In the work of Oya and Hagino [12] an error signal between the desired trajectory and the actual response is defined and an adaptive compensation input is determined so as to reduce the effect of uncertainties. These robust controllers consist of a fixed gain controller and a variable gain one, and the variable gain controller is adjusted by updating rules.

In this paper, we consider a design problem of an adaptive gain robust controller for polytopic uncertain systems. The proposed adaptive gain robust controller is composed of a state feedback law with a fixed gain and a compensation input.

The compensation input is defined as a state feedback with a fixed gain and one with adaptive gains tuned by updating rules. The advantage of our new adaptive gain robust control is that, for the case that conventional quadratic stabilizing controller based on Lyapunov criterion cannot be obtained, the proposed design method may be able to design stabilizing controller. This paper is organized as follows. In Section 2, we show notations which are used in this paper, and Section 3 contains our main result. We show that sufficient conditions for the existence of the proposed robust controller are given in terms of LMIs. Finally, simple numerical examples are included to show the effectiveness of the proposed robust controller design approach.

2. Notations

In this section, we introduce notations which are used in this paper as well as the existing works (e.g., see [13]).

In the paper, the following notations are used. For a matrix $\mathscr{A}$, the transpose of the matrix $\mathscr{A}$ and its inverse are denoted by $\mathscr{A}^T$ and $\mathscr{A}^{-1}$, respectively. Additionally $H_e\{\mathscr{A}\}$ and I_n mean $\mathscr{A}+\mathscr{A}^T$ and n-dimensional identity matrix, respectively, and the notation $\mathrm{diag}(\mathscr{A}_1,\dots,\mathscr{A}_{\mathscr{M}})$ represents a block diagonal matrix composed of matrices $\mathscr{A}_i$ for $i = 1,\dots,\mathscr{M}$. For real symmetric matrices $\mathscr{A}$ and $\mathscr{B}$, $\mathscr{A} > \mathscr{B}$ (resp., $\mathscr{A} \geq \mathscr{B}$) means that $\mathscr{A} - \mathscr{B}$ is positive (resp., nonnegative) definite matrix. For a vector $\alpha \in \mathbb{R}^n$, $\|\alpha\|$ denotes the standard Euclidian norm and, for a matrix $\mathscr{A}$, $\|\mathscr{A}\|$ represents its induced norm. The symbols "$\triangleq$" and "$\star$" mean equality by definition and symmetric blocks or symmetric elements in matrices, respectively.

3. Problem Formulation

Consider the following uncertain linear system:

$$\frac{d}{dt}x(t) = \left(A + \sum_{k=1}^{\mathscr{N}}\theta_k\mathscr{D}_k\right)x(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the vectors of state (assumed to be available for feedback) and the control input, respectively. In (1), the matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the nominal values of system parameters, and the matrix $\mathscr{D}_k \in \mathbb{R}^{n\times n}$ $(k = 1,\dots,\mathscr{N})$ denotes the structure of uncertainties. Besides, $\theta_k \in \mathbb{R}^1$ represents unknown parameters which satisfy the relation

$$\sum_{k=1}^{\mathscr{N}}\theta_k = 1, \quad \theta_k \geq 0. \tag{2}$$

Namely, the uncertain system of (1) is a class of polytopic uncertain systems. Additionally, we assume that the pairs (A, B) and $(A + \mathscr{D}_k, B)$ are stabilizable.

The nominal system, ignoring the unknown parameters in (1), is given by

$$\frac{d}{dt}\overline{x}(t) = A\overline{x}(t) + B\overline{u}(t). \tag{3}$$

In this paper, first of all, we consider the standard linear quadratic control problem for the nominal system of (3). Namely, we define the following quadratic cost function for the nominal system of (3):

$$\mathscr{J} = \int_0^\infty \left(\overline{x}^T(t)\mathscr{Q}\overline{x}(t) + \overline{u}^T(t)\mathscr{R}\overline{u}(t)\right)dt, \tag{4}$$

where the matrices $\mathscr{Q} \in \mathbb{R}^{n\times n}$ and $\mathscr{R} \in \mathbb{R}^{m\times m}$ which are selected by designers are semipositive and positive definite. It is well-known that the optimal control input minimizing the quadratic cost function of (4) is given by $\overline{u}(t) = K\overline{x}(t)$, where $K \in \mathbb{R}^{m\times n}$ represents the optimal control gain matrix for the nominal system of (3). Note that the closed-loop system matrix $A_K \triangleq A+BK$ is stable (the design parameter $\mathscr{Q} \in \mathbb{R}^{n\times n}$ is selected such that the pair $(A, \mathscr{H})$ is detectable, where $\mathscr{H}$ is any matrix satisfying $\mathscr{Q} = \mathscr{H}\mathscr{H}^T$, and then the closed-loop system matrix A_K is stable) and the optimal feedback gain matrix $K \in \mathbb{R}^{m\times n}$ can be computed as $K = -\mathscr{R}^{-1}B^T\mathscr{P}$, where $\mathscr{P} \in \mathbb{R}^{n\times n}$ is the unique solution of the algebraic Riccati equation:

$$H_e\left\{A^T\mathscr{P}\right\} - \mathscr{P}B\mathscr{R}^{-1}B^T\mathscr{P} + \mathscr{Q} = 0. \tag{5}$$

Now, by using the optimal feedback gain matrix $K \in \mathbb{R}^{m\times n}$ for the nominal system of (3), we consider the following control input:

$$u(t) \triangleq Kx(t) + \xi\left(x, \widehat{\theta}_k, t\right), \tag{6}$$

where $\xi(x, \widehat{\theta}_k, t) \in \mathbb{R}^m$ is a compensation input so as to compensate the effect of unknown parameters [12]. The compensation input $\xi(x, \widehat{\theta}_k, t)$ is defined as

$$\xi\left(x, \widehat{\theta}_k, t\right) \triangleq \mathscr{K}x(t) + \sum_{k=1}^{\mathscr{N}}\widehat{\theta}_k(t)K_kx(t), \tag{7}$$

where $\mathscr{K} \in \mathbb{R}^{m\times n}$ and $K_k \in \mathbb{R}^{m\times n}$ $(k = 1,\dots,\mathscr{N})$ are fixed gain matrices and $\widehat{\theta}_k(t)$ $(k = 1,\dots,\mathscr{N})$ are time-varying adjustable parameters. The decision method of fixed gain matrices $\mathscr{K}$ and K_k and adjustable parameters $\widehat{\theta}_k(t)$ are shown in Section 4. From (1), (6), and (7), we have the uncertain closed-loop system described as

$$\begin{aligned}\frac{d}{dt}x(t) &= \left(A + \sum_{k=1}^{\mathscr{N}}\theta_k\mathscr{D}_k\right)x(t) \\ &\quad + B\left(Kx(t) + \mathscr{K}x(t) + \sum_{k=1}^{\mathscr{N}}\widehat{\theta}_kK_k(t)x(t)\right) \\ &= \left(A_K + B\mathscr{K} + \sum_{k=1}^{\mathscr{N}}\theta_k\mathscr{D}_k\right)x(t) \\ &\quad + \sum_{k=1}^{\mathscr{N}}\widehat{\theta}_k(t)BK_kx(t).\end{aligned} \tag{8}$$

By introducing the complementary variables $\widetilde{\theta}_k(t) \triangleq \widehat{\theta}_k(t) - \theta_k$ $(k = 1, \cdots, \mathscr{N})$, the uncertain closed-loop system of (8) can be rewritten as

$$\begin{aligned} \frac{d}{dt}x(t) &= \sum_{k=1}^{\mathscr{N}} \theta_k \left(A_K + \mathscr{D}_k + B\mathscr{K}\right) x(t) \\ &\quad + \sum_{k=1}^{\mathscr{N}} \left(\widetilde{\theta}_k(t) + \theta_k\right) BK_k x(t) \\ &= \sum_{k=1}^{\mathscr{N}} \theta_k \left(A_K + \mathscr{D}_k + BK_k + B\mathscr{K}\right) x(t) \\ &\quad + \sum_{k=1}^{\mathscr{N}} \widetilde{\theta}_k(t) BK_k x(t). \end{aligned} \tag{9}$$

Here we have used the relation of (2). In addition, defining the time-varying matrix

$$\Gamma_K(x,t) \triangleq \left(BK_1 x(t), BK_2 x(t), \ldots, BK_{\mathscr{N}} x(t)\right), \tag{10}$$

we obtain the following uncertain closed-loop system:

$$\frac{d}{dt}x(t) = \sum_{k=1}^{\mathscr{N}} \theta_k \left(A_{\mathscr{D}_k} + B\mathscr{K}\right) x(t) + \Gamma_K(x,t)\widetilde{\theta}(t), \tag{11}$$

where $A_{\mathscr{D}_k}$ and $\widetilde{\theta}(t)$ are given by

$$\begin{aligned} A_{\mathscr{D}_k} &= A_K + \mathscr{D}_k + BK_k, \\ \widetilde{\theta}(t) &= \left(\widetilde{\theta}_1(t), \ldots, \widetilde{\theta}_{\mathscr{N}}(t)\right)^T. \end{aligned} \tag{12}$$

From the above, our control objective in this paper is to design the compensation input of (7) such that the uncertain closed-loop system of (11) is asymptotically stable, that is, deriving the fixed gain matrices $\mathscr{K} \in \mathbb{R}^{m\times n}$ and $K_k \in \mathbb{R}^{m\times n}$ and updating rules of adjustable parameters $\widehat{\theta}_k(t)$ which stabilize the uncertain closed-loop system of (11).

4. Main Results

In this section, we show a design method of the proposed adaptive gain robust controller such that the uncertain closed-loop system of (11) is asymptotically stable. The following theorem shows a sufficient condition for the existence of the proposed robust control system.

Theorem 1. *Consider the uncertain linear system of (1) and the control input of (6).*

Firstly, in order to design the fixed gain matrices $K_k \in \mathbb{R}^{m\times n}$ we solve the following LMIs for the symmetric positive definite matrix $\mathscr{X}_k \in \mathbb{R}^{n\times n}$ and the matrix $\mathscr{Y}_k \in \mathbb{R}^{m\times n}$:

$$H_e\left\{\left(A_K + \mathscr{D}_k\right)\mathscr{X}_k + B\mathscr{Y}_k\right\} + \mathscr{Q}_k < 0, \tag{13}$$

where $\mathscr{Q}_k \in \mathbb{R}^{n\times n}$ $(k = 1, \ldots, \mathscr{N})$ are symmetric positive definite matrices selected by designers. If there exist the symmetric positive definite matrix $\mathscr{X}_k \in \mathbb{R}^{n\times n}$ and the matrix $\mathscr{Y}_k \in \mathbb{R}^{m\times n}$ satisfying the LMIs of (13), the fixed gain matrices $K_k \in \mathbb{R}^{m\times n}$ are determined as $K_k = \mathscr{Y}_k\mathscr{X}_k^{-1}$.

Next, the fixed gain matrix $\mathscr{K} \in \mathbb{R}^{m\times n}$ and the updating rule of adjustable parameters $\widehat{\theta}_k(t) \in \mathbb{R}^{\mathscr{N}}$ are designed as

$$\begin{aligned} \mathscr{K} &= \mathscr{W}\mathscr{S}^{-1}, \\ \frac{d}{dt}\widehat{\theta}(t) &= -\Sigma_\theta^{-1}\Gamma_K^T(x,t)\mathscr{S}^{-1}x(t) \end{aligned} \tag{14}$$

provided that the following LMIs for the symmetric positive definite matrix $\mathscr{S} \in \mathbb{R}^{n\times n}$ and the matrix $\mathscr{W} \in \mathbb{R}^{m\times n}$ are feasible:

$$H_e\left\{A_{\mathscr{D}_k}\mathscr{S} + B\mathscr{W}\right\} < 0 \quad (k = 1, \ldots, \mathscr{N}), \tag{15}$$

where $\Sigma_\theta \in \mathbb{R}^{\mathscr{N}\times\mathscr{N}}$ in (14) is the symmetric positive definite matrix selected by designers. Then asymptotical stability of the uncertain closed-loop system of (11) is guaranteed.

Proof. First of all, for the design parameters $\mathscr{Q}_k \in \mathbb{R}^{n\times n}$ if the LMIs of (13) are feasible, then, by solving the LMIs of (13), the fixed gain matrices $K_k \in \mathbb{R}^{m\times n}$ are derived as $K_k = \mathscr{Y}_k\mathscr{X}_k^{-1}$. Note that then the matrices $A_{\mathscr{D}_k} + B\mathscr{K}$ $(k = 1, \ldots, \mathscr{N})$ are stable.

Next, we introduce the following quadratic function as a Lyapunov function candidate:

$$\mathscr{V}(x,t) \triangleq x^T(t)\mathscr{X}x(t) + \widetilde{\theta}^T(t)\Sigma_\theta\widetilde{\theta}(t), \tag{16}$$

where $\mathscr{X} \in \mathbb{R}^{n\times n}$ and $\Sigma_\theta \in \mathbb{R}^{\mathscr{N}\times\mathscr{N}}$ are symmetric positive definite matrices. Note that the matrix $\mathscr{X} \in \mathbb{R}^{n\times n}$ satisfies $\mathscr{X} = \mathscr{S}^{-1}$. Namely, the quadratic function $\mathscr{V}(x,t)$ is positive definite and radially unbounded. The time derivative of the quadratic function $\mathscr{V}(x,t)$ along the trajectory of the uncertain closed-loop system of (11) can be expressed as:

$$\begin{aligned} \frac{d}{dt}\mathscr{V}(x,t) &= x^T(t)\left[H_e\left\{\sum_{k=1}^{\mathscr{N}}\theta_k\mathscr{X}\left(A_{\mathscr{D}_k} + B\mathscr{K}\right)\right\}\right]x(t) \\ &\quad + H_e\left\{x^T(t)\mathscr{X}\Gamma_K(x,t)\widetilde{\theta}(t)\right\} \\ &\quad + \left(\frac{d}{dt}\widetilde{\theta}(t)\right)^T\Sigma_\theta\widetilde{\theta}(t) + \widetilde{\theta}^T(t)\Sigma_\theta\left(\frac{d}{dt}\widetilde{\theta}(t)\right). \end{aligned} \tag{17}$$

Thus from the updating rule of (14) and the relation of (17) we have

$$\frac{d}{dt}\mathscr{V}(x,t) = x^T(t)\left[H_e\left\{\sum_{k=1}^{\mathscr{N}}\theta_k\mathscr{X}\left(A_{\mathscr{D}_k} + B\mathscr{K}\right)\right\}\right]x(t). \tag{18}$$

Namely, if the matrix inequality

$$H_e\left\{\sum_{k=1}^{\mathscr{N}}\theta_k\mathscr{X}\left(A_{\mathscr{D}_k} + B\mathscr{K}\right)\right\} < 0 \tag{19}$$

is satisfied, then the following relation for $\forall x(t) \neq 0$ holds:

$$\frac{d}{dt}\mathscr{V}(x,t) < 0; \tag{20}$$

that is, the uncertain closed-loop system of (11) is asymptotically stable.

By introducing the matrix $\mathscr{W} = \mathscr{K}\mathscr{S}$ one can easily see that the condition of (19) is equivalent to

$$H_e\left\{\sum_{k=1}^{\mathscr{N}}\theta_k\left(A_{\mathscr{D}_k}\mathscr{S} + B\mathscr{W}\right)\right\} < 0. \tag{21}$$

Since the unknown parameters θ_k satisfy the relation of (2), we find that the inequality condition of (21) is equivalent to the LMIs of (15).

Thus the proof of Theorem 1 is accomplished. □

Remark 2. It is well known that if the LMIs of (A.1) are not feasible then the conventional quadratic stabilizing controller cannot be designed. Furthermore, the conventional design method does not have design parameters. On the other hand, in this paper, by introducing the adjustable parameter, the control input is constructed as (6) and (7). One can see from stabilizability of the pair $(A + \mathscr{D}_k, B)$ that the LMIs of (13) are feasible. Namely, the fixed gain matrices $K_k \in \mathbb{R}^{m\times n}$ can always be derived and the matrix $A_{\mathscr{D}_k} = A_K + \mathscr{D}_k + BK_k$ in (12) is stable, because the fixed gain matrices K_k are derived by using the solution of LMIs of (13). Besides, LMIs of (15) may be feasible provided that the LMIs of (A.1) are not feasible (see Section 5). Therefore, for the case that the conventional quadratic stabilizing controller cannot be derived, the proposed design approach has possibility of robust stabilizing controller design. Note that the analysis of conservativeness for the proposed robust controller is one of our future research subjects.

Remark 3. In this paper, by introducing adjustable parameters $\widehat{\theta}_k(t)$ $(k = 1,\ldots,\mathscr{N})$, we have proposed an adaptive gain robust controller. The adaptive action of the time-varying parameters $\widehat{\theta}_k(t)$ can be adjusted by selecting the design parameter $\Sigma_\theta \in \mathbb{R}^{\mathscr{N}\times\mathscr{N}}$ in (14). In order to construct an adaptive robust control system, the adjustable parameters $\widehat{\theta}_k(t)$ $(k = 1,\ldots,\mathscr{N})$ are introduced and the time-varying adjustable parameter $\widehat{\theta}_k(t)$ differs from an estimate for the unknown parameter. Namely, we introduce the adjustable parameter $\widehat{\theta}_k(t)$ so as to reduce the effect for the uncertainties adaptively and determining $\widehat{\theta}(t)$ in (14) is not to estimate the unknown parameter θ_k [14]. Besides, the value of the adjustable parameter $\widehat{\theta}_k(t)$ depends on its initial value and the design parameter $\Sigma_\theta \in \mathbb{R}^{\mathscr{N}\times\mathscr{N}}$ in (14) and the adjustable parameters $\widehat{\theta}_k(t)$ do not satisfy the relation of (2) usually (see Example 1 in Section 5).

Remark 4. In this paper, we consider the polytopic uncertain system of (1) which has uncertainties in the state matrix only. The proposed design method of the adaptive gain robust controller can also be applied to the case that the uncertainties are included in both the system matrix and the input one. By introducing additional actuator dynamics and constituting an augmented system, the uncertainties in the input matrix are embedded in the system matrix of the augmented system [15]. Therefore the same design procedure can be applied.

5. Numerical Examples

In order to demonstrate the efficiency of the proposed control scheme, we have run two simple numerical examples. The control problems considered here are not necessarily practical. However, the simulation results stated below illustrate the distinct feature of the proposed controller design.

Example 1. Consider the uncertain linear system

$$\frac{d}{dt}x(t) = \begin{pmatrix} -2.0+\delta_1 & 1.0 \\ 0.0 & 1.0+\delta_2 \end{pmatrix} x(t) + \begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix} u(t), \tag{22}$$

where δ_1 and δ_2 are unknown parameters and these parameters are supposed to vary within the intervals $[-1.5, 1.5]$ and $[-1.0, 1.0]$, respectively. Namely, one can see that the matrices $\mathscr{D}_k$ $(k = 1,\ldots,4)$ in (1) can be described as

$$\begin{aligned} \mathscr{D}_1 &= \begin{pmatrix} -1.5 & 0.0 \\ \star & -1.0 \end{pmatrix}, \qquad \mathscr{D}_2 = \begin{pmatrix} 1.5 & 0.0 \\ \star & -1.0 \end{pmatrix}, \\ \mathscr{D}_3 &= \begin{pmatrix} -1.5 & 0.0 \\ \star & 1.0 \end{pmatrix}, \qquad \mathscr{D}_4 = \begin{pmatrix} 1.5 & 0.0 \\ \star & 1.0 \end{pmatrix}. \end{aligned} \tag{23}$$

Now we select the weighting matrices $\mathscr{Q} \in \mathbb{R}^{2\times 2}$ and $\mathscr{R} \in \mathbb{R}^{1\times 1}$ such as $\mathscr{Q} = 1.0 \times I_2$ and $\mathscr{R} = 3.0$ in (5). Then, solving the algebraic Riccati equation of (5), we obtain

$$\begin{aligned} K &= \begin{pmatrix} -1.8142\times 10^{-1} & -3.0887 \end{pmatrix} \times 10^{-1}, \\ \mathscr{P} &= \begin{pmatrix} 7.4177\times 10^{-1} & 1.8142\times 10^{-1} \\ \star & 3.0887 \end{pmatrix}. \end{aligned} \tag{24}$$

Next, we solve LMIs of (13). The design parameters $\mathscr{Q}_k \in \mathbb{R}^{2\times 2}$ in (13) are set at $\mathscr{Q}_1 = 1.0\times I_2$, $\mathscr{Q}_2 = 3.0\times I_2$, $\mathscr{Q}_3 = 1.0\times 10^1\times I_2$, and $\mathscr{Q}_4 = 7.0\times I_2$ and then we can obtain symmetric positive definite matrices $\mathscr{X}_k \in \mathbb{R}^{2\times 2}$ such as

$$\mathscr{X}_1 = \begin{pmatrix} 1.8171\times 10^{3} & 1.3633\times 10^{2} \\ \star & 2.7714\times 10^{3} \end{pmatrix},$$

$$\mathscr{X}_2 = \begin{pmatrix} 2.6483\times 10^{3} & 8.2403\times 10^{-1} \\ \star & 2.6475\times 10^{3} \end{pmatrix},$$

$$\mathcal{X}_3 = \begin{pmatrix} 1.8185 \times 10^3 & 1.3610 \times 10^2 \\ \star & 2.7712 \times 10^3 \end{pmatrix},$$

$$\mathcal{X}_4 = \begin{pmatrix} 2.6497 \times 10^3 & -2.1580 \\ \star & 2.6475 \times 10^3 \end{pmatrix} \tag{25}$$

and matrices $\mathcal{Y}_k \in \mathbb{R}^{1\times 2}$ given by

$$\begin{aligned} \mathcal{Y}_1 &= (-1.5440 \quad 7.2608) \times 10^3, \\ \mathcal{Y}_2 &= (-2.1701 \quad 6.8521) \times 10^3, \\ \mathcal{Y}_3 &= (-1.8187 \quad 1.7131) \times 10^3, \\ \mathcal{Y}_4 &= (-2.1704 \quad 1.5548) \times 10^3. \end{aligned} \tag{26}$$

Thus we have the following gain matrices

$$\begin{aligned} K_1 &= (-1.0501 \quad 2.6715), \\ K_2 &= (-8.1862 \times 10^{-1} \quad 2.5879), \\ K_3 &= (-1.0502 \quad 6.6976 \times 10^{-1}), \\ K_4 &= (-8.1866 \quad 5.8659) \times 10^{-1}. \end{aligned} \tag{27}$$

Besides, solving the LMIs of (15), we obtain

$$\begin{aligned} \mathcal{S} &= \begin{pmatrix} 1.3942 & -1.0158 \\ \star & 4.3169 \end{pmatrix}, \\ \mathcal{W} &= (-2.9440 \quad -1.3793), \\ \mathcal{K} &= (-2.8294 \quad -9.8528 \times 10^{-1}). \end{aligned} \tag{28}$$

From the above, the proposed adaptive gain robust controller can be obtained.

Now, in this example, the initial value for the uncertain system of (22) and a design parameter for the updating law of (14) are selected as $x(0) = (1.0 \quad -2.0)^T$ and $\Sigma_\theta = 1.0 \times 10^1 \times I_4$, respectively, and the initial value of the adjustable parameter is set at $\widehat{\theta}(0) = (0.0 \quad 0.0 \quad 0.0 \quad 0.0)^T$. Besides, we consider the following two cases for the unknown parameters δ_k $(k = 1, 2)$:

(i) Case 1: $\delta_1 = -1.5$ and $\delta_2 = -1.0$;

(ii) Case 2: $\delta_1 = -1.5$ and $\delta_2 = 1.0$;

that is, the unknown parameters θ_k are given as follows.

(i) Case 1: $\theta_1 = 1.0, \theta_2 = \theta_3 = \theta_4 = 0.0$.

(ii) Case 2: $\theta_1 = \theta_2 = \theta_4 = 0.0, \theta_3 = 1.0$.

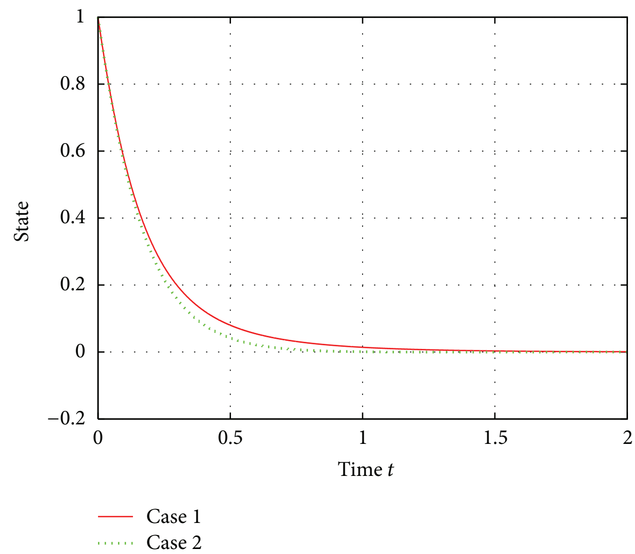

FIGURE 1: Time histories of the state $x_1(t)$.

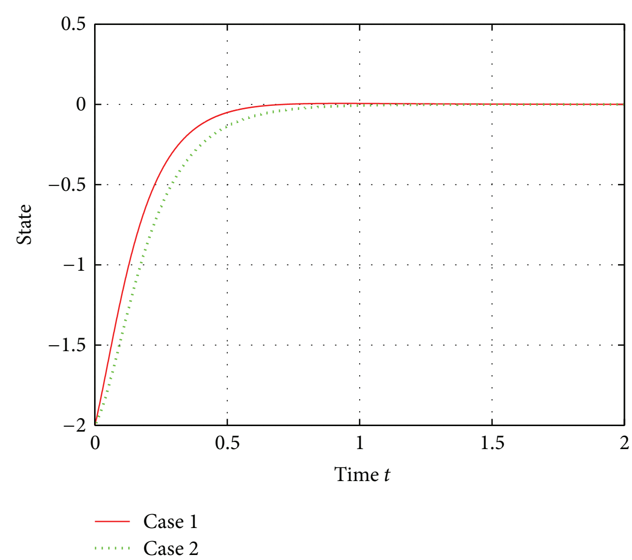

FIGURE 2: Time histories of the state $x_2(t)$.

The results of the simulation of this example are depicted in Figures 1–5. In Figures 1–3, "Case 1" and "Case 2" represent the time histories of the state variables $x_1(t)$ and $x_2(t)$ and the control input $u(t)$. Figures 4 and 5 show the time histories of the adjustable parameters $\widehat{\theta}_k(t)$ $(k = 1, \ldots, 4)$.

From Figures 1–5, we find that the proposed adaptive gain robust controller stabilizes the linear system of (22) in spite of plant uncertainties. In addition, one can see from Figures 4 and 5 that in this example the parameters $\widehat{\theta}_1(t)$ and $\widehat{\theta}_2(t)$ take negative value; that is, $\widehat{\theta}_k(t)$ does not satisfy the relation of (2) and the time-varying adjustable parameter $\widehat{\theta}_k(t)$ does not mean an estimate of unknown parameter $\theta_k \in \mathbb{R}^1$.

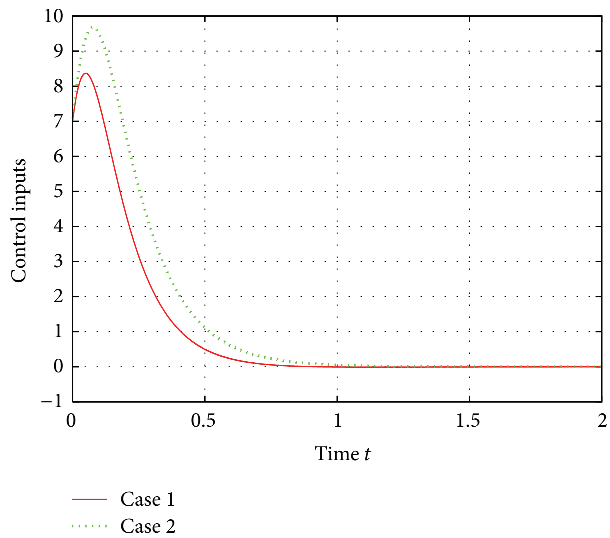

FIGURE 3: Time histories of the control input $u(t)$.

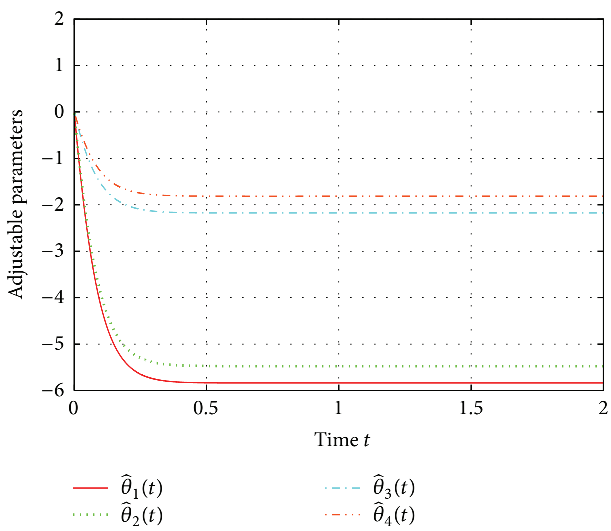

FIGURE 4: Time histories of adjustable parameters $\widehat{\theta}_k(t)$: Case 1.

Example 2. Consider the linear system with coefficient matrices of

$$A = \begin{pmatrix} 1.5 & 1.0 & 0.0 \\ 0.0 & 1.0 & 2.0 \\ -2.0 & -1.0 & -1.0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.0 \\ 0.0 \\ 1.0 \end{pmatrix},$$

$$\mathscr{D}_1 = \begin{pmatrix} 0.0 & -9.5 \times 10^{-1} & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{pmatrix},$$

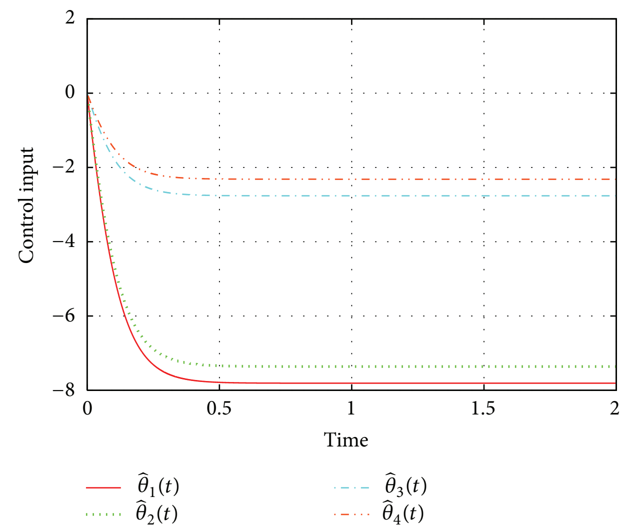

FIGURE 5: Time histories of adjustable parameters $\widehat{\theta}_k(t)$: Case 2.

$$\mathscr{D}_2 = \begin{pmatrix} 0.0 & 9.5 \times 10^{-1} & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{pmatrix},$$

$$\mathscr{D}_3 = \begin{pmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -1.5 \times 10^{-1} \\ 0.0 & 0.0 & 0.0 \end{pmatrix},$$

$$\mathscr{D}_4 = \begin{pmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.5 \times 10^{-1} \\ 0.0 & 0.0 & 0.0 \end{pmatrix}. \tag{29}$$

By selecting the design parameters in (13) such as $\mathscr{Q} = 3.0 \times I_3$, $\mathscr{R} = 4.0$, $\mathscr{Q}_1 = 3.0 \times I_3$, $\mathscr{Q}_2 = 9.0 \times I_3$, $\mathscr{Q}_3 = 1.0 \times 10^1 \times I_3$, and $\mathscr{Q}_4 = 8.0 \times I_3$, respectively, and solving the algebraic Riccati equation of (5), we have

$$K = \begin{pmatrix} -1.1174 \times 10^1 & -9.5394 & -5.3173 \end{pmatrix},$$
$$\mathscr{P} = \begin{pmatrix} 2.2507 \times 10^2 & 1.2893 \times 10^2 & 4.4696 \times 10^1 \\ \star & 8.9733 \times 10^1 & 3.8158 \times 10^1 \\ \star & \star & 2.1269 \times 10^1 \end{pmatrix}. \tag{30}$$

Additionally, the solution of LMIs of (13) and fixed gain matrices $K_k \in \mathbb{R}^{1\times 3}$ $(K = 1, \ldots, 4)$ can be computed as

$$\mathscr{X}_1 = \begin{pmatrix} 1.1711 \times 10^1 & -4.0285 \times 10^2 & 9.3258 \times 10^1 \\ \star & 1.8472 \times 10^4 & -1.1168 \times 10^4 \\ \star & \star & 2.5536 \times 10^4 \end{pmatrix},$$

$$\mathscr{Y}_1 = \left(-3.0832\times10^3 \quad -7.8885\times10^4 \quad 3.8937\times10^4\right),$$

$$\mathscr{X}_2 = \begin{pmatrix} 9.1439\times10^3 & -8.3256\times10^3 & 1.7793\times10^2 \\ \star & 1.1592\times10^4 & -7.5956\times10^3 \\ \star & \star & 2.5293\times10^4 \end{pmatrix},$$

$$\mathscr{Y}_2 = \left(4.8391 \quad -7.8488 \quad 7.6165\right)\times10^4,$$

$$\mathscr{X}_3 = \begin{pmatrix} 1.0718\times10^3 & -1.8219\times10^3 & 2.6448\times10^3 \\ \star & 3.7569\times10^3 & -8.8775\times10^3 \\ \star & \star & 4.7077\times10^4 \end{pmatrix},$$

$$\mathscr{Y}_3 = \left(1.6538\times10^4 \quad -5.5147\times10^4 \quad 2.3278\times10^5\right),$$

$$\mathscr{X}_4 = \begin{pmatrix} 4.2455\times10^3 & -7.737\times10^3 & -1.8843\times10^2 \\ \star & 2.1156\times10^4 & -7.4184\times10^3 \\ \star & \star & 1.9875\times10^4 \end{pmatrix},$$

$$\mathscr{Y}_4 = \left(1.9074 \quad 1.2019 \quad 3.9039\right)\times10^4, \tag{31}$$

$$K_1 = \left(-3.8863\times10^2 \quad -3.2965 \quad 1.5024\right),$$

$$K_2 = \left(-8.6679\times10^{-1} \quad -6.7430 \quad 9.9241\times10^{-1}\right),$$

$$K_3 = \left(-4.4573\times10^1 \quad -3.3716\times10^1 \quad 1.0909\right), \tag{32}$$

$$K_4 = \left(-7.9847 \quad -1.9438 \quad 1.1646\right).$$

Finally, by using these gain parameters of (32), we can obtain the following solution of the LMIs of (15) and a fixed gain matrix $\mathscr{K} \in \mathbb{R}^{1\times3}$ of (34):

$$\mathscr{S} = \begin{pmatrix} 5.4616\times10^{-4} & -1.6403\times10^{-2} & 4.7503\times10^{-3} \\ \star & 8.9260\times10^{-1} & -2.5601\times10^1 \\ \star & \star & 2.3132\times10^3 \end{pmatrix},$$

$$\mathscr{W} = \left(9.2471\times10^{-1} \quad 9.1270\times10^{-2} \quad 1.400110^{-5}\right). \tag{33}$$

Thus the following gain matrix can be computed:

$$\mathscr{K} = \left(-6.0417\times10^5 \quad -2.0255\times10^4 \quad -2.8346\times10^2\right). \tag{34}$$

On the other hand, the uncertain system with the coefficient matrices of (29), the conventional quadratic stabilizing controller based on Lyapunov criterion, cannot be designed; that is, the LMIs of (A.1) are not feasible. This result shows that, for the case such that the robust controller based on Lyapunov criterion cannot be obtained, the proposed controller design method has possibility of robust control system design. Therefore, the proposed design approach is very useful.

6. Conclusions

This paper has dealt with a design problem of adaptive gain robust controllers for polytopic uncertain systems. The proposed adaptive gain robust controller consists of fixed gain parameters and time-varying adjustable ones. In this paper, we show an LMI-based design algorithm of the proposed robust controller and simple numerical examples are given for illustration of the proposed controller design method. The simulation result has shown that the closed-loop system is well stabilized in spite of plant uncertainties for the case such that the robust stabilizing controller based on quadratic stabilization cannot be derived.

The proposed adaptive gain robust controller can easily be obtained by solving LMIs; that is, the proposed design method is very simple. Besides, one can see that, for the case that conventional quadratic stabilizing controller based on Lyapunov criterion cannot be obtained, the proposed controller design method has possibility of robust controller design. Therefore, the proposed design approach is very useful.

The future research subjects are extensions of the proposed adaptive gain robust controller to such a broad class of systems as uncertain time-delay systems, uncertain discrete-time systems, and so on.

Appendix

In this appendix, the conventional quadratic stabilizing controller via Lyapunov criterion is shown.

One can easily see that the following theorem shows the LMI-based design method of quadratic stabilizing controllers for the polytopic uncertain system of (1).

Theorem A.1. *Consider the polytopic uncertain system of (1) and the control input $u(t) = Kx(t)$.*

There exists the state feedback gain matrix K such that the control law $u(t) = Kx(t)$ is a quadratic stabilizing control, if there exist a symmetric positive definite matrix $\mathscr{X} \in \mathbb{R}^{n\times n}$ and a matrix $\mathscr{Y} \in \mathbb{R}^{m\times n}$ satisfying the following LMI conditions:

$$\left(A + \mathscr{D}_k\right)\mathscr{X} + B\mathscr{Y} < 0 \quad (k = 1, \dots, \mathscr{N}). \tag{A.1}$$

If the solution of the LMI of (A.1) exists, then the state feedback gain matrix $K \in \mathbb{R}^{m\times n}$ is obtained as $K = \mathscr{Y}\mathscr{X}^{-1}$.

Proof. See the studies by Boyd et al. [16] and Oya et al. [17] for details. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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