On the Optimality and Performance of PID Controller for Robotic Manipulators

Youngjin Choi[*]   and   Wan Kyun Chung[†]

Department of Mechanical Engineering
Pohang University of Science & Technology(POSTECH), KOREA.
Tel: +82-54-279-2844, Fax: +82-54-279-5899

Abstract-*This paper suggests an inverse optimal PID control design method for mechanical manipulators. We find the Lyapunov function and the control law satisfying the disturbance input-to-state stability by using the characteristics of Lagrange system. Also, we show that the inverse optimal PID controller satisfies the Hamilton-Jacobi-Isaacs equation. Hence, the inverse optimality of the closed-loop system dynamics has been acquired through the PID controller, if some conditions for the control law are satisfied. Also, simple coarse/fine performance tuning laws are suggested based on the analysis for performance limitation of the inverse optimal PID controller.*

1   Introduction

The optimal control theory has been progressed recently for nonlinear mechanical systems. On the other hand, industries stick to use conventional PID controller in spite of the recent development of optimal control. Why does the industry insist on using PID controller instead of the optimal controller which may guarantee given performance level and robustness? There are some reasons. First one is the easy applicability, second one is that each term of PID controller has clear physical meanings as the present, past and predictive, and third one is that it can be utilized irrespective of the system dynamics. To overcome the gap between the industry and the academy, hence, it would be worth while to reveal the relationship between the optimal control and PID. In this paper, we analyze the optimality and performance of PID controller, especially for Lagrange systems.

Most industrial mechanical systems can be described by Lagrange equation of motion and its controller consists of the conventional PID type. The PID controller has been shown in practice to be effective for position control of Lagrange systems. But unfortunately it lacks an asymptotic stability proof until now. Under some conditions for PID gains, globally (or semi-globally) asymptotic stability of PID set-point tracking controller was proved in [4, 7, 11, 13] for the robot systems without considering the external disturbances. However, they did not consider the effect of PID gains on system performances in view of optimality.

In optimal control theories, nonlinear $\mathcal{H}_\infty$ controllers assuring the stability and prescribed performance have been proposed and progressed during the last decade. The basic theories were suggested through two papers[3, 14], where the one dealt with the control law for full state

feedback case, and the other for output feedback case. However, we still have a problem of solving the Hamilton-Jacobi-Isaacs(HJI) equation to acquire the nonlinear $\mathcal{H}_\infty$ controller, which is indeed a hard problem because it is a partial differential equation. There have been some trials to solve HJI equation: The first utilizes the approximation method, e.g., the approximated solution of HJI equation for Lagrange systems was obtained as a feasible form in [1]. The second is using the concept of extended disturbances including system error dynamics. The feasible solution of HJI equation for Lagrange systems has been acquired in [8, 10].

As a matter of fact, we need another notion to deal with disturbances. When there exist unknown bounded disturbances such as perturbations and external disturbances acting on systems, the behavior of the system should remain bounded. Also, when the set of inputs including the control and disturbance go to the zero, the behavior of system tends toward the equilibrium point. This notion for the stability is called input-to-state stability(ISS) [5, 6, 12]. ISS notion is helpful to understand the effect of disturbances on system states. The basic characteristics and properties on the ISS are summarized in the followings. For future notation, the $\mathcal{L}_2$ norm is defined by $\|\boldsymbol{x}(t)\| = \sqrt{\int_0^t \boldsymbol{x}(\tau)^T \boldsymbol{x}(\tau) d\tau}$ and the Euclidian norm is defined by $|\boldsymbol{x}(t)| = \sqrt{\boldsymbol{x}(t)^T \boldsymbol{x}(t)}$.

Now we consider a non-autonomous nonlinear system given by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t, \boldsymbol{w}) \qquad (1)$$

where $\boldsymbol{f}$ is piecewise continuous in t and locally Lipschitz in $\boldsymbol{x}$ and $\boldsymbol{w}$. The system is said to be disturbance input-to-state stable(ISS) if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that the solution for (1) exists for all $t \geq 0$ and satisfies

$$|\boldsymbol{x}(t)| \leq \beta(|\boldsymbol{x}(0)|, t) + \gamma \left(\sup_{0 \leq \tau \leq t} |\boldsymbol{w}(\tau)| \right),$$

for an initial state vector $\boldsymbol{x}(0)$ and for a disturbance vector $\boldsymbol{w}(\cdot)$ piecewise continuous and bounded on $[0, \infty)$. Especially, the ISS becomes available through the Lyapunov function based on ISS. For the system (1), there exist a smooth positive definite radially unbounded function $V(\boldsymbol{x}, t)$, a class $\mathcal{K}_\infty$ function γ_1 and a class $\mathcal{K}$ function γ_2 such that the following dissipativity inequality is satisfied:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}^T} \boldsymbol{f}(\boldsymbol{x}, t, \boldsymbol{w}) \leq -\gamma_1(|\boldsymbol{x}|) + \gamma_2(|\boldsymbol{w}|), \qquad (2)$$

if and only if the system is ISS. Also, suppose that for the system (1) there exists a function $V(\boldsymbol{x}, t)$ such that

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[*]Graduate Student, yjchoi@postech.edu
[†]Professor, wkchung@postech.edu

for all $\boldsymbol{x}$ and $\boldsymbol{w}$,

$$|\boldsymbol{x}| \geq \rho(|\boldsymbol{w}|) \quad \Rightarrow \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}^T} \boldsymbol{f}(\boldsymbol{x}, t, \boldsymbol{w}) \leq -\gamma_3(|\boldsymbol{x}|),$$

where ρ and γ_3 are class $\mathcal{K}_\infty$ functions. Then, the system is ISS and even we can say globally asymptotically stability(GAS) if the unknown disturbance input satisfies the condition $|\boldsymbol{x}| \geq \rho(|\boldsymbol{w}|)$ indeed. However, we do not know whether the disturbance $\boldsymbol{w}$ satisfies the condition or not, hence only ISS is proved. Above properties on ISS will be utilized in following sections.

2 State-Space Description of Lagrange Systems

In general, mechanical systems can be described by Lagrange mechanics. If a mechanical system with n degrees of freedom is represented by n generalized configuration coordinates $\boldsymbol{q} = [q_1, q_2, \cdots, q_n]^T \in \Re^n$, then the Lagrange system is described as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{d}(t) = \boldsymbol{\tau} \tag{3}$$

where $\boldsymbol{M}(\boldsymbol{q}) \in \Re^{n \times n}$ is Inertia matrix, $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in \Re^n$ Coriolis and centrifugal torque vector, $\boldsymbol{g}(\boldsymbol{q}) \in \Re^n$ gravitational torque vector, $\boldsymbol{\tau} \in \Re^n$ the control input torque vector and $\boldsymbol{d}(t)$ unknown external disturbances. Disturbances exerted on the system can be caused by the friction nonlinearity and so on. Also, in the trajectory tracking control case, the extended disturbance can be defined by including the external disturbance as following form:

$$\begin{aligned} \boldsymbol{w}\left(t, \dot{\boldsymbol{e}}, \boldsymbol{e}, \int \boldsymbol{e}\right) &= \boldsymbol{M}(\boldsymbol{q})\left(\ddot{\boldsymbol{q}}_d + \boldsymbol{K}_P \dot{\boldsymbol{e}} + \boldsymbol{K}_I \boldsymbol{e}\right) + \\ \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &\left(\dot{\boldsymbol{q}}_d + \boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_I \int \boldsymbol{e}\right) + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{d}(t), \end{aligned} \tag{4}$$

where $\boldsymbol{K}_P, \boldsymbol{K}_I$ are the diagonal constant matrices, $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$ is the configuration error vector and desired configurations $(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d)$ are the function of time, hence, the extended disturbance $\boldsymbol{w}$ is the function of time and configuration normal/differentiation/integration error vectors because $\boldsymbol{q}(= \boldsymbol{q}_d - \boldsymbol{e})$ and $\dot{\boldsymbol{q}}(= \dot{\boldsymbol{q}}_d - \dot{\boldsymbol{e}})$ are the function of time and configuration normal/differentiation error vectors. If the extended disturbance defined above is used in the Lagrange system of (3), then the system model can be rewritten as

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{s} = \boldsymbol{w}\left(t, \dot{\boldsymbol{e}}, \boldsymbol{e}, \int \boldsymbol{e}\right) + \boldsymbol{u}. \tag{5}$$

where $\boldsymbol{u} = -\boldsymbol{\tau}$ and $\boldsymbol{s} = \dot{\boldsymbol{e}} + \boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_I \int \boldsymbol{e} dt$.

Now, let us define the state vector as $\boldsymbol{x} = \left[\left(\int \boldsymbol{e}\right)^T, \boldsymbol{e}^T, \dot{\boldsymbol{e}}^T\right]^T$, then the state space representation of (5) can be given by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x}, t)\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{x}, t)\boldsymbol{w} + \boldsymbol{B}(\boldsymbol{x}, t)\boldsymbol{u} \tag{6}$$

where

$$\boldsymbol{A}(\boldsymbol{x}, t) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{K}_I & -\boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{K}_P - \boldsymbol{K}_I & -\boldsymbol{M}^{-1}\boldsymbol{C} - \boldsymbol{K}_P \end{bmatrix},$$

and

$$\boldsymbol{B}(\boldsymbol{x}, t) = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{M}^{-1} \end{bmatrix}.$$

This is one of generic forms for Lagrange system defined by Park et al[8, 10]. An available characteristics for Lagrange system is that the equality $(\dot{\boldsymbol{M}} = \boldsymbol{C} + \boldsymbol{C}^T)$ is always satisfied. This characteristics offers the clue to solve the inverse optimal problem for the generic Lagrange system.

3 ISS and Optimality of PID Control

Among the stability theories, the notion of disturbance input-to-state stability(ISS) is more convenient to deal with the disturbance input than other theories. Moreover, [5, 6] showed that the backstepping controller designed using ISS notion is inverse optimal for the performance index found from the controller. This has offered a useful insight from which we can show the optimality of PID controller for Lagrange systems. The following section suggests the general control Lyapunov function satisfying ISS for Lagrange systems.

3.1 ISS-CLF for Lagrange Systems

Using the input-to-state stability, Freeman defined the input-to-state stabilizable control Lyapunov function, in short, ISS-CLF[5]. The regular definition for ISS-CLF is as follows: a smooth positive definite radially unbounded function $V(\boldsymbol{x}, t) : \Re^{3n} \times \Re_+ \to \Re_+$ is called an ISS-CLF for (6) if there exists a class $\mathcal{K}_\infty$ function ρ such that the following implication holds for all $\boldsymbol{x} \neq \boldsymbol{0}$ and all $\boldsymbol{w}$:

$$|\boldsymbol{x}| \geq \rho(|\boldsymbol{w}|) \Rightarrow \inf_{\boldsymbol{u}} \{V_t + \boldsymbol{V}_x \boldsymbol{A}\boldsymbol{x} + \boldsymbol{V}_x \boldsymbol{B}\boldsymbol{w} + \boldsymbol{V}_x \boldsymbol{B}\boldsymbol{u}\} < 0. \tag{7}$$

The following Theorem suggests the ISS-CLF for Lagrange system. The Lyapunov function for the generic form of Lagrange systems was suggested in [8]. Here, we show that it satisfies the ISS and ISS-CLF under some conditions on the control law.

Theorem 1 Let $\boldsymbol{s} \triangleq \dot{\boldsymbol{e}} + \boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_I \int \boldsymbol{e} dt$. If the following control law

$$\boldsymbol{u} = -\alpha \boldsymbol{K} \boldsymbol{s} - \rho^{-1}(|\boldsymbol{x}|)\frac{\boldsymbol{s}}{|\boldsymbol{s}|} \tag{8}$$

is utilized with the condition $\alpha = \frac{1}{2}$, then $V(\boldsymbol{x}, t) = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{P}(\boldsymbol{x}, t)\boldsymbol{x}$ is an ISS-CLF for the Lagrange system (6), where

$$\boldsymbol{P}(\boldsymbol{x}, t) = \begin{bmatrix} \boldsymbol{K}_I \boldsymbol{M} \boldsymbol{K}_I + \boldsymbol{K}_I \boldsymbol{K}_P \boldsymbol{K} & \boldsymbol{K}_I \boldsymbol{M} \boldsymbol{K}_P + \boldsymbol{K}_I \boldsymbol{K} & \boldsymbol{K}_I \boldsymbol{M} \\ \boldsymbol{K}_P \boldsymbol{M} \boldsymbol{K}_I + \boldsymbol{K}_I \boldsymbol{K} & \boldsymbol{K}_P \boldsymbol{M} \boldsymbol{K}_P + \boldsymbol{K}_P \boldsymbol{K} & \boldsymbol{K}_P \boldsymbol{M} \\ \boldsymbol{M} \boldsymbol{K}_I & \boldsymbol{M} \boldsymbol{K}_P & \boldsymbol{M} \end{bmatrix} \tag{9}$$

under the following two conditions for $\boldsymbol{P}$:

1. $\boldsymbol{K}, \boldsymbol{K}_P, \boldsymbol{K}_I > \boldsymbol{0}$ constant diagonal matrices

2. $\boldsymbol{K}_P^2 > 2\boldsymbol{K}_I$.

Proof. First, if $\boldsymbol{K}_P^2 > 2\boldsymbol{K}_I > \boldsymbol{0}$ and $\boldsymbol{K} > \boldsymbol{0}$, then the proof that the suggested Lyapunov matrix $\boldsymbol{P} > \boldsymbol{0}$

can be found in [8]. Second, the components of (7) can be calculated using $\dot{\boldsymbol{M}} - \boldsymbol{C}^T - \boldsymbol{C} = \boldsymbol{0}$ as follows:

$$
\begin{aligned}
V_t + \boldsymbol{V}_x \boldsymbol{A} \boldsymbol{x} &= \frac{1}{2} \boldsymbol{x}^T \left(\dot{\boldsymbol{P}} + \boldsymbol{P} \boldsymbol{A} + \boldsymbol{A}^T \boldsymbol{P} \right) \boldsymbol{x} \\
&= \frac{1}{2} \left(\boldsymbol{s}^T \boldsymbol{K} \boldsymbol{s} - \int \boldsymbol{e}^T \boldsymbol{K}_I^2 \boldsymbol{K} \int \boldsymbol{e} \right. \\
&\quad \left. - \boldsymbol{e}^T (\boldsymbol{K}_P^2 - 2\boldsymbol{K}_I) \boldsymbol{K} \boldsymbol{e} - \dot{\boldsymbol{e}}^T \boldsymbol{K} \dot{\boldsymbol{e}} \right) (10)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{V}_x \boldsymbol{B} &= \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{B} \\
&= \boldsymbol{x}^T \left[\boldsymbol{K}_I \,,\; \boldsymbol{K}_P \,,\; \boldsymbol{I} \right]^T = \boldsymbol{s}^T . \quad (11)
\end{aligned}
$$

Now, the right hand side of (7) is calculated by using (10) and (11) as

$$
V_t + \boldsymbol{V}_x \boldsymbol{A} \boldsymbol{x} + \boldsymbol{V}_x \boldsymbol{B} \boldsymbol{w} + \boldsymbol{V}_x \boldsymbol{B} \boldsymbol{u} < 0
$$
$$
\rightarrow \frac{1}{2} \left(\boldsymbol{s}^T \boldsymbol{K} \boldsymbol{s} - \int \boldsymbol{e}^T \boldsymbol{K}_I^2 \boldsymbol{K} \int \boldsymbol{e} - \boldsymbol{e}^T (\boldsymbol{K}_P^2 - 2\boldsymbol{K}_I) \boldsymbol{K} \boldsymbol{e} \right.
$$
$$
\left. - \dot{\boldsymbol{e}}^T \boldsymbol{K} \dot{\boldsymbol{e}} \right) + \boldsymbol{s}^T \boldsymbol{w} + \boldsymbol{s}^T \boldsymbol{u} < 0
$$
$$
\rightarrow \frac{1}{2} \boldsymbol{s}^T \boldsymbol{K} \boldsymbol{s} + \boldsymbol{s}^T \boldsymbol{w} + \boldsymbol{s}^T \boldsymbol{u} < \frac{1}{2} \left(\int \boldsymbol{e}^T \boldsymbol{K}_I^2 \boldsymbol{K} \int \boldsymbol{e} \right.
$$
$$
\left. + \boldsymbol{e}^T (\boldsymbol{K}_P^2 - 2\boldsymbol{K}_I) \boldsymbol{K} \boldsymbol{e} + \dot{\boldsymbol{e}}^T \boldsymbol{K} \dot{\boldsymbol{e}} \right) . \quad (12)
$$

Let us consider only the right hand side of (12). Then we can see that it is always positive definite except $\boldsymbol{x} = \boldsymbol{0}$ under conditions 1 and 2. Also, if the control input (8) is utilized, then the left hand side of (12) becomes negative semi-definite for $\alpha \geq \frac{1}{2}$:

$$
|\boldsymbol{x}| \geq \rho(|\boldsymbol{w}|)
$$
$$
\Downarrow
$$
$$
\begin{aligned}
\frac{1}{2} \boldsymbol{s}^T \boldsymbol{K} \boldsymbol{s} + \boldsymbol{s}^T \boldsymbol{w} + \boldsymbol{s}^T \boldsymbol{u} &\leq \frac{1}{2} \boldsymbol{s}^T \boldsymbol{K} \boldsymbol{s} + |\boldsymbol{s}||\boldsymbol{w}| + \boldsymbol{s}^T \boldsymbol{u} \\
&\leq \frac{1}{2} \boldsymbol{s}^T \boldsymbol{K} \boldsymbol{s} + |\boldsymbol{s}| \rho^{-1}(|\boldsymbol{x}|) + \boldsymbol{s}^T \boldsymbol{u} \\
&= -\left(\alpha - \frac{1}{2} \right) \boldsymbol{s}^T \boldsymbol{K} \boldsymbol{s} .
\end{aligned}
$$

Therefore, (12) is always satisfied for all $\boldsymbol{x} \neq \boldsymbol{0}$, and $\alpha \geq \frac{1}{2}$. In other words, the Lagrange system is input-to-state stable(ISS) and the infimum for the control input is achieved when $\alpha = \frac{1}{2}$. Third, the suggested $V(\boldsymbol{x}, t)$ is differentiable and radially unbounded function because $V(\boldsymbol{x}, t) \rightarrow \infty$ as $\boldsymbol{x} \rightarrow \infty$. Therefore, we can conclude that $V(\boldsymbol{x}, t)$ under the control input (8) is an ISS-CLF for Lagrange systems. □

The condition of $\alpha \geq \frac{1}{2}$ guarantees the ISS and the controller based on the ISS-CLF is obtained at $\alpha = \frac{1}{2}$ because the infimum for the control input is achieved at $\alpha = \frac{1}{2}$ as shown in Theorem 1. An important characteristics of the controller (8) based on ISS is that it has the PID control type as follows:

$$
\begin{aligned}
\boldsymbol{u} &= -\left(\alpha \boldsymbol{K} + \frac{\rho^{-1}(|\boldsymbol{x}|)}{|\boldsymbol{s}|} \boldsymbol{I} \right) \left(\dot{\boldsymbol{e}} + \boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_I \int \boldsymbol{e} dt \right) \\
&\quad \text{for all} \quad \alpha \geq \frac{1}{2} .
\end{aligned}
$$

As a matter of fact, the conventional PID is a full state feedback controller which guarantees the disturbance input-to-state stability, if conditions in Theorem 1 can be satisfied. Though we obtained the PID controller based on ISS, we can not directly recognize the relationship between PID and optimality. To reveal the optimality of

PID control law, we rewrite the above PID controller as the optimal control type of $\boldsymbol{u} = -\boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{P} \boldsymbol{x}$ by letting

$$
\boldsymbol{R}(\boldsymbol{x}) = \left(\alpha \boldsymbol{K} + \frac{\rho^{-1}(|\boldsymbol{x}|)}{|\boldsymbol{s}|} \boldsymbol{I} \right)^{-1} ,
$$

where $\boldsymbol{B}^T \boldsymbol{P} \boldsymbol{x} = \dot{\boldsymbol{e}} + \boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_I \int \boldsymbol{e} dt$ as shown in (11).

3.2 Optimality of PID Control Law

To state the optimality of a control system, the Hamilton-Jacobi-Isaacs(HJI) equation should be solved for the given performance index, however, its solution is too hard to obtain for general systems including the Lagrange system. To overcome the difficulty of direct optimal problem, Kristic et al showed that the inverse optimal problem is solvable if the system is input-to-state stable(ISS)[5]. Also, Park et al showed that the nonlinear $\mathcal{H}_\infty$ control problem for robotic manipulators can be solved using the characteristics of Lagrange system[8, 10]. In [8], the Hamilton-Jacobi-Isaacs(HJI) equation and its analytic solution for Lagrange systems were suggested, but it dealt with the computed torque controller form, not a conventional PID control type.

Now, we solve the optimal control problem for Lagrange systems by using the suggested control law based on ISS-CLF in Theorem 1. Consider the general $\mathcal{H}_\infty$ performance index(PI) as following form:

$$
\begin{aligned}
PI(t, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) = \lim_{t \to \infty} &[2V(\boldsymbol{x}(t), t) \\
&+ \int_0^t \left(\boldsymbol{x}^T \boldsymbol{Q}(\boldsymbol{x}) \boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{R}(\boldsymbol{x}) \boldsymbol{u} - \gamma^2 \boldsymbol{w}^T \boldsymbol{w} \right) d\tau] .
\end{aligned} \quad (13)
$$

Also, the HJI inequality for the above cost function subject to the Lagrange system of (6) is given as follows:

$$
\begin{aligned}
HJI_\gamma = \dot{\boldsymbol{P}} + \boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A} \\
- \boldsymbol{P} \boldsymbol{B} \boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{P} + \frac{1}{\gamma^2} \boldsymbol{P} \boldsymbol{B} \boldsymbol{B}^T \boldsymbol{P} + \boldsymbol{Q} \leq \boldsymbol{0} .
\end{aligned} \quad (14)
$$

In the case of nonlinear $\mathcal{H}_\infty$ control, the above inequality plays an important role which gives the optimality and stability to the control systems. In the next Theorem, we reveal the relationship between the cost function and optimal PID controller. It is an inverse optimal problem, not direct optimal, in that both $\boldsymbol{Q}(\boldsymbol{x})$ and $\boldsymbol{R}(\boldsymbol{x})$ can be found from the gains of controller and even the HJI equation can be obtained from $\boldsymbol{Q}(\boldsymbol{x})$.

*Theorem **2*** *For a given Lagrange system (6), suppose that there exists the ISS-CLF in Theorem 1 and that the suggested PID controller of (8) is utilized with conditions*

  1. $\alpha = 1$

  2. $\rho^{-1}(|\boldsymbol{x}|) \geq \frac{1}{\gamma^2} |\boldsymbol{s}|$,

then the following control law

$$
\boldsymbol{u} = -\boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{P} \boldsymbol{x} \quad (15)
$$

is a solution of the minimization problem for the cost function (13) when using

$$
\boldsymbol{Q}(\boldsymbol{x}) = -\left(\dot{\boldsymbol{P}} + \boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A} - \boldsymbol{P} \boldsymbol{B} \boldsymbol{K} \boldsymbol{B}^T \boldsymbol{P} \right) \quad (16)
$$

$$
\boldsymbol{R}(\boldsymbol{x}) = \left(\boldsymbol{K} + \frac{\rho^{-1}(|\boldsymbol{x}|)}{|\boldsymbol{s}|} \boldsymbol{I} \right)^{-1} . \quad (17)
$$

Proof.   First, we show that the matrix $Q(x)$ of (16) is positive definite and constant matrix. Let us obtain Q using (10) and (11) in the proof of Theorem 1, then Q is acquired as the following constant matrix

$$Q = \begin{bmatrix} K_I^2 K & 0 & 0 \\ 0 & (K_P^2 - 2K_I)K & 0 \\ 0 & 0 & K \end{bmatrix}. \quad (18)$$

Hence, Q is positive definite and constant matrix. This was proved in [8] for the first time. Second, we prove the optimality of PID controller (15) by showing the minimum of the performance index. The condition of $\alpha = 1$, not $\frac{1}{2}$, makes it possible to solve the optimization problem, in other words, the optimal α is two times the value based on ISS-CLF. This fact was proved in [5] for the first time. If we put Q into the performance index (13) and use $K = R^{-1} - \frac{\rho^{-1}(|x|)}{|s|}I$ of (17), then we have the followings:

$$
\begin{aligned}
PI(t, x, u, w) &= \lim_{t \to \infty} [2V(x(t), t) \\
&- \int_0^t x^T \left[\dot{P} + A^T P + PA - PBKB^T P \right] x d\tau \\
&+ \int_0^t \left(u^T R(x)u - \gamma^2 w^T w \right) d\tau] \\
&= \lim_{t \to \infty} [2V(x(t), t) - \\
&\int_0^t \left(x^T \left[\dot{P} + A^T P + PA \right] x + 2x^T PBu + 2x^T PBw \right) d\tau \\
&+ \int_0^t \left(u^T R(x)u + 2x^T PBu + x^T PBKB^T Px \right. \\
&\left. + 2x^T PBw - \gamma^2 w^T w \right) d\tau] \\
&= \lim_{t \to \infty} \left[2V(x(t), t) - 2\int_0^t \dot{V} d\tau \right. \\
&+ \int_0^t (u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px) d\tau \\
&- \gamma^2 \int_0^t \left(w - \frac{1}{\gamma^2}B^T Px \right)^T \left(w - \frac{1}{\gamma^2}B^T Px \right) d\tau \\
&\left. - \int_0^t \left(\frac{\rho^{-1}(|x|)}{|s|} - \frac{1}{\gamma^2} \right) x^T PBB^T Px d\tau \right] \\
&= 2V(x(0), 0) \\
&+ \int_0^\infty (u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px) d\tau \\
&- \gamma^2 \int_0^\infty \left| w - \frac{1}{\gamma^2}B^T Px \right|^2 d\tau - \int_0^\infty \left(\frac{\rho^{-1}(|x|)}{|s|} - \frac{1}{\gamma^2} \right) |s|^2 d\tau.
\end{aligned}
$$
$$(19)$$

From (19), we can see that the minimum for the $\mathcal{H}_\infty$ performance index is achieved in the case that the control law is (15). In this case, the worst case disturbance is given by

$$w^* = \frac{1}{\gamma^2} B^T Px$$

and $|w^*| = \frac{1}{\gamma^2}|s|$. Also, the condition of $\rho^{-1}(|x|) \geq |w^*| = \frac{1}{\gamma^2}|s|$ should be satisfied for the minimization of the performance index. Therefore, we conclude that the control law (15) satisfying the conditions in Theorems 1 and 2 minimizes the cost function of (13).   □

Although the HJI equation is not explicitly utilized to show the optimality of PID control law in Theorem 2, the Q of (16) contains implicitly the HJI equation as following form:

$$
\begin{aligned}
HJI_\rho &= \dot{P} + A^T P + PA \\
&- PBR^{-1}B^T P + \frac{\rho^{-1}(|x|)}{|s|}PBB^T P + Q = 0.
\end{aligned}
$$
$$(20)$$

From the inequality of (14) and equality of (20), we can perceive the following relation:

$$HJI_\gamma \leq HJI_\rho = 0.$$

4 Performance of Inverse Optimal PID Control

Till now, we showed that the inverse optimality of PID controller type for a general $\mathcal{H}_\infty$ performance index could be achieved through Theorem 1 and 2. This type of controller (15) is not practical, however, since it contains the gain dependent on the unknown function of state vector. To be a static PID control type, we take the function $\rho^{-1}(|x|)$ as the lowest bound value $\frac{1}{\gamma^2}|s|$. If the PID controller is designed according to conditions in the following Theorem, then we can also show that the PID control system is ISS.

Theorem 3 If the inverse optimal PID controller

$$
\begin{aligned}
\tau &= R^{-1}B^T Px \\
&= (K + \frac{1}{\gamma^2}I)\left(\dot{e} + K_P e + K_I \int e \right) \quad (21)
\end{aligned}
$$

satisfying the following conditions:

1. $K, K_P, K_I > 0$, *constant diagonal matrices*

2. $K_P^2 > 2K_I$,

3. $\gamma > 0$

is applied to (6), then the closed-loop control system is disturbance input-to-state stable(ISS).

Proof.   If the suggested inverse optimal PID controller is applied to the Lagrange system (6), then HJI of (14) is satisfied. Therefore, along the solution trajectory of (6) with the control law (21), we get the time derivative of Lyapunov function:

$$
\begin{aligned}
\dot{V} &= V_t + V_x Ax + V_x Bu + V_x Bw \\
&= \frac{1}{2}x^T \left(\dot{P} + A^T P + PA \right) x \\
&\quad - x^T PBR^{-1}B^T Px + x^T PBw,
\end{aligned}
$$

where $u = -\tau$. If we rearrange the above equation using the HJI (14) and $x^T PBw \leq \frac{1}{\gamma^2}|x^T PB|^2 + \gamma^2|w|^2$, then we get the following similar to (2):

$$\dot{V} \leq -\frac{1}{2}x^T \left(Q + PBKB^T P \right) x + \gamma^2|w|^2. \quad (22)$$

Since the right hand side of above inequality (22) is unbounded function for x and w respectively, hence, the Lagrange system with the inverse optimal PID controller (21) is disturbance input-to-state stable(ISS).   □

4.1 PID Does Not Give GAS

Through Theorem 3, we showed that the inverse optimal PID controller brings the disturbance input-to-state stability(ISS). However, it does not give the global asysmptotic stability(GAS). This fact is explained by the

characteristics of extended disturbance in this section. The extended disturbance of (4) is expressed as a function of time and state vector as following form:

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{H}(\boldsymbol{x},t)\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x},t) \qquad (23)$$

where

$$\begin{aligned}
\boldsymbol{H}(\boldsymbol{x},t) &= [\ \boldsymbol{C}\boldsymbol{K}_I\ ,\ \boldsymbol{M}\boldsymbol{K}_I + \boldsymbol{C}\boldsymbol{K}_P\ ,\ \boldsymbol{M}\boldsymbol{K}_P\ ] \\
\boldsymbol{h}(\boldsymbol{x},t) &= \boldsymbol{M}\ddot{\boldsymbol{q}}_d + \boldsymbol{C}\dot{\boldsymbol{q}}_d + \boldsymbol{g} + \boldsymbol{d}.
\end{aligned}$$

Now, consider the Euclidian norm of the extended disturbance of (23). Then we get the insight such that the extended disturbance can be bounded by the function of Euclidian norm of state vectors under the following two assumptions:

(A1): the configuration velocity $\dot{\boldsymbol{q}}$ is bounded
(A2): the external disturbance $\boldsymbol{d}(t)$ is bounded.

The first assumption is not a hard condition to be satisfied if and only if the applied controller can stabilize the system. Also, we think that the second assumption is a minimal information for the unknown external disturbance. By the boundedness of $\dot{\boldsymbol{q}}$, the Coriolis and centrifugal matrix $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ can be bounded, e.g., $|\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})| \leq c_0|\dot{\boldsymbol{q}}|$. Additionally, we know that the gravitational torque $\boldsymbol{g}(\boldsymbol{q})$ is bounded if the system stays at the earth, and the Inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ is bounded by its own maximum eigenvalue, e.g., $|\boldsymbol{M}(\boldsymbol{q})| \leq m$. Also, the desired configuration normal/velocity/acceleration vectors are specified as bounded values. Therefore, we can derive the following relationship from above assumptions:

$$\begin{aligned}
|\boldsymbol{w}|^2 &= \boldsymbol{x}^T(\boldsymbol{H}^T\boldsymbol{H})\boldsymbol{x} + 2(\boldsymbol{h}^T\boldsymbol{H})\boldsymbol{x} + (\boldsymbol{h}^T\boldsymbol{h}) \\
&\leq c_1|\boldsymbol{x}|^2 + c_2|\boldsymbol{x}| + c_3 \qquad (24)
\end{aligned}$$

where c_1, c_2 and c_3 are some positive constants. Under above assumptions, we know that the Euclidian norm of the extended disturbance can be upper bounded by the function of Euclidian norm of the state vector, conversely, the Euclidian norm of the state vector can be lower bounded by the inverse function of that of the extended disturbance

$$|\boldsymbol{w}| \leq \rho_o^{-1}(|\boldsymbol{x}|) \qquad \leftrightharpoons \qquad \rho_o(|\boldsymbol{w}|) \leq |\boldsymbol{x}|,$$

where $\rho_o(|\boldsymbol{w}|) = 0$ for $0 \leq |\boldsymbol{w}| \leq \sqrt{c_3}$ because when $0 \leq |\boldsymbol{w}| \leq \sqrt{c_3}$, necessarily $\boldsymbol{x} = \boldsymbol{0}$. Also, the constant c_3 of (24) can not be zero either in the case of trajectory tracking or in the presence of external disturbances. Though the function ρ_o must be a continuous and increasing function and $\rho_o(|\boldsymbol{w}|) \to \infty$ as $|\boldsymbol{w}| \to \infty$, the function ρ_o is not a class $\mathcal{K}_\infty$ function because it is not strictly increasing as shown in Figure 1.

Since ρ_o is not a class $\mathcal{K}_\infty$ function and especially c_3 is an unknown coefficient, the GAS can not be proved in this setup. On the other hand, if there exist no external disturbances ($\boldsymbol{d}(t) = \boldsymbol{0}$) and gravity torques ($\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0}$), then the GAS can be proved in the case of the set-point
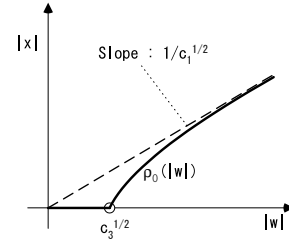
Figure 1: Function $\rho_o(|\boldsymbol{w}|)$

tracking control ($\ddot{\boldsymbol{q}}_d = \boldsymbol{0}, \dot{\boldsymbol{q}}_d = \boldsymbol{0}$) because c_3 of (24) is zero and the function ρ_o becomes a class $\mathcal{K}_\infty$. For the first time, this fact was proved for mechanical systems in [13]. However, either in the trajectory tracking or in the existence of external disturbance, the inverse optimal PID control system does not ensure the GAS. Therefore, the inverse optimal PID controller guarantees only ISS. Also, the fact that ρ_o is not a class $\mathcal{K}_\infty$ function brings a performance limitation of the inverse optimal PID control system. The analysis on the performance limitation will be given in the following section.

4.2 Performance Limitation and Tuning

The control performance is determined by gain values of controller. Hence, it is important to perceive the relation between gain values and the error. This relationship can be found by examining the point that the time derivative of Lyapunov function is equal to zero. Since the inverse optimal PID controller can not achieve GAS, the performance limitation of control system appears. The following Theorem suggests the mathematical expression for the Euclidian norm of state vector as a measure for the performance limitation.

Theorem 4 Let $\boldsymbol{K} = k\boldsymbol{I}, \boldsymbol{K}_P = k_P\boldsymbol{I}$ and $\boldsymbol{K}_I = k_I\boldsymbol{I} \in \Re^{n \times n}$. Suppose that λ_{min} is the minimum eigenvalue of the following matrix

$$\boldsymbol{Q}_K = \boldsymbol{Q} + \boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\boldsymbol{B}^T\boldsymbol{P},$$

and that $|\boldsymbol{x}|_{P.L}$ is the performance limitation satisfying $\dot{V} = 0$. If the inverse optimal PID controller in Theorem 3 is applied to the Lagrange system of (6) and λ_{min} is chosen sufficiently large and γ sufficiently small so that $\lambda_{min} - 2\gamma^2 c_1 > 0$ can be satisfied, then its performance limitation is upper bounded by

$$|\boldsymbol{x}|_{P.L} \leq \frac{\gamma^2}{\lambda_{min} - 2\gamma^2 c_1}\left[c_2 + \sqrt{c_2^2 + \frac{2c_3}{\gamma^2}(\lambda_{min} - 2\gamma^2 c_1)}\right]$$

(25)

where c_1, c_2, c_3 are coefficients for the upper bound of the extended disturbance (24) and the minimum eigenvalue of $\boldsymbol{Q}_K$ is determined by

$$\lambda_{min} \geq \min\left\{k, (k_P^2 - 2k_I)k, k_I^2 k\right\}. \qquad (26)$$

This equation (25) can be regarded as the performance prediction equation which can predict the performance of the closed-loop system as gain changes.
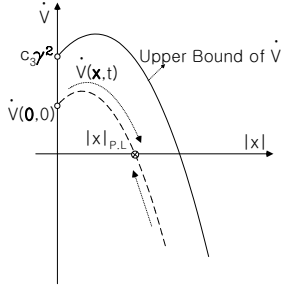
Figure 2: Relation b/w $|\boldsymbol{x}|_{P.L}$ and the upper bound of $\dot{V}$

Proof. First, to show the performance limitation, we examine the point that the time derivative of Lyapunov function (22) stays at zero:

$$\dot{V}(\boldsymbol{x},t) \leq -\frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}_K\boldsymbol{x} + \gamma^2|\boldsymbol{w}|^2$$
$$\leq -\frac{1}{2}\lambda_{min}|\boldsymbol{x}|^2 + c_1\gamma^2|\boldsymbol{x}|^2 + c_2\gamma^2|\boldsymbol{x}| + c_3\gamma^2. \quad (27)$$

Here, we know that the error can not be further reduced when $\dot{V} = 0$, and it brings the expression (25) for the performance limitation. Second, let us consider the minimum eigenvalue λ_{min} of the matrix $\boldsymbol{Q}_K$:

$$\boldsymbol{Q}_K = \begin{bmatrix} kk_I^2\boldsymbol{I} & 0 & 0 \\ 0 & k(k_P^2 - 2k_I)\boldsymbol{I} & 0 \\ 0 & 0 & k\boldsymbol{I} \end{bmatrix}$$
$$+ k\begin{bmatrix} k_I\boldsymbol{I} \\ k_P\boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix}[k_I\boldsymbol{I}, k_P\boldsymbol{I}, \boldsymbol{I}]$$
$$= \boldsymbol{Q} + k\boldsymbol{Z}\boldsymbol{Z}^T$$

where $\boldsymbol{Q}$ is the diagonal positive definite matrix and $\boldsymbol{Z}\boldsymbol{Z}^T$ is a symmetric positive semi-definite matrix. Therefore, the following inequality is always satisfied by Weyl's Theorem in [2]

$$\lambda_{min}(\boldsymbol{Q}) + \lambda_{min}(k\boldsymbol{Z}\boldsymbol{Z}^T) \leq \lambda_{min} = \lambda_{min}(\boldsymbol{Q}_K). \quad (28)$$

Since $\lambda_{min}(k\boldsymbol{Z}\boldsymbol{Z}^T)$ is zero, the minimum eigenvalue of $\boldsymbol{Q}_K$ is not smaller than the minimum value among diagonal entries of $\boldsymbol{Q}$. □

Above analysis can be explained easily as follows: In the case of trajectory tracking control for robot manipulators, we start the simulation/experiment with zero error $\boldsymbol{x} = \boldsymbol{0}$ after adjusting initial conditions. The value of Lyapunov function is zero at the start time, however, the error increases to some extent because $\dot{V}(\boldsymbol{0}, 0)$ may have any positive constant smaller than $c_3\gamma^2$ as shown in Figure 2. This Figure depicts the upper bound of $\dot{V}$ vs. $|\boldsymbol{x}|$ of the equation (27). Since the suggested performance limit $|\boldsymbol{x}|_{P.L}$ is the convergent point as we can see in Figure 2, the Euclidian norm of error tends to stay at this point. This analysis can naturally illustrate the gain tuning.

The PID gain tuning has been an important subject, however, it has not been much investigated till now. Recently, the noticeable tuning method was suggested as the name of "square law" by Park et al[9]. They showed that the square law is a good tuning method through their experiments for the industrial robot manipulator. Theoretically, we can confirm once more that the square law is a good tuning method by showing that the performance limitation of (25) can be written approximately as the following form:

$$|\boldsymbol{x}|_{P.L} \quad \propto \quad \gamma^2,$$

where the square law means that the error is approximately reduced to the square times of the reduction ratio for γ values. Though the square law is a good performance tuning method, it is not always exact or applicable. The exact performance tuning measure is the performance limitation of (25) in Theorem 4, however, the coefficients c_1, c_2, c_3 are unknowns. To develop the available and more exact tuning method, we rewrite the performance limitation (25) as follows:

$$|\boldsymbol{x}|_{P.L} \leq \left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right)^2\left[c_2 + \sqrt{c_2^2 + 2c_3\left(\frac{\sqrt{\lambda_\gamma}}{\gamma}\right)^2}\right]$$
$$\leq \left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right)^2\left[2c_2 + \sqrt{2c_3}\left(\frac{\sqrt{\lambda_\gamma}}{\gamma}\right)\right]$$
$$= 2c_2\left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right)^2 + \sqrt{2c_3}\left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right) \quad (29)$$

where $\lambda_\gamma = \lambda_{min} - 2\gamma^2 c_1$. Since the value of γ can be chosen sufficiently small, we assume that $\lambda_\gamma \approx \lambda_{min}$. Also the values of $k_P^2 - 2k_I$ and k_I are selected such that its values are bigger than 1, in other words, $k_P^2 - 2k_I > 1$ and $k_I > 1$. Then, the value of λ_{min} is lower bounded by k, $\lambda_{min} \geq k$, from (26). By letting $\lambda_\gamma \approx k$ and defining the tuning variable as $(\gamma/\sqrt{k})$, the performance limitation of (29) can be expressed as the 2nd order function of tuning variable $(\gamma/\sqrt{k})$. In the case of a large tuning variable, since the second order term governs the inequality (29), the following square tuning law is approximately obtained from (29):

$$|\boldsymbol{x}|_{P.L} \propto \gamma^2, \quad \text{for a small } \sqrt{k}. \quad (30)$$

Also, if the tuning variable $(\gamma/\sqrt{k})$ is small, then we can perceive another linear tuning law because the first order term of (29) becomes dominant

$$|\boldsymbol{x}|_{P.L} \propto \gamma, \quad \text{for a large } \sqrt{k}. \quad (31)$$

Here, we propose two tuning methods; one is the coarse tuning which brings the square relation of (30) and the other is the fine tuning which brings the linear relation of (31). Roughly speaking, the coarse tuning is achieved for a small k value and the fine tuning for a large k. One more thing is that, when γ is small, the validity of coarse/fine tuning method is more obvious because the

robustness for uncertainties is enhanced. The effect of the uncertain parameter c_1 of $\lambda_\gamma = \lambda_{min} - 2\gamma^2 c_1$ in the performance limitation (29) fades out to zero when γ is very small. Therefore, the coarse/fine tunings of (30) and (31) are more valuable when γ is sufficiently small.

4.3 Numerical Examples

To show the optimality and performance limitation of the inverse optimal PID controller numerically, we utilize two link robot manipulator. The masses and moments of inertia of each link are set all to 1 and even each link length is set to 1 to simplify the simulation procedures. The desired trajectory is the line trajectory and it consists of the fifth order polynomial function of time so that the initial/final velocity and acceleration can be set to zero. The start and end points of Cartesian coordinates (X, Z) in vertical plane are set to $(1.414, 0.0)$ and $(0.0, 1.414)$. The execution time is 5 seconds. After executing many simulations using the inverse optimal PID controller for various k and γ values, the data in Table 1 were obtained for the fixed constant gains $k_P = 20$ and $k_I = 100$. The $\mathcal{L}_2$ norm performance as the mean performance is evaluated by

$$\|\boldsymbol{x}\| = \sqrt{\int_0^5 \left[\dot{\boldsymbol{e}}^T \dot{\boldsymbol{e}} + \boldsymbol{e}^T \boldsymbol{e} + \int \boldsymbol{e}^T \int \boldsymbol{e} \right] \, dt}.$$

Finally, the inverse optimal PID controller is

$$\boldsymbol{\tau} = \left(k + \frac{1}{\gamma^2} \right) \left(\dot{\boldsymbol{e}} + 20\boldsymbol{e} + 100 \int \boldsymbol{e} dt \right)$$

and it is optimal for the $\mathcal{H}_\infty$ performance index of (13) with

$$\boldsymbol{Q} = \begin{bmatrix} 10^4 k \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 200k\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & k\boldsymbol{I} \end{bmatrix}$$

$$\boldsymbol{R} = \left(k + \frac{1}{\gamma^2} \right)^{-1} \boldsymbol{I}.$$

In Table 1, the coarse performance tuning is shown in the upper part of Table, whenever γ is halved for a small $\sqrt{k}$, the $\mathcal{L}_2$ norm of error is reduced to a quarter approximately by the square tuning law. On the other hand, the fine performance tuning is shown in the lower part of Table, whenever γ is halved for a large $\sqrt{k}$, the $\mathcal{L}_2$ norm of error is reduced to a half approximately by the linear tuning law.

5 Concluding Remarks

The inverse optimality of PID controller type for Lagrange systems was proved analytically using the ISS concept. Also, the performance of inverse optimal PID controller was analyzed using the analysis for performance limitation. Through the analysis, we proposed the performance prediction equation including the coarse/fine performance tuning rules.

k	γ	$\|\boldsymbol{x}\|$	$\|\boldsymbol{x}\|_u/\|\boldsymbol{x}\|_l$	Expected
10.0	0.2	0.09985	4.32	
	0.1	0.02310	4.00	
	0.05	0.00577		4.0
20.0	0.2	0.06832	3.26	
	0.1	0.02098	3.73	
	0.05	0.00563		
100.0	0.2	0.02007	1.65	
	0.1	0.01213	2.57	
	0.05	0.00471		2.0
200.0	0.2	0.01072	1.35	
	0.1	0.00795	2.03	
	0.05	0.00392		

Table 1: The simulation results for various γ and k, where the upper part of Table complies with the square tuning law and the lower part complies with the linear tuning law

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