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PURE MATHEMATICS | RESEARCH ARTICLE

A study on the stability of a modified Degasperis–Procesi equation

Jing Chen^{1*}**Abstract:** A modified Degasperis–Procesi equation is investigated. The local existence and uniqueness of the strong solution for the equation are established in the Sobolev space $H^s(R)$ with $s > \frac{3}{2}$. The $L^1(R)$ local stability for the strong solution is obtained under certain assumptions on the initial data.**Subjects:** Applied Mathematics; Mathematics & Statistics; Non-Linear Systems; Science**Keywords:** modified Degasperis–Procesi equation; Sobolev space; $L^1(R)$ local stability**Mathematics subject classifications:** 35G25; 35L05**1. Introduction and main results**

The Degasperis–Procesi (DP) equation of the form

$$\phi_t - \phi_{txx} + 4\phi\phi_x = 3\phi_x\phi_{xx} + \phi\phi_{xxx}, \quad t > 0, x \in R, \quad (1)$$

which represents a model for shallow water dynamics, has been investigated by many scholars (see Coclite & Karlsen, 2006, 2007; Degasperis, Holm, & Hone, 2002; Escher, Liu, & Yin, 2006; Lai, Yan, Chen, & Wang, 2014; Lai & Wu, 2010; Lin & Liu, 2009; Lenells, 2005; Yin, 2003). Coclite and Karlsen (2006) established existence and L^1 stability results for entropy weak solutions of Equation (3) in the space $L^1 \cap BV$ and extended the results to a kind of generalized Degasperis–Procesi equations. Escher et al. (2006) discussed several qualitative properties of the Degasperis–Procesi equation. The existence and uniqueness of global weak solutions for Equation (1) have been established, provided that the initial data satisfy appropriate conditions in Escher (2006). Lenells (2005) dealt with the travelling wave solutions of Equation (1) and classified all weak travelling wave solutions of the Degasperis–Procesi equation. Recently, Lai et al. (2014) studied the generalized Degasperis–Procesi equation

$$\phi_t - \phi_{txx} + k\phi_x + m\phi\phi_x = 3\phi_x\phi_{xx} + \phi\phi_{xxx}, \quad (2)$$

where $k \geq 0$ and $m > 0$ are constants. Lai et al. (2014) derived the $L^2(R)$ conservation law and established the $L^1(R)$ stability of local strong solutions to Equation (2) by assuming that its initial

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Jing Chen was born in Sichuan, China, on July 1978. She received her MS degree in mathematics from Sichuan Normal University, in 2007. She has been working on the research of the existence and stability of solutions of partial differential equations and has published many articles in international journals. At present, she is teaching at Southwest University of Science and Technology.

PUBLIC INTEREST STATEMENT

In this paper, we have studied the generalized Degasperis–Procesi equation. Using the partial differential operator, the equivalent form Equation (6) has been derived. In the previous sections, we have used the Kato Theorem and double variables method to establish existence and stability of the solution for Equation (6). The approaches presented in this paper can be summarized to discuss other partial differential equations with initial value.

value belongs to the space $H^s(R)$ with $s > \frac{3}{2}$. For other approaches to study the DP equation and related partial differential equations, the reader is referred to Kato (1975), Kruzkov (1970), Rodriguez-Blanco (2001) and the references therein.

The objective of this work is to investigate the modified Degasperis–Procesi equation in the form

$$\phi_t - \phi_{txx} + \beta f'(\phi)\phi_x = f'''(\phi)\phi_x^3 + 3f''(\phi)\phi_x\phi_{xx} + f'(\phi)\phi_{xxx} + \alpha m[(m-1)\phi^{m-2}\phi_x^2 + \phi^{m-1}\phi_{xx}] - \alpha\phi^m, \tag{3}$$

where $\alpha, \beta \in R, m \geq 2$, function $f(\phi)$ is a polynomial of order $n(n \geq 2)$. Letting $\alpha = 0, \beta = 4, f(\phi) = \frac{\phi^2}{2}$, Equation (3) reduces to the Degasperis–Procesi Equation (1). Applying the operator $(1 - \partial_x^2)^{-1}$ to Equation (3), we obtain its equivalent form

$$\phi_t + f'(\phi)\phi_x + \alpha\phi^m + (\beta - 1)(1 - \partial_x^2)^{-1}\partial_x f(\phi) = 0, \tag{4}$$

where $(1 - \partial_x^2)^{-1}I(t, x) = \frac{1}{2} \int_R e^{-|x-y|} I(t, y) dy$. Assuming that the initial value $\phi(0, x)$ of Equation (4) belongs to $H^s(R)$ ($s > \frac{3}{2}$), we will prove the existence and uniqueness of local solution for Equation (4) using the Kato Theorem (see Kato, 1975) in the space $C([0, \infty); H^s(R)) \cap C^1([0, \infty); H^{s-1}(R))$. Furthermore, we will use the approaches presented in Kruzkov (1970) to establish the $L^1(R)$ local stability of the solution for the modified Degasperis–Procesi Equation (4). The results obtained in this paper extend parts of results presented in Lai et al. (2014).

We let $H^s(R)$ (where s is a real number) denote the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_R (1 + |\xi|^2)^s |\hat{h}(t, \xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty,$$

where $\hat{h}(t, \xi) = \int_R e^{-ix\xi} h(t, x) dx$. For $T > 0$ and $s \geq 0$, we let $C([0, \infty); H^s(R))$ denote the Fréchet space of all continuous H^s -valued functions on $[0, T)$. Set $\Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$. For simplicity, we let c denote any positive constants.

We consider the Cauchy problem of Equation (3)

$$\begin{cases} \phi_t - \phi_{txx} + \beta f'(\phi)\phi_x = f'''(\phi)\phi_x^3 + 3f''(\phi)\phi_x\phi_{xx} + f'(\phi)\phi_{xxx} \\ + \alpha m[(m-1)\phi^{m-2}\phi_x^2 + \phi^{m-1}\phi_{xx}] - \alpha\phi^m, \\ \phi(0, x) = \phi_0(x), \end{cases} \tag{5}$$

which is equivalent to the problem

$$\begin{cases} \phi_t + f'(\phi)\phi_x + \alpha\phi^m + (\beta - 1)\Lambda^{-2}\partial_x f(\phi) = 0, \\ \phi(0, x) = \phi_0(x). \end{cases} \tag{6}$$

Now we state the main results of our work.

THEOREM 1.1 Let $\phi_0(x) \in H^s(R)$ with $s > \frac{3}{2}$. There exists a $T > 0$ depending on $\|\phi_0\|_{H^s(R)}$ such that problem (5) or (6) has a unique solution $\phi(t, x) \in C([0, T); H^s(R)) \cap C^1([0, T); H^{s-1}(R))$.

THEOREM 1.2 Assume that $\phi(t, x)$ and $\psi(t, x)$ are two local strong solutions of problem (5) or (6) with initial data $\phi_0(x), \psi_0(x) \in L^1(R) \cap H^s(R)$ ($s > \frac{3}{2}$), respectively. Let $T > 0$ be the maximum existence time of $\phi(t, x)$ and $\psi(t, x)$. For any $t \in [0, T)$, it holds that

$$\|\phi(t, x) - \psi(t, x)\|_{L^1} \leq e^{ct} \int_{-\infty}^{+\infty} |\phi_0(x) - \psi_0(x)| dx,$$

where c is a positive constant depending on $\|\phi_0\|_{L^\infty(\mathbb{R})}$ and $\|\psi_0\|_{L^\infty(\mathbb{R})}$.

This paper is organized as follows. Section 2 gives the proof of Theorem 1.1. The proof of Theorem 1.2 is given in Section 3.

2. Proof of Theorem 1.1

Firstly, we introduce the abstract quasi-linear evolution equation

$$\begin{cases} \frac{du}{dt} + A(u)u = W(u), & t \geq 0, \\ u(0) = u_0. \end{cases} \quad (7)$$

Let X and Y be Hilbert spaces where Y is continuously and densely embedded in X , and $Q: Y \rightarrow X$ be a topological isomorphism. We define $L(Y, X)$ as the space of all bounded linear operators from Y to X . We denote $L(X, X)$ by $L(X)$. Note that ρ_1, ρ_2, ρ_3 and ρ_4 in the following are constants and depend on $\max\{\|y\|_Y, \|z\|_Y\}$.

(I) $A(y) \in L(Y, X)$ for $y \in X$ with

$A(y) \in G(X, 1, \xi)$ ($\xi > 0$) and uniform on bounded sets in Y .

$$\|(A(y) - A(z))w\|_X \leq \rho_1 \|y - z\|_X \|w\|_Y, \quad y, z, w \in Y,$$

(II) $QA(y)Q^{-1} = A(y) + B(y)$, in which $B(y) \in L(X)$ is bounded and uniform on bounded sets in Y and

$$\|(B(y) - B(z))w\|_X \leq \rho_2 \|y - z\|_X \|w\|_Y, \quad y, z \in Y, \quad w \in X.$$

(III) $W: Y \rightarrow Y$ extends to a map from X to X , is bounded on bounded sets in Y and satisfies

Kato Theorem (1975). Assume that (I), (II) and (III) hold. If $u_0 \in Y$, there is a maximal $T > 0$ depending only on $\|u_0\|_Y$ and a unique solution u to problem (7) such that

$$\|W(y) - W(z)\|_Y \leq \rho_3 \|y - z\|_Y, \quad y, z \in Y,$$

$$\|W(y) - W(z)\|_X \leq \rho_4 \|y - z\|_X, \quad y, z \in Y.$$

$$u = u(\cdot, u_0) \in C([0, T]; Y) \cap C^1([0, T]; X).$$

Moreover, the map $u_0 \rightarrow u(\cdot, u_0)$ is a continuous map from Y to the space $C([0, T]; Y) \cap C^1([0, T]; X)$.

For problem (6), we set $A(\phi) = f'(\phi)\partial_x$, $Y = H^s(\mathbb{R})$, $X = H^{s-1}(\mathbb{R})$, $W(\phi) = -\alpha\phi^m - (\beta - 1)\Lambda^{-2}\partial_x f(\phi)$, and $Q = \Lambda$. Then, we will verify that $A(\phi)$ and $W(\phi)$ satisfy conditions (I)–(III). We cite several conclusions presented in Rodriguez-Blanco (2001).

LEMMA 2.1 The operator $A(\phi) = f'(\phi)\partial_x$ with $\phi \in H^s(\mathbb{R})$ ($s > \frac{3}{2}$), belongs to $G(H^{s-1}, 1, \xi)$.

LEMMA 2.2 For $\phi, z, w \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, $A(\phi) \in L(H^s, H^{s-1})$, it holds that

$$\|(A(\phi) - A(z))w\|_{H^{s-1}} \leq \rho_1 \|\phi - z\|_{H^{s-1}} \|w\|_{H^s}.$$

LEMMA 2.3 For $\phi, z \in H^s(\mathbb{R})$ and $w \in H^{s-1}(\mathbb{R})$ ($s > \frac{3}{2}$), it holds that $B(\phi) = [\Lambda, f'(\phi)\partial_x]\Lambda^{-1} \in L(H^{s-1})$ and

$$\|(B(\phi) - B(z))w\|_{H^{s-1}} \leq \rho_2 \|\phi - z\|_{H^s} \|w\|_{H^{s-1}}.$$

LEMMA 2.4 (Kato, 1975). Let r and q be real numbers such that $-r < q \leq r$. Then,

$$\|\phi\psi\|_{H^q} \leq c\|\phi\|_{H^r}\|\psi\|_{H^q}, \quad \text{if } r < \frac{1}{2},$$

$$\|\phi\psi\|_{H^{r+q-\frac{1}{2}}} \leq c\|\phi\|_{H^r}\|\psi\|_{H^q}, \quad \text{if } r < \frac{1}{2}.$$

LEMMA 2.5 Let $\phi, z \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ and $W(\phi) = -\alpha\phi^m - (\beta - 1)\Lambda^{-2}\partial_x f(\phi)$. Then, W is bounded on bounded sets in H^s and satisfies

$$\|W(\phi) - W(z)\|_{H^s} \leq \rho_3 \|\phi - z\|_{H^s}, \tag{8}$$

$$\|W(\phi) - W(z)\|_{H^{s-1}} \leq \rho_4 \|\phi - z\|_{H^{s-1}}. \tag{9}$$

Proof For $s > \frac{3}{2}$, we have $\|\phi\|_{L^\infty} \leq c\|\phi\|_{H^s}$ and $\|\phi\|_{H^{s-1}} \leq c\|\phi\|_{H^s}$. Applying the algebra property of $H^s(\mathbb{R})$ and Lemma 2.4, we get

$$\begin{aligned} \|W(\phi) - W(z)\|_{H^s} &\leq |\alpha| \|\phi^m - z^m\|_{H^s} + |\beta - 1| \|\Lambda^{-2}\partial_x(f(\phi) - f(z))\|_{H^s} \\ &\leq c\|\phi^m - z^m\|_{H^s} + c\|f(\phi) - f(z)\|_{H^{s-2}} \\ &\leq c\|\phi - z\|_{H^s} \|\phi^{m-1} + \phi^{m-2}z + \dots + z^{m-1}\|_{H^s} \\ &\quad + c\|f(\phi) - f(z)\|_{H^s} \\ &\leq \rho_3 \|\phi - z\|_{H^s}, \end{aligned}$$

which completes the proof of (8). Similarly, we get

$$\begin{aligned} \|W(\phi) - W(z)\|_{H^{s-1}} &\leq \|\alpha\phi^m - \alpha z^m\|_{H^{s-1}} + |\beta - 1| \|\Lambda^{-2}\partial_x(f(\phi) - f(z))\|_{H^{s-1}} \\ &\leq c\|\phi^m - z^m\|_{H^{s-1}} + c\|f(\phi) - f(z)\|_{H^{s-3}} \\ &\leq c\|\phi - z\|_{H^{s-1}} \|\phi^{m-1} + \phi^{m-2}z + \dots + z^{m-1}\|_{H^{s-1}} \\ &\quad + c\|f(\phi) - f(z)\|_{H^{s-1}} \\ &\leq \rho_4 \|\phi - z\|_{H^{s-1}}, \end{aligned}$$

which completes the proof of (9). □

Proof of Theorem 1.1 Using Lemmas 2.1–2.3 and Lemma 2.5, we know that the conditions (I), (II) and (III) hold. Applying the Kato Theorem, we find that problem (5) or (6) has a unique local solution

$$\phi = \phi(t, x) \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R})),$$

where $T > 0$ depends on $\|\phi_0\|_{H^s}$. □

Remark 2.6 Let $T > 0$ be described in Theorem 1.1. Using the Sobolev embedding Theorem, we ensure the boundedness of solution $\phi(t, x)$ to problem (6) in the domain $[0, T] \times \mathbb{R}$. Namely, provided that $\phi_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, we have $\|\phi\|_{L^\infty(\mathbb{R})} \leq M_T$, where M_T is a positive constant.

3. Proof of Theorem 1.2

Let $P(t, x, \phi) = \alpha\phi^m + (\beta - 1)\Lambda^{-2}\partial_x f(\phi)$ in the first equation of (6); we get,

$$\begin{cases} \phi_t + f'(\phi)\phi_x + P(t, x, \phi) = 0, \\ \phi(0, x) = \phi_0(x). \end{cases} \tag{10}$$

Assume that $\phi(t, x)$ and $\psi(t, x)$ are solutions of problem (10) in the domain $[0, T] \times \mathbb{R}$ with initial functions $\phi_0(x)$ and $\psi_0(x) \in H^s(\mathbb{R})$ ($s > \frac{3}{2}$). Now we give several lemmas.

LEMMA 3.1 Let $\phi(t, x)$ be the solution of problem (10) and $\phi_0(x) \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Then, $\|\phi\|_{L^\infty(\mathbb{R})} \leq M_T$ and $\|P(t, x, \phi)\|_{L^\infty(\mathbb{R})} \leq cM_T^k$,

where positive constant c depends on $\alpha, \beta, m, n, \|\phi_0\|_{L^\infty}$ and $k = \max\{m, n\}$.

Proof We have

$$\begin{aligned} |P(t, x, \phi)| &= \left| \alpha \phi^m(t, x) + (\beta - 1) \partial_x \Lambda^{-2} f(\phi) \right| \\ &= \left| \alpha \phi^m(t, x) + \frac{\beta - 1}{2} \int_R e^{-|x-y|} \operatorname{sign}(y-x) f(\phi) \, dy \right| \\ &\leq |\alpha| \left| \phi^m(t, x) \right| + \frac{|\beta - 1|}{2} \int_R e^{-|x-y|} \left| f(\phi) \right| \, dy. \end{aligned}$$

Applying Remark 2.6 and the integral $\int_R e^{-|x-y|} \, dy = 2$, we complete the proof. □

LEMMA 3.2 Assume that $\phi(t, x)$ and $\psi(t, x)$ are solutions of problem (10) in the domain $[0, T] \times R$ with initial functions $\phi_0(x)$ and $\psi_0(x) \in H^s(R)$ ($s > \frac{3}{2}$), respectively. Then,

$$\int_{-\infty}^{+\infty} |P(t, x, \phi) - P(t, x, \psi)| \, dx \leq c \int_{-\infty}^{+\infty} |\phi - \psi| \, dx, \tag{11}$$

where $c > 0$ depends on $\alpha, \beta, m, n, \|\phi_0\|_{L^\infty(R)}, \|\psi_0\|_{L^\infty(R)}$ and T .

Proof Using the property of the operator Λ^{-2} and Remark 2.6, we get

$$\begin{aligned} \int_{-\infty}^{+\infty} |P(t, x, \phi) - P(t, x, \psi)| \, dx &\leq |\alpha| \int_{-\infty}^{+\infty} |\phi^m - \psi^m| \, dx + \frac{|\beta - 1|}{2} \int_{-\infty}^{+\infty} \left| \partial_x \Lambda^{-2} (f(\phi) - f(\psi)) \right| \, dx \\ &\leq c \int_{-\infty}^{+\infty} |\phi - \psi| \, dx + c \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \left| e^{-|x-y|} \operatorname{sign}(y-x) (f(\phi) - f(\psi)) \right| \, dy \\ &\leq c \int_{-\infty}^{+\infty} |\phi - \psi| \, dx + c \int_{-\infty}^{+\infty} |f(\phi) - f(\psi)| \, dx \int_{-\infty}^{+\infty} e^{-|x-y|} \, dy \\ &\leq c \int_{-\infty}^{+\infty} |\phi - \psi| \, dx, \end{aligned}$$

in which we apply the Tonelli Theorem to complete the proof. □

We introduce a function $\delta(\sigma)$ which is infinitely differential on $(-\infty, +\infty)$. Note that $\delta(\sigma) \geq 0$, $\delta(\sigma) \equiv 0$ for $|\sigma| \geq 1$, $\int_{-\infty}^{+\infty} \delta(\sigma) \, d\sigma = 1$. Let $\delta_\varepsilon(\sigma) = \delta(\varepsilon^{-1}\sigma)\varepsilon$, where ε is an arbitrary positive constant. It is found that $\delta_\varepsilon(\sigma) \in C_0^\infty(-\infty, +\infty)$ and

$$\begin{cases} \delta_\varepsilon(\sigma) \geq 0, & \delta_\varepsilon(\sigma) = 0 & \text{for } |\sigma| \geq \varepsilon, \\ |\delta_\varepsilon(\sigma)| \leq \frac{c}{\varepsilon}, & \int_{-\infty}^{+\infty} \delta_\varepsilon(\sigma) \, d\sigma = 1. \end{cases} \tag{12}$$

Let the function $v(x)$ be defined and locally integrable on $(-\infty, +\infty)$. Let $v^\varepsilon(x)$ denote the approximation function of $v(x)$ as

$$v^\varepsilon(x) = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \delta\left(\frac{x-y}{\varepsilon}\right) v(y) \, dy. \tag{13}$$

We call x_0 a Lebesgue point of the function $v(x)$ if

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} |v(x) - v(x_0)| dx = 0.$$

At any Lebesgue point x_0 , we get

$$\lim_{\varepsilon \rightarrow 0} v^\varepsilon(x_0) = v(x_0).$$

Since the set of points which are not Lebesgue points of $v(x)$ has measure zero, we have $v^\varepsilon(x) \rightarrow v(x)$ as $\varepsilon \rightarrow 0$ almost everywhere.

For any $T_1 \in [0, T]$, we denote the band $\{(t, x): [0, T_1] \times R\}$ by π_{T_1} . Let $K_r = \{x: |x| \leq r\}$ and

$$\Pi = \left\{ (t, x, \tau, y): \left| \frac{t - \tau}{2} \right| \leq \varepsilon, \quad \rho \leq \frac{t + \tau}{2} \leq T_1 - \rho, \right. \\ \left. \left| \frac{x - y}{2} \right| \leq \varepsilon, \quad \left| \frac{x + y}{2} \right| \leq r - \rho \right\},$$

where $r > 0, \rho > 0$.

LEMMA 3.3 (Kruzkov, 1970). Let the function $v(t, x)$ be bounded and measurable in cylinder $[0, T_1] \times K_r$. If for any $\rho \in (0, \min[r, T_1])$ and any $\varepsilon \in (0, \rho)$, the function

$$V_\varepsilon = \frac{1}{\varepsilon^2} \int \int \int_{\Pi} |v(t, x) - v(\tau, y)| dt dx d\tau dy$$

satisfies

$$\lim_{\varepsilon \rightarrow 0} V_\varepsilon = 0.$$

LEMMA 3.4 (Kruzkov, 1970). If $\frac{\partial F(u)}{\partial u}$ is bounded, then the function $H(u, v) = \text{sign}(u - v)(F(u) - F(v))$ satisfies the Lipschitz condition in u and v .

We state the concept of a characteristic cone. Let T be described in Theorem 1.1 and $\|\phi\|_{L^\infty(R)} \leq M_T$. For any $T_1 \in [0, T]$ and $R_1 > 0$, we define

$$N > \max_{(t, x) \in [0, T_1] \times K_{R_1}} |f'(\phi)|.$$

Let Ω represent the cone $\{(t, x): |x| \leq R_1 - Nt, 0 \leq t \leq T_0 = \min(T_1, R_1 N^{-1})\}$ and S_τ designate the cross section of the cone Ω by the plane $t = \tau, \tau \in [0, T_0]$.

LEMMA 3.5 Let $\phi(t, x)$ be the solution of problem (10) on $\pi_{T_1}, g(t, x) \in C_0^\infty(\pi_{T_1})$; it holds that

$$\int \int_{\pi_{T_1}} \{ |\phi - k| g_t + \text{sign}(\phi - k) [f(\phi) - f(k)] g_x \\ - \text{sign}(\phi - k) g(t, x) P(t, x, \phi) \} dt dx = 0, \tag{14}$$

where k is an arbitrary constant.

Proof Suppose that $\Phi(\phi)$ is a twice differential function. Multiplying the first equation of problem (10) by $\Phi'(\phi)g(t, x)$ and integrating over π_{T_1} , we get

$$\int \int_{\pi_{T_1}} \{ \Phi'(\phi)g\phi_t + \Phi'(\phi)gf'(\phi)\phi_x + \Phi'(\phi)gP(t, x, \phi) \} dt dx = 0. \tag{15}$$

Using the method of integration by parts, we get

$$\int \int_{\pi_{T_1}} \Phi'(\phi)g\phi_t dt dx = - \int \int_{\pi_{T_1}} \Phi(\phi)g_t dt dx. \tag{16}$$

Notice that

$$\left(\int_k^\phi \Phi'(z)f'(z) dz \right)' = \Phi'(\phi)f'(\phi)\phi_x.$$

Thus,

$$\int_{-\infty}^{+\infty} \left(\int_k^\phi \Phi'(z)f'(z) dz \right) g_x dx = - \int_{-\infty}^{+\infty} \Phi'(\phi)f'(\phi)\phi_x g dx.$$

Then, we have

$$\begin{aligned} \int \int_{\pi_{T_1}} \Phi'(\phi)f'(\phi)\phi_x g dt dx &= - \int \int_{\pi_{T_1}} \left(\int_k^\phi \Phi'(z)f'(z) dz \right) g_x dt dx \\ &= - \int \int_{\pi_{T_1}} \{ \Phi'(\phi)[f(\phi) - f(k)] \\ &\quad - \int_k^\phi \Phi''(z)[f(z) - f(k)] dz \} g_x dt dx. \end{aligned} \tag{17}$$

Substitute Equations (16) and (17) into Equation (15). Let $\Phi^\epsilon(\phi)$ be an approximation of the function $\Phi(\phi) = |\phi - k|$. When $\epsilon \rightarrow 0$, $\Phi^\epsilon(\phi) \rightarrow \Phi(\phi)$, we obtain Equation (14). \square

We will give the proof of Theorem 1.2. Set function $g(t, x) \in C_0^\infty(\pi_{T_1})$, $g(t, x) \equiv 0$ outside the cylinder $\varpi = \{(t, x) = [\rho, T_1 - 2\rho] \times K_{r-2\rho}\}$, where $K_{r-2\rho} = \{|x|: |x| \leq r - 2\rho\}$, $r > 0$, $0 < 2\rho < \min(T_1, r)$.

Proof of Theorem 1.2.

We define

$$F(t, x, \tau, y) = g\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right) \delta_\epsilon\left(\frac{t-\tau}{2}, \frac{x-y}{2}\right) = g(\dots)\lambda_\epsilon(\cdot), \tag{18}$$

in which $(\dots) = \left(\frac{t+\tau}{2}, \frac{x+y}{2}\right)$, $(\cdot) = \left(\frac{t-\tau}{2}, \frac{x-y}{2}\right)$. Thus, we obtain

$$F_t + F_\tau = g_t(\dots)\lambda_\epsilon(\cdot), \quad F_x + F_y = g_x(\dots)\lambda_\epsilon(\cdot).$$

Using Lemma 3.5 and setting $k = \psi(\tau, y)$, $g(t, x) = F(t, x, \tau, y)$, we get

$$\begin{aligned} & \int \int \int \int_{\pi_{\tau_1} \times \pi_{\tau_1}} \{ |\phi(t, x) - \psi(\tau, y)| F_t + \text{sign}(\phi(t, x) - \psi(\tau, y)) \\ & \times [f(\phi(t, x)) - f(\psi(\tau, y))] F_x - \text{sign}(\phi(t, x) - \psi(\tau, y)) \\ & \times P(t, x, \phi(t, x)) F \} dt dx d\tau dy = 0. \end{aligned} \tag{19}$$

Similarly, we have

$$\begin{aligned} & \int \int \int \int_{\pi_{\tau_1} \times \pi_{\tau_1}} \{ |\psi(\tau, y) - \phi(t, x)| F_\tau + \text{sign}(\psi(\tau, y) - \phi(t, x)) \\ & \times [f(\psi(\tau, y)) - f(\phi(t, x))] F_y - \text{sign}(\psi(\tau, y) - \phi(t, x)) \\ & \times P(\tau, y, \psi(\tau, y)) F \} dt dx d\tau dy = 0. \end{aligned} \tag{20}$$

Adding (19) and (20), we obtain

$$\begin{aligned} 0 \leq & \int \int \int \int_{\pi_{\tau_1} \times \pi_{\tau_1}} \{ |\phi(t, x) - \psi(\tau, y)| g_t \lambda_\epsilon + \text{sign}(\phi(t, x) - \psi(\tau, y)) \\ & \times [f(\phi(t, x)) - f(\psi(\tau, y))] g_x \lambda_\epsilon \} dt dx d\tau dy + \left| \int \int \int \int_{\pi_{\tau_1} \times \pi_{\tau_1}} \{ \text{sign}(\phi(t, x) - \psi(\tau, y)) \right. \\ & \left. \times [P(t, x, \phi(t, x)) - P(\tau, y, \psi(\tau, y))] g \lambda_\epsilon \} dt dx d\tau dy \right|. \end{aligned} \tag{21}$$

We note that the first two terms of the integrand of (21) have the form

$$K_\epsilon = K(t, x, \tau, y, \phi(t, x), \psi(\tau, y)) \lambda_\epsilon(\cdot), \tag{22}$$

where K satisfies the Lipschitz condition in all its variables. Then,

$$\begin{aligned} \int \int \int \int_{\pi_{\tau_1} \times \pi_{\tau_1}} K_\epsilon dt dx d\tau dy &= \int \int \int \int_{\pi_{\tau_1} \times \pi_{\tau_1}} K(t, x, \tau, y, \phi(t, x), \psi(\tau, y)) \lambda_\epsilon dt dx d\tau dy \\ &= \int \int \int \int_{\pi_{\tau_1} \times \pi_{\tau_1}} \{ K(t, x, \tau, y, \phi(t, x), \psi(\tau, y)) \\ & \quad - K(t, x, t, x, \phi(t, x), \psi(t, x)) \} \lambda_\epsilon dt dx d\tau dy \\ & \quad + \int \int \int \int_{\pi_{\tau_1} \times \pi_{\tau_1}} K(t, x, t, x, \phi(t, x), \psi(t, x)) \lambda_\epsilon dt dx d\tau dy \\ &= J_{11}(\epsilon) + J_{12} \end{aligned} \tag{23}$$

Note that $K_\epsilon = 0$ outside the region Π . Applying the estimate $\lambda_\epsilon(\cdot) \leq \frac{c}{\epsilon^2}$ and Lemma 3.4, we get

$$|J_{11}(\epsilon)| \leq c \left[\epsilon + \frac{1}{\epsilon^2} \int \int \int \int_{\Pi} |\psi(t, x) - \psi(\tau, y)| dt dx d\tau dy \right],$$

where c is a positive constant independent of ϵ . Using Lemma 3.3, we know $J_{11}(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

For the term J_{12} , we substitute $t = \alpha_1, \frac{t-\tau}{2} = \beta_1, x = \eta, \frac{x-y}{2} = \gamma$. Combining with the identity

$$\int_{-\epsilon}^{\epsilon} \int_{-\infty}^{+\infty} \lambda_\epsilon(\beta_1, \gamma) d\beta_1 d\gamma = 1,$$

we obtain

$$\begin{aligned}
 J_{12} &= 2^2 \int_{\pi_{\tau_1} \times \pi_{\tau_1}} \int_{\pi_{\tau_1}} K(\alpha_1, \eta, \alpha_1, \eta, \phi(\alpha_1, \eta), \psi(\alpha_1, \eta)) \left(\int_{-\epsilon}^{\epsilon} \int_{-\infty}^{+\infty} \lambda_{\epsilon}(\beta_1, \gamma) d\beta_1 d\gamma \right) d\alpha_1 d\eta \\
 &= 4 \int_{\pi_{\tau_1}} \int K(t, x, t, x, \phi(t, x), \psi(t, x)) dt dx.
 \end{aligned}$$

Thus, we have

$$\lim_{\epsilon \rightarrow 0} \int_{\pi_{\tau_1} \times \pi_{\tau_1}} \int_{\pi_{\tau_1}} K_{\epsilon} = 4 \int_{\pi_{\tau_1}} K(t, x, t, x, \phi(t, x), \psi(t, x)) dt dx. \tag{23}$$

Similarly, the integrand of the third term in (21) can be represented as

$$\begin{aligned}
 \bar{K}_{\epsilon} &= \text{sign}(\phi(t, x) - \psi(\tau, y)) [P(t, x, \phi(t, x)) - P(\tau, y, \psi(\tau, y))] g \lambda_{\epsilon} \\
 &= \bar{K}(t, x, \tau, y, \phi(t, x), \psi(\tau, y)) \lambda_{\epsilon}(\cdot).
 \end{aligned} \tag{24}$$

Then,

$$\begin{aligned}
 \int_{\pi_{\tau_1} \times \pi_{\tau_1}} \int_{\pi_{\tau_1}} \int_{\pi_{\tau_1}} \bar{K}_{\epsilon} dt dx d\tau dy &= \int_{\pi_{\tau_1} \times \pi_{\tau_1}} \int_{\pi_{\tau_1}} \left\{ \bar{K}(t, x, \tau, y, \phi(t, x), \psi(\tau, y)) \right. \\
 &\quad \left. - \bar{K}(t, x, t, x, \phi(t, x), \psi(t, x)) \lambda_{\epsilon} \right\} dt dx d\tau dy \\
 &\quad + \int_{\pi_{\tau_1} \times \pi_{\tau_1}} \int_{\pi_{\tau_1}} \bar{K}(t, x, t, x, \phi(t, x), \psi(t, x)) \lambda_{\epsilon} dt dx d\tau dy \\
 &= J_{21}(\epsilon) + J_{22}.
 \end{aligned}$$

Using Lemma 3.4, we have

$$|J_{21}(\epsilon)| \leq c \left[\epsilon + \frac{1}{\epsilon^2} \int_{\Pi} \int_{\Pi} \int_{\Pi} |\psi(t, x) - \psi(\tau, y)| dt dx d\tau dy \right].$$

Using Lemma 3.3, it yields $J_{21}(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Repeating the steps as before, we have

$$J_{22} = 4 \int_{\pi_{\tau_1}} \int \bar{K}(t, x, t, x, \phi(t, x), \psi(t, x)) dt dx. \tag{25}$$

From (21) to (25), we get

$$\begin{aligned}
 &\int_{\pi_{\tau_1}} \int \left\{ |\phi(t, x) - \psi(t, x)| g_t + \text{sign}(\phi(t, x) - \psi(t, x)) [f(\phi) - f(\psi)] g_x \right\} dt dx \\
 &\quad + \left| \int_{\pi_{\tau_1}} \int \text{sign}(\phi(t, x) - \psi(t, x)) [P(t, x, \phi) - P(t, x, \psi)] g dt dx \right| \geq 0.
 \end{aligned} \tag{26}$$

We set

$$h(t) = \int_{-\infty}^{+\infty} |\phi(t, x) - \psi(t, x)| dx \tag{27}$$

and

$$\mu_\varepsilon(\sigma) = \int_{-\infty}^{\sigma} \delta_\varepsilon(\sigma) d\sigma. \tag{28}$$

Take two numbers $\rho, \tau \in (0, T_0)$ and $\rho < \tau$. In (26), we let

$$g(t, x) = [\mu_\varepsilon(t - \rho) - \mu_\varepsilon(t - \tau)] \chi(t, x), \quad \varepsilon < \min(\rho, T_0 - \tau), \tag{29}$$

in which

$$\chi(t, x) = \chi_\theta(t, x) = 1 - \mu_\theta(|x| + Nt - R_1 + \theta), \tag{30}$$

where θ is a small positive constant and $\chi(t, x) = 0$ outside the cone Ω . When $\theta \rightarrow 0, R_1 \rightarrow +\infty$, we observe that $\chi_\theta \rightarrow 1$. By the definition of the number N , we have

$$0 = \chi_t + N|\chi_x| \geq \chi_t + N\chi_x, \quad (t, x) \in \Omega.$$

Applying (26)–(30), we get

$$\begin{aligned} & \int \int_{\pi_{T_0}} \{ |\phi(t, x) - \psi(t, x)| [\delta_\varepsilon(t - \rho) - \delta_\varepsilon(t - \tau)] \chi_\theta(t, x) \} dt dx \\ & + \int_0^{T_0} dt \int_{-\infty}^{+\infty} \{ |P(t, x, \phi) - P(t, x, \psi)| \\ & \times [\mu_\varepsilon(t - \rho) - \mu_\varepsilon(t - \tau)] \chi_\theta(t, x) \} dx \geq 0. \end{aligned} \tag{31}$$

In (31), sending $\theta \rightarrow 0, R_1 \rightarrow +\infty$ and using Lemma 3.2, we obtain

$$\int_0^{T_0} [\delta_\varepsilon(t - \rho) - \delta_\varepsilon(t - \tau)] h(t) dt + c \int_0^{T_0} [\mu_\varepsilon(t - \rho) - \mu_\varepsilon(t - \tau)] h(t) dt \geq 0, \tag{32}$$

where c is independent of ε .

Applying the properties of the function δ_ε for $\varepsilon \leq \min(\rho, T_0 - \rho)$, we get

$$\begin{aligned} \left| \int_0^{T_0} \delta_\varepsilon(t - \rho) h(t) dt - h(\rho) \right| &= \left| \int_0^{T_0} \delta_\varepsilon(t - \rho) [h(t) - h(\rho)] dt \right| \\ &\leq \frac{c}{\varepsilon} \int_{\rho - \varepsilon}^{\rho + \varepsilon} |h(t) - h(\rho)| dt. \end{aligned}$$

Then,

$$\int_0^{T_0} \delta_\varepsilon(t - \rho) h(t) dt \rightarrow h(\rho) \quad \text{as } \varepsilon \rightarrow 0. \tag{33}$$

Let

$$G(\rho) = \int_0^{T_0} \mu_\varepsilon(t - \rho) h(t) dt = \int_0^{T_0} dt \int_{-\infty}^{t - \rho} \delta_\varepsilon(\sigma) h(t) d\sigma.$$

We observe that

$$G'(\rho) = - \int_0^{T_0} \delta_\varepsilon(t - \rho) h(t) dt.$$

Let $\varepsilon \rightarrow 0$; it derives that

$$G'(\rho) \rightarrow -h(\rho)$$

and

$$G(\rho) \rightarrow G(0) - \int_0^\rho h(t) dt, \quad G(\tau) \rightarrow G(0) - \int_0^\tau h(t) dt.$$

Therefore, we have

$$G(\rho) - G(\tau) \rightarrow \int_\rho^\tau h(t) dt \quad \text{as } \varepsilon \rightarrow 0. \tag{34}$$

From (32)–(34), we obtain inequality

$$h(\rho) + c \int_\rho^\tau h(t) dt \geq h(\tau).$$

Let $\rho \rightarrow 0, \tau \rightarrow t$; we get

$$\begin{aligned} & \int_{-\infty}^{+\infty} |\phi(0, x) - \psi(0, x)| dx + c \int_0^t dt \int_{-\infty}^{+\infty} |\phi(t, x) - \psi(t, x)| dx \\ & \geq \int_{-\infty}^{+\infty} |\phi(t, x) - \psi(t, x)| dx. \end{aligned}$$

Applying the Gronwall inequality, we complete the proof of Theorem 1.2. □

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