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A CLASS OF ANTIPERSISTENT PROCESSES

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Abstract. We introduce a class of stationary processes characterized by the behaviour of their infinite moving average parameters. We establish the asymptotic behaviour of the covariance function and the behaviour around zero of the spectral density of these processes, showing their antipersistent character. Then, we discuss the existence of an infinite autoregressive representation for this family of processes, and we present some consequences for fractional autoregressive moving average models.

Keywords. Antipersistent process; FARIMA process; moving average parameters; autoregressive expansion.
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1. INTRODUCTION

Antipersistent time series are defined by Dittmann and Granger (2002) as those covariance-stationary processes with zero spectral density at the origin. They have been used to model time series from disciplines as diverse as finance and economics (Barkoulas and Baum, 1996; McLeod, 1998; Henry, 2002), oceanography (Ausloos and Ivanova, 2001), and biology (Liebovitch and Yang, 1997). In turn, there has been a considerable amount of theoretical developments on antipersistent processes during the last decades. For instance, estimation techniques have been discussed by Beran (1995), and Beran and Feng (2002); the behaviour of partial autocorrelations and optimal prediction error variance have been studied by Inoue (2000), and Inoue and Kasahara (2004); interpolation of missing data has been investigated by Wilson *et al.* (2003); nonlinear transformations of these processes have been analysed by Robinson (2001), and Dittmann and Granger (2002).

In this article, we introduce a class of processes whose infinite moving average parameters satisfy two conditions and we prove that these conditions imply the antipersistent character defined by Dittmann and Granger (2002). Then we study the invertibility of antipersistent processes, and as a particular result, we show that the fractional autoregressive integrated moving average (FARIMA) model is invertible if and only if its degree of differencing belongs to $(-1, 1/2)$, extending the interval $(-1/2, 1/2)$ which is usually given in the literature.

This paper is structured as follows. Section 2 concerns preliminaries, Section 3 contains the main results and Section 4 is devoted to the invertibility of antipersistent FARIMA processes. Proofs of the theorems are given in the Appendix.

2. PRELIMINARIES

Let $(X_t)_{t\in\mathbb{Z}}$ be a real, zero-mean, weakly stationary process defined on a probability space $(\Omega,\mathcal{F},P)$. Here, E stands for the expectation operator, $L^2 = L^2(\Omega,\mathcal{F},P)$ denotes the Hilbert space with inner product $\langle X,Y\rangle = E(XY)$ and norm $\|X\| = \sqrt{E(X^2)}$. For a collection S of random variables in L^2, the subspace of all (finite) linear combinations of elements of S is denoted by sp S and its closure in L^2 by $\overline{\text{sp}}\ S$. In particular, $\overline{\text{sp}}\{X_s; s \le t\}$ is the past and present subspace of the process (X_t) up to time t, and the orthogonal projection of X_t onto $\overline{\text{sp}}\{X_s;\ s \le t-1\}$, that we shall denote by $\hat{X}_t$, is the best mean square infinite past linear predictor of X_t.

Let $L^p(\lambda)$, $1 \le p \le \infty$, be the Banach space for the Lebesgue measure λ on $(-\pi,\pi]$. The process (X_t) is assumed to be purely nondeterministic which means that its spectral measure is absolutely continuous with respect to λ, and that its spectral density f satisfies $\ln f \in L^1(\lambda)$. In this case, the function

$$h(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\lambda}+z}{e^{i\lambda}-z}\ln f(\lambda)\mathrm{d}\lambda\right\}, \quad z \in \mathbb{C},\ |z| < 1, \tag{1}$$

is an outer function in the Hardy space $H^2(\lambda)$, does not vanish for $|z| < 1$, and satisfies $f(\lambda) = |h(e^{i\lambda})|^2$ (see, for instance, Rozanov, 1967, Ch. II). Let $g(z) = h(z)/h(0)$, and $\mathbb{N}$ the set of non-negative integers. The MA (∞) parameters $(c_k)_{k\in\mathbb{N}}$ of (X_t) are defined by

$$g(z) = \sum_{k=0}^{\infty} c_k z^k, \quad z \in \mathbb{C},\ |z| < 1, \tag{2}$$

and the AR(∞) parameters $(a_k)_{k\in\mathbb{N}}$ by

$$\frac{-1}{g(z)} = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{C},\ |z| < 1.$$

Both sequences (c_k) and (a_k) are real, and we have

$$c_0 = -a_0 = 1 \quad \text{and} \quad \sum_{k=0}^{\infty} c_k^2 < \infty.$$

The Wold decomposition theorem states that (X_t) has the L^2-convergent series representation

$$X_t = \sum_{k=0}^{\infty} c_k \epsilon_{t-k}, \tag{3}$$

where $\epsilon_t = X_t - \hat{X}_t$ is the innovation process of (X_t) and is zero-mean, uncorrelated and weakly stationary with variance

$$\sigma_\epsilon^2 = 2\pi \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln f(\lambda) \mathrm{d}\lambda \right\}. \tag{4}$$

It follows from eqn (3) that the covariance function $(\gamma_k)_{k \in \mathbb{Z}}$ of (X_t) satisfies,

$$\gamma_n = \sum_{k=0}^{\infty} c_k c_{n+k}, \quad n \geq 0, \tag{5}$$

and then, asymptotic properties of (γ_k) follow directly from equivalent properties of (c_k). For instance, if (c_k) converges to zero at least exponentially, i.e. $c_n = O(\alpha^n)$ for some $\alpha \in (0, 1)$, then $\gamma_n = O(\alpha^n)$, and this behaviour is typical of short-memory processes like autoregressive moving average (ARMA) models. Similarly, if $c_n \sim C_1 n^{d-1}$ when $n \to \infty$, where $C_1 \in \mathbb{R}\backslash\{0\}$ and $d \in (0,1/2)$, then eqn (5) implies that $\gamma_n \sim C_2 n^{2d-1}$ where $C_2 \in \mathbb{R}\backslash\{0\}$ (see Lemma 2), and this behaviour is characteristic of long-memory processes like FARIMA time series.

Let Γ be the Gamma function, and $\mathcal{R}_0$ be the set of slowly varying functions at infinity: the set of positive, measurable functions l, defined on some neighbourhood $[A, \infty)$ of infinity, such that $\lim_{x\to\infty} l(\lambda x)/l(x) = 1$ for any $\lambda > 0$ (Bingham *et al.*, 1987 Ch. I).

3. A CLASS OF ANTIPERSISTENT PROCESSES

We introduce the class of purely nondeterministic processes whose MA(∞) parameters satisfy the two conditions

(A1) $c_{n+1} - c_n \sim n^{d-2} l(n) \Gamma(d-1)^{-1}$ *when* $n \to \infty$, *where* $l \in \mathcal{R}_0$ *and* $d \in (-1, 0)$,
(A2) $\sum_{k=0}^{\infty} c_k = 0$.

This class is nonempty and contains, in particular, the FARIMA model with a negative degree of differencing (see Remark 3). We establish that the processes satisfying (A1) and (A2) are antipersistent in the sense that (γ_k) tends to zero hyperbolically but is summable. More precisely, we prove in Theorem 1 that (A1) and (A2) imply that $\gamma_n \sim C_1 n^{2d-1} l(n)^2$ when $n \to \infty$ and $f(\lambda) \sim C_2 \lambda^{-2d} l(1/\lambda)^2$ when $\lambda \to 0_+$, where $C_1, C_2 \in \mathbb{R}\backslash\{0\}$. We show in Theorem 2 that condition (A2), which may seem restrictive, actually holds when (c_k) and (a_k) satisfy, respectively,

(A3) $\sum c_k$ *is convergent, and*
(A4) (a_k) *is eventually negative and* $\sum_{k=1}^{\infty} a_k k^{-\epsilon} = -\infty$ *for some* $\epsilon > 0$.

Since (A1) implies (A3) (see Lemma 1), the processes satisfying (A1) and (A4) are antipersistent.

Then we discuss the invertibility of the Wold decomposition (3) in the sense of the existence of an L^2-convergent series representation

$$\epsilon_t = -\sum_{k=0}^{\infty} a_k X_{t-k}. \tag{6}$$

Observe that eqn (6) is equivalent to the existence of a series representation for the predictor $\hat{X}_t$ itself, which is of fundamental importance in prediction theory and time series analysis (see Pourahmadi, 2001, Ch. 6). We show in Theorem 3 that the series in eqn (6) converges in L^2 whenever $f(\lambda) \sim \lambda^{-2d} l(1/\lambda)$ when $\lambda \to 0_+$, where $l \in \mathcal{R}_0$ and $d \in (-1, 1/2)$, and f is sufficiently smooth outside zero. Lemma 1 is useful in the following.

Lemma 1. *Let $l \in \mathcal{R}_0$ and $d \in (-1, 0)$. Condition* (A1) *implies that*

$$c_n \sim n^{d-1} l(n) \Gamma(d)^{-1}, \quad n \to \infty, \tag{7}$$

and eqn (7) *implies* (A3) and

$$\sum_{k=n}^{\infty} c_k \sim -n^d l(n) \Gamma(d+1)^{-1}, \quad n \to \infty. \tag{8}$$

Proof. Since $d < 1$, we deduce from (A1) and Bingham *et al.* (1987, Propn 1.5.10) that the sequence $(c_{k+1} - c_k)$ is summable and we have,

$$\lim_{k\to\infty} c_k - c_n = \sum_{k=n}^{\infty} (c_{k+1} - c_k) \sim -n^{d-1} l(n) \Gamma(d)^{-1}, \quad n \to \infty.$$

Since (c_k) is square summable, $\lim_{k\to\infty} c_k = 0$, which gives eqn (7). Similarly, since $d < 0$, eqn (7) implies (A3) and eqn (8). □

Theorem 1. *Let $l \in \mathcal{R}_0$ and $d \in (-1, 0)$. Conditions* (A1) *and* (A2) *imply that*

$$\gamma_n \sim \frac{n^{2d-1} l(n)^2 \Gamma(1-2d) \sin(\pi d)}{\pi}, \quad n \to \infty. \tag{9}$$

Furthermore, condition (A2) *and eqn* (9) *imply that*

$$f(\lambda) \sim \frac{\lambda^{-2d} l(1/\lambda)^2}{2\pi}, \quad \lambda \to 0_+. \tag{10}$$

Theorem 2. *Conditions* (A3) *and* (A4) *imply* (A2).

Corollary 1 is an immediate consequence of Lemma 1 and Theorems 1 and 2.

COROLLARY 1. *Let* $l \in \mathcal{R}_0$ *and* $d \in (-1, 0)$. *Conditions* (A1) *and* (A4) *imply eqns* (9) *and* (10).

Theorem 3 gives sufficient spectral conditions for the existence of the series representation (6).

THEOREM 3. *If* f *satisfies the three conditions:*

(i) $f(\lambda) \sim \lambda^{-2d} l(1/\lambda)$ *when* $\lambda \to 0_+$, *where* $l \in \mathcal{R}_0$ *and* $d \in (-1, 1/2)$,
(ii) f *is bounded on* $[\epsilon, \pi]$ *for every* $\epsilon > 0$,
(iii) f^{-1} *is locally integrable on* $(0, \pi]$,

then the series representation (6) *converges in* L^2.

REMARK 1. Theorem 3 does not hold if $d > 1/2$ because in that case, $f \notin L^1(\lambda)$ (Bingham *et al.*, 1987, Propn 1.5.8) and hence cannot be a spectral density function. Theorem 3 does not hold if $d \leq -1$. Indeed, consider the MA(1) process (X_t) defined by $X_t = Z_t - Z_{t-1}$ where (Z_t) is a sequence of uncorrelated random variables in L^2 with unit variance. We have $f(\lambda) = 2 \sin (\lambda/2)^2/\pi$, and then f satisfies the three conditions of Theorem 3 with l a constant function and $d = -1$. Nevertheless, in this case, the series representation (6) does not converge in L^2 (see, Topsøe, 1977, p. 52) or Theorem 4(ii).

4. INVERTIBILITY OF FARIMA PROCESSES

The FARIMA model was introduced by Granger and Joyeux (1980) and Hosking (1981) and has been used to describe long-memory phenomena in a wide variety of scientific disciplines, from hydrology to economics (see, e.g. Doukhan *et al.*, 2003, and references therein). More precisely, (X_t) is called a FARIMA(p, d, q) process if (X_t) satisfies the difference equation

$$(1 - \phi_1 B - \cdots - \phi_p B^p) X_t = (1 + \theta_1 B + \cdots + \theta_q B^q)(1 - B)^{-d} Z_t, \tag{11}$$

where B is the backward shift operator $BX_t = X_{t-1}$, (Z_t) a sequence of zero-mean uncorrelated random variables in L^2 with the same variance σ_Z^2, $d \in (-\infty, 1/2)$, and the polynomials $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ and $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ with real coefficients have no common zeros and neither $\phi(z)$ nor $\theta(z)$ has zeros in the closed unit disc $\{z \in \mathbb{C}: |z| \leq 1\}$. If $d \in -\mathbb{N}$, $(1 - B)^{-d}$ is a polynomial in B and (X_t) is an ARMA process. If $d \notin -\mathbb{N}$, the process $(1 - B)^{-d} Z_t$ is defined by

$$(1 - B)^{-d} Z_t = \sum_{k=0}^{\infty} b_k Z_{t-k}, \tag{12}$$

where $(b_k)_{k\in\mathbb{N}}$ are the coefficients in the Taylor series expansion of $(1 - z)^{-d}$ for $|z| < 1$, i.e.

$$b_k = \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)}, \quad k \in \mathbb{N}. \tag{13}$$

By Stirling's formula, $b_k \sim k^{d-1}/\Gamma(d)$ when $k \to \infty$, and then the series in eqn (12) converges in L^2 whenever $d < 1/2$. According to Kokoszka and Taqqu (1995, Thm 2.1), the unique causal moving average satisfying eqn (11) is the process

$$X_t = \sum_{k=0}^{\infty} \varphi_k Z_{t-k}, \tag{14}$$

where $(\varphi_k)_{k\in\mathbb{N}}$ are the coefficients in the Taylor series expansion of $\varphi(z) = (1 - z)^{-d}\theta(z)/\phi(z)$ for $|z| < 1$. This process has the spectral density

$$f(\lambda) = \frac{\sigma_Z^2 |\varphi(\mathrm{e}^{\mathrm{i}\lambda})|^2}{2\pi}, \quad \ln f \in L^1(\lambda),$$

and we deduce from eqn (4) by easy calculations that the variance of the innovation process of (X_t) is $\sigma_\epsilon^2 = \sigma_Z^2$. Therefore, the function Q defined by $Q(z) = \sigma_\epsilon \varphi(z)/\sqrt{2\pi}$ is analytic in the open unit disc $\{z \in \mathbb{C}: |z| < 1\}$ and satisfies

$$|Q(\mathrm{e}^{\mathrm{i}\lambda})|^2 = f(\lambda), \quad Q(0) = \frac{\sigma_\epsilon}{\sqrt{2\pi}}.$$

According to Hannan (1970, Thm 5, p. 142), Q coincides with the outer function h defined by eqn (1). Hence, $g = \varphi$ in eqn (2), the MA (∞) parameters $(c_k)_{k\in\mathbb{N}}$ of (X_t) are the coefficients $(\varphi_k)_{k\in\mathbb{N}}$, and the AR (∞) parameters $(a_k)_{k\in\mathbb{N}}$ of (X_t) are the coefficients in the Taylor series expansion of $-1/\varphi(z)$ for $|z| < 1$. Let $\tilde{\epsilon}(\lambda)$, $\tilde{Z}(\lambda)$ and $\tilde{X}(\lambda)$ be the random spectral measure of (ϵ_t), (Z_t) and (X_t), respectively. According to eqns (3) and (14), we have

$$X_t = \int_{(-\pi,\pi]} \mathrm{e}^{\mathrm{i}t\lambda} \varphi(\mathrm{e}^{-\mathrm{i}\lambda}) \mathrm{d}\tilde{\epsilon}(\lambda) = \int_{(-\pi,\pi]} \mathrm{e}^{\mathrm{i}t\lambda} \varphi(\mathrm{e}^{-\mathrm{i}\lambda}) \mathrm{d}\tilde{Z}(\lambda). \tag{15}$$

Since (ϵ_t) and (Z_t) are white noises, their spectral distribution functions are continuous at zero. As $\varphi(\mathrm{e}^{-\mathrm{i}\lambda}) \neq 0$ for $\lambda \neq 0$, it results from eqn (15) and Brockwell and Davis (1991, Thm 4.10.1) that

$$\epsilon_t = \int_{(-\pi,\pi]} \mathrm{e}^{\mathrm{i}t\lambda} \varphi(\mathrm{e}^{-\mathrm{i}\lambda})^{-1} \mathrm{d}\tilde{X}(\lambda) = Z_t.$$

Therefore, (Z_t) is the innovation process of (X_t) and eqn (14) is the Wold decomposition of (X_t).

We show in Theorem 4 that (Z_t) has the L^2-Abel-convergent series representation (16) for any $d < 1/2$, and the L^2-convergent series representation (17) if and only if $d \in (-1, 1/2)$.

THEOREM 4. *Let (X_t) be the FARIMA process defined by eqn (11) where the polynomials $\phi(z)$ and $\theta(z)$ have no common zeros and do not have zeros in the closed unit disc $\{z \in \mathbb{C}:|z| \leq 1\}$. Then*

(i) *(Z_t) has the L^2-Abel-convergent series representation*

$$Z_t = -\lim_{r\to 1_-} \sum_{k=0}^{\infty} r^k a_k X_{t-k}, \tag{16}$$

for any $d \in (-\infty,1/2)$;

(ii) *(Z_t) has the L^2-convergent series representation*

$$Z_t = -\sum_{k=0}^{\infty} a_k X_{t-k}, \tag{17}$$

if and only if $d \in (-1,\ 1/2)$.

REMARK 2. Theorem 4 generalizes Hosking (1981, Thm 2) where d is restricted to the range $(-1/2,\ 1/2)$. It is interesting to notice that it is generally thought that the series in eqn (17) converges in L^2 only when $d \in (-1/2,\ 1/2)$ (see, for instance, Hosking, 1981, p. 170, and Brockwell and Davis, 1991, Rmk 7, p. 526). Bloomfield (1985, p. 231) seems to be the first who noticed that the series in eqn (17) converges in L^2 whenever $d \in (-1,\ 1/2)$ and (X_t) is a fractional noise ($\phi(z) = \theta(z) = 1$).

REMARK 3. The MA (∞) parameters (c_k) of any FARIMA process with $d \in (-\infty,0)\backslash -\mathbb{N}$ satisfy (A1) and (A2) with $l(x) = \theta(1)/\phi(1)$. Indeed, since (c_k) are the coefficients in the Taylor series expansion of $\varphi(z)$ for $|z| < 1$, it follows from Inoue (2002, Lem. 2.1) that (A1) and eqn (7) hold for any $d \in (-\infty,1/2)\backslash -\mathbb{N}$ and $l(x) = \theta(1)/\phi(1)$. Therefore, if $d < 0$, $\sum c_k$ is convergent and we deduce from eqn (23) that

$$\sum_{k=0}^{\infty} c_k = \lim_{x\to 1_-} \varphi(x) = 0.$$

APPENDIX

To prove Theorem 1, we need Lemma 2 which is an immediate consequence of Inoue (1997, Propn 4.3).

LEMMA 2. *Let $r, s \in \mathbb{R}$ with $s < 1 < r + s$, $l_1, l_2 \in \mathcal{R}_0$, and $(f_k)_{k\in\mathbb{N}}$, $(g_k)_{k\in\mathbb{N}}$ two real sequences satisfying $f_k \sim k^{-r} l_1(k)$, $g_k \sim k^{-s} l_2(k)$ when $k \to \infty$. For any $n \in \mathbb{N}$, the sequence $(f_{n+k} g_k)_{k\in\mathbb{N}}$ is summable, and we have*

$$\sum_{k=0}^{\infty} f_{n+k} g_k \sim n^{-(r+s-1)} l_1(n) l_2(n) B(r+s-1, 1-s), \quad n \to \infty,$$

where $B(\cdot,\cdot)$ is the beta function.

Proof of Theorem 1. According to Lemma 1, (A3) holds. For all non-negative integers k, n, N, let $g_k = \sum_{l=k+1}^{\infty} c_l$, and

$$\gamma_{n,N} = \sum_{k=0}^{N} g_k (c_{n+k+1} - c_{n+k}).$$

Using (A2), we get

$$\gamma_{n,N} = g_N c_{n+N+1} + \sum_{k=0}^{N} c_k c_{n+k},$$

and then, according to eqn (5), $\lim_{N \to \infty} \gamma_{n,N} = \gamma_n$, i.e.

$$\gamma_n = \sum_{k=0}^{\infty} g_k (c_{n+k+1} - c_{n+k}). \tag{18}$$

Since $d \in (-1, 0)$, using eqn (8) and (A1) in Lemma 2, it follows from eqn (18) that

$$\gamma_n \sim -n^{2d-1} l(n)^2 \Gamma(d-1)^{-1} \Gamma(d+1)^{-1} B(1-2d, d+1), \quad n \to \infty.$$

Using simple manipulations and the formula $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$, this equation simplifies to eqn (9). According to eqn (9), (γ_k) is summable, and then f is given by the absolutely convergent series

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-ik\lambda} = \frac{1}{2\pi} \left[\gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos(k\lambda) \right].$$

Since (c_k) is summable, it results from eqn (5) that

$$\sum_{n=-\infty}^{\infty} \gamma_n = 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} c_k c_{n+k} + \sum_{k=0}^{\infty} c_k^2 = \left(\sum_{k=0}^{\infty} c_k \right)^2.$$

Hence, (A2) implies that

$$\gamma_0 = -2 \sum_{k=1}^{\infty} \gamma_k,$$

and we get

$$f(\lambda) = -\frac{2}{\pi} \sum_{k=1}^{\infty} \gamma_k \sin\left(\frac{k\lambda}{2} \right)^2. \tag{19}$$

For all positive integers k, n, let $a_{n,k} = \sin(k/2n)^2/k$. Fix $\rho \in (-2, 0)$, we have

$$\sum_{k=1}^{n} a_{n,k}\left(\frac{k}{n}\right)^{\rho} \leq \frac{1}{4n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{\rho+1} = O(1), \quad n \to \infty,$$
$$\sum_{k=n+1}^{\infty} a_{n,k}\left(\frac{k}{n}\right)^{\rho} \leq \frac{1}{n}\sum_{k=n+1}^{\infty}\left(\frac{k}{n}\right)^{\rho-1} = O(1), \quad n \to \infty.$$

Hence,

$$M_{\rho} = \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\left(\frac{k}{n}\right)^{\rho} < \infty. \tag{20}$$

Let $\eta > 0$ with $-2 < \rho - \eta$ and $\rho + \eta < 0$, and let $\delta \in (0, 1)$. We have

$$\left(\sum_{k\leq n\delta} + \sum_{k>n/\delta}\right) a_{n,k}\left(\frac{k}{n}\right)^{\rho} \leq \delta^{\eta}\left(\sum_{k\leq n\delta} a_{n,k}\left(\frac{k}{n}\right)^{\rho-\eta} + \sum_{k>n/\delta} a_{n,k}\left(\frac{k}{n}\right)^{\rho+\eta}\right).$$

Hence,

$$\lim_{\delta\to 0_+}\limsup_{n\to\infty}\left(\sum_{k\leq n\delta} + \sum_{k>n/\delta}\right) a_{n,k}\left(\frac{k}{n}\right)^{\rho} \leq (M_{\rho-\eta} + M_{\rho+\eta})\lim_{\delta\to 0_+}\ \delta^{\eta} = 0. \tag{21}$$

Now, let

$$\alpha_n^{(\rho)}(t) = \sum_{k=1}^{[nt]} a_{n,k}\left(\frac{k}{n}\right)^{\rho} = \frac{1}{n}\sum_{k=1}^{[nt]} u\left(\frac{k}{n}\right), \quad t > 0,$$

where $u(x) = \sin(x/2)^2 x^{\rho-1}\mathbb{1}_{\mathbb{R}_+}(x)$. Since the function u is continuous on $(0, t]$, it may be shown that the sequence $(u_n)_{n\geq 1}$ defined by

$$u_n(x) = \sum_{k=1}^{[nt]} u\left(\frac{k}{n}\right)\mathbb{1}_{(\frac{k-1}{n},\frac{k}{n}]}(x) + u(t)\mathbb{1}_{(\frac{[nt]}{n},t]}(x), \quad 0 < x \leq t,$$

converges simply to u on $(0, t]$. Furthermore, for all $n \geq 1$ and for all $x \in (0, t]$, $|u_n(x)| \leq v(x)$ where v is integrable on $(0, t]$. Take for instance, $v(x) = t^{\rho+1}/4$ if $\rho \geq -1$ and $v(x) = x^{\rho+1}/4$ if $\rho \in (-2, -1)$. Therefore, it results from the Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty}\int_0^t u_n(x)\ \mathrm{d}x = \int_0^t u(x)\mathrm{d}x.$$

Since

$$\int_0^t u_n(x)\mathrm{d}x = \alpha_n^{(\rho)}(t) + O\left(\frac{1}{n}\right),$$

$\lim_{n\to\infty}\alpha_n^{(\rho)}(t)$ exists and is given by

$$\alpha^{(\rho)}(t) = \lim_{n\to\infty}\alpha_n^{(\rho)}(t) = \int_0^t \sin\left(\frac{x}{2}\right)^2 x^{\rho-1}\mathrm{d}x, \quad t > 0. \tag{22}$$

Equations (20), (21) and the existence of $\alpha^{(\rho)}(t)$ are conditions (4.0.8), (4.0.9b) and (4.0.11) in Bingham *et al.* (1987, p. 196) respectively. These conditions imply that the sequence $(a_{n,k})$ is ρ-radial for every $\rho \in (-2, 0)$, i.e. $(a_{n,k})$ is $(-2, 0)$-radial. Let $s_k = k\gamma_k$. According to eqn (9), we have $s_n \sim n^{\rho} l'(n)$ when $n \to \infty$, where $l' \in \mathcal{R}_0$ and $\rho = 2d \in (-2, 0)$. It results from Bingham *et al.* (1987, Cor. 4.2.3) that

$$t_n := \sum_{k=1}^{\infty} a_{n,k} s_k$$

exists for all large n and $t_n \sim \mathring{\alpha}(\rho) n^{\rho} l'(n)$ when $n \to \infty$, where

$$\mathring{\alpha}(\rho) = \int_0^{\infty} \mathrm{d}\alpha^{(\rho)}(t).$$

Integrating by part, we deduce from eqn (22) that

$$\mathring{\alpha}(\rho) = -\frac{1}{2\rho} \int_0^{\infty} x^{\rho} \sin x \,\mathrm{d}x = -\frac{\pi}{4\Gamma(1-\rho)\sin(\pi\rho/2)},$$

where the last equality follows from Bingham *et al.* (1987, eqn 4.3.1a). Therefore, $t_n \sim -n^{2d} l(n)^2/4$ when $n \to \infty$, and it results from eqn (19) that

$$f(1/n) = -\frac{2t_n}{\pi} \sim \frac{n^{2d} l(n)^2}{2\pi}, \quad n \to \infty.$$

As in Bingham *et al.* (1987, p. 207), we may replace $1/n$ by λ, and let $\lambda \to 0_+$ through continuous values, which gives eqn (10). □

Proof of Theorem 2. It results from eqn (2) and (A3) that

$$\lim_{x \to 1_-} g(x) = \sum_{k=0}^{\infty} c_k, \tag{23}$$

(see, for instance, Rudin, 1976, Thm 8.2). According to (A4), there exists $k_0 > 2$ such that $a_k < 0$ for all $k \geq k_0$. Let

$$C = \sum_{k=0}^{k_0-1} |a_k|,$$

then

$$\frac{1}{g(x)} \geq -C - \sum_{k=k_0}^{\infty} a_k x^k, \quad -1 < x < 1.$$

Let $\epsilon > 0$ such that (A4) holds, and let $M > 0$. Then there exists $k_1 \geq k_0$ such that

$$-\sum_{k=k_0}^{k_1} a_k k^{-\epsilon} \geq C + M.$$

Let $\alpha = 1 - \exp(-\epsilon \ln k_1/k_1)$. Since $k_1 > 1$, $\alpha \in (0, 1)$, and for all $x \in [1-\alpha, 1)$, we have $\ln x + \epsilon \ln k_1/k_1 \geq 0$. Since the function $\ln t/t$ is decreasing for $t > e$ and $k_0 > e$, we have for all $k \in [k_0, k_1]$ and for all $x \in [1-\alpha, 1)$, $\ln x + \epsilon \ln k/k \geq 0$, i.e. $x^k k^{\epsilon} \geq 1$. Hence,

$$\frac{1}{g(x)} \geq -C - \sum_{k=k_0}^{k_1} a_k x^k \geq -C - \sum_{k=k_0}^{k_1} a_k k^{-\epsilon} \geq M, \quad 1-\alpha \leq x < 1,$$

which implies that $\lim_{x \to 1_-} 1/g(x) = \infty$. Combining with eqn (23), we get (A2). □

Proof of Theorem 3. The parameter d can be written as $d = d_1 + d_2$ where $d_1 \in (-1/2, 0)$ and $d_2 \in (-1/2, 1/2)$. Let $f_2(\lambda) = |1-e^{i\lambda}|^{-2d_2}$ and $f_1(\lambda) = f(\lambda)/f_2(\lambda)$. Then $f_1(\lambda) \sim \lambda^{-2d_1}l(1/\lambda)$ when $\lambda \to 0_+$, and f_1 satisfies (ii) and (iii). It results from Helson and Szegö (1960 Thm 1 and Corollary) that f_2 satisfies condition A_2 in Bloomfield (1985, Thm 3). Now, there exists $\epsilon > 0$ such that $f_1(\lambda)^{-1} \leq 2\lambda^{2d_1}l_1(1/\lambda)$ for all $\lambda \in (0,\epsilon)$, where $l_1 = l^{-1} \in \mathcal{R}_0$. If $\epsilon < \pi$, we have

$$\int_0^{\pi} f_1(\lambda)^{-1}\mathrm{d}\lambda \leq 2\int_0^{\epsilon} \lambda^{2d_1} l_1(1/\lambda)\mathrm{d}\lambda + \int_{\epsilon}^{\pi} f_1(\lambda)^{-1}\mathrm{d}\lambda. \tag{24}$$

Since f_1 satisfies (iii), the last term on the right-hand side of eqn (24) is finite. On the other hand,

$$\int_0^{\epsilon} \lambda^{2d_1} l_1\left(\frac{1}{\lambda}\right)\mathrm{d}\lambda = \int_{1/\epsilon}^{\infty} x^{-2(d_1+1)} l_1(x)\mathrm{d}x,$$

which is finite because $d_1 > -1/2$ (Bingham *et al.*, 1987, Propn 1.5.10). Therefore, the left-hand side of eqn (24) is finite. If $\epsilon \geq \pi$, we replace ϵ by π in eqn (24), and we get the same conclusion. Thus $f_1^{-1} \in L^1(\lambda)$. Since $d_1 < 0$, $\lim_{\lambda\to 0_+} f_1(\lambda) = 0$ (Bingham *et al.*, 1987, Propn 1.3.6). Moreover, f_1 satisfies (ii), and then $f_1 \in L^{\infty}(\lambda)$. Therefore, taking $p = 1$ in Bloomfield (1985, Thm 4), we get the result. □

Proof of Theorem 4. (i) Consider the class of spectral densities f satisfying the A_∞ condition

$$\int_E f(\lambda)\mathrm{d}\lambda \leq C\left[\frac{|E|}{|I|}\right]^{\epsilon}\int_I f(\lambda)\mathrm{d}\lambda,$$

in which $\epsilon > 0$ is fixed and $C > 0$ is independent of the interval $I \subset (-\pi,\pi]$ and its measurable subsets E. According to Huang *et al.* (1997, Example 3.1) the density $g(\lambda) = |1 - e^{i\lambda}|^{-2d}$ is in A_∞ for any $d < 1/2$. For any $z \in \mathbb{C}$ with $|z| \leq 1$, we have

$$0 < c_\theta := \prod_{i=1}^{q}(1 - |\xi_i|^{-1}) \leq |\theta(z)| \leq 1 + \sum_{i=1}^{q}|\theta_i| := C_\theta < \infty,$$

where (ξ_i) are the non-necessarily distinct zeros of $\theta(z)$. Analogously,

$$0 < c_\phi \leq |\phi(z)| \leq C_\phi < \infty, \quad z \in \mathbb{C}, \ |z| \leq 1.$$

Therefore, the spectral density f of (X_t) satisfies $f = gf_1$ where $f_1, f_1^{-1} \in L^{\infty}(\lambda)$, and since g is in A_∞, f is in A_∞. Then, it follows from Huang *et al.* (1997, Thm 3.1) that $\hat{X}_t$ has the L^2-Abel-convergent series representation

$$\hat{X}_t = \lim_{r\to 1_-}\sum_{k=1}^{\infty} r^k a_k X_{t-k},$$

which is equivalent to eqn (16). (ii) Since $f = gf_1$ where $f_1, f_1^{-1} \in L^{\infty}(\lambda)$, f satisfies the conditions of Theorem 3 whenever $d \in (-1, 1/2)$, and eqn (17) holds. To complete the proof, we now show that if $d \in (-\infty,-1]$, then $a_n \nrightarrow 0$ when $n \to \infty$ which implies that $a_n X_{t-n} \nrightarrow 0$ in L^2 when $n \to \infty$ and therefore, the series representation (6) cannot converge in L^2. For any $\alpha \in \mathbb{R}$, let $(\lambda_k(\alpha))_{k\in\mathbb{N}}$ be the coefficients in the Taylor series expansion of $(1 - z)^{\alpha}\phi(z)/\theta(z)$ for $|z| < 1$, and $(g_k)_{k\in\mathbb{N}}$ the coefficients in the Taylor series expansion of

$\phi(z)/\theta(z)$ for $|z| < R$, where $R = \min(|\xi_i|) > 1$. We have $a_k = -\lambda_k(d)$. For any non-negative integer m, $(\lambda_k(\alpha+m))_{k\in\mathbb{N}}$ are therefore the coefficients in the Taylor series expansion of $(1-z)^m \sum_{k=0}^{\infty} \lambda_k(\alpha) z^k$ for $|z| < 1$, and satisfy

$$\lambda_n(\alpha+m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \lambda_{n-k}(\alpha), \quad n \geq m. \tag{25}$$

Let $d \in (-\infty,-1]$, $\lfloor -d \rfloor$ the greatest integer less than or equal to $-d$, and $\delta = d + \lfloor -d \rfloor$. Then $\lfloor -d \rfloor \geq 1$, $\delta = 0$ if d is an integer, and $\delta \in (-1, 0)$ if d is a noninteger. Suppose d is an integer, and take $\alpha = d$ and $m = -d - 1$ in eqn (25). Then

$$\lambda_n(-1) = \sum_{k=0}^{-d-1} (-1)^k \binom{m}{k} \lambda_{n-k}(d), \quad n \geq -d-1. \tag{26}$$

On the other hand,

$$\lambda_n(-1) = \sum_{k=0}^{n} g_k, \quad n \geq 0. \tag{27}$$

Suppose that $\lim_{n\to\infty}\lambda_n(d) = 0$. Then according to eqn (26), $\lim_{n\to\infty}\lambda_n(-1) = 0$. But according to eqn (27),

$$\lim_{n\to\infty} \lambda_n(-1) = \sum_{k=0}^{\infty} g_k = \frac{\phi(1)}{\theta(1)} \neq 0.$$

Therefore, $\lambda_n(d) = -a_n \nrightarrow 0$ when $n \to \infty$. Suppose now that d is a noninteger. Since the sequence (g_k) converges to zero at least exponentially and $\delta \in (-1, 0)$, it results from Inoue (2002, Lem. 2.1) that

$$\lambda_n(\delta) \sim \frac{\sum_{k=0}^{\infty} g_k}{\Gamma(-\delta)} n^{-\delta-1}, \quad n \to \infty. \tag{28}$$

Since $\delta < 0$, eqn (28) implies that the series $\sum \lambda_n(\delta)$ is divergent. Taking $m = 1$ and $\alpha = \delta - 1$ in eqn (25), we see that $\lambda_n(\delta) = \lambda_n(\delta-1) - \lambda_{n-1}(\delta-1)$ for all $n \geq 1$, which implies that

$$\lambda_n(\delta-1) = \sum_{k=0}^{n} \lambda_k(\delta),$$

since $\lambda_0(\alpha) = 1$ for any $\alpha \in \mathbb{R}$. Then the sequence $(\lambda_n(\delta-1))$ is divergent. Suppose that $\lim_{n\to\infty}\lambda_n(d) = 0$. Then, taking $\alpha = d$ and $m = \lfloor -d \rfloor - 1$ in eqn (25), we see that $\lim_{n\to\infty}\lambda_n(\delta-1) = 0$, which leads to a contradiction. Then $\lambda_n(d) = -a_n \nrightarrow 0$ when $n \to \infty$. $\square$

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