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Generalized vector variational-like inequalities and vector optimization

Qamrul Hasan Ansari · Mahboubeh Rezaie · Jafar Zafarani

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Abstract In this paper, we consider different kinds of generalized vector variational-like inequality problems and a vector optimization problem. We establish some relationships between the solutions of generalized Minty vector variational-like inequality problem and an efficient solution of a vector optimization problem. We define a perturbed generalized Stampacchia vector variational-like inequality problem and discuss its relation with generalized weak Minty vector variational-like inequality problem. We establish some existence results for solutions of our generalized vector variational-like inequality problems.

Keywords Generalized vector variational-like inequalities · Vector optimization problems · Pseudoinvexity · Quasimonotonicity · Properly quasimonotonicity · Clarke's generalized subdifferential

1 Introduction

In 1998, Giannessi [8] first used, so called, Minty type vector variational inequality (in short, MVVI) to establish the necessary and sufficient conditions for a point to be an efficient

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Q. H. Ansari (✉)
Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India
e-mail: qhansari@gmail.com

M. Rezaie · J. Zafarani
Department of Mathematics, University of Isfahan, Isfahan 81745-163, Iran
e-mail: mrezaie@sci.ui.ac.ir

J. Zafarani
e-mail: jzar@sci.ui.ac.ir; jzaf@zafarani.ir

J. Zafarani
University of Sheikhbahaee, Isfahan, Iran

solution of a vector optimization problem (in short, VOP) for differentiable and convex functions. Since then, several researchers have studied VOP by using different kinds of MVVI under different assumptions. See, for example, [1,3,5,7,9,11,13–16,19,20] and references therein. Yang et al. [19] extended the results of Giannessi [8] for differentiable but pseudoconvex functions. Recently, Yang and Yang [16] gave some relationships between Minty vector variational-like inequality problem (MVVLIP) and VOP for differentiable but pseudoinvex vector-valued functions. In particular, they extended the results of Giannessi [8] and Yang et al. [19] for differentiable but pseudoinvex vector-valued functions. Very recently, Al-Homidan and Ansari [1] and Rezaie and Zafarani [15] considered different kinds of generalized Minty vector variational-like inequality problems, generalized Stampacchia vector variational-like inequality problems and a nonsmooth vector optimization problem under nonsmooth invexity assumption. In [1], we studied the relationship among these problems under nonsmooth invexity assumption. We also considered the weak formulations of a generalized Minty vector variational-like inequality problem and a generalized Stampacchia vector variational-like inequality problem and gave some relationships between the solutions of these problems and a weak efficient solution of a vector optimization problem.

In this paper, we consider generalized Minty vector variational-like inequality problems, generalized Stampacchia vector variational-like inequality problems and nonsmooth vector optimization problems under nonsmooth pseudoinvexity assumption. We study the relationships among these problems under nonsmooth pseudoinvexity assumption. We also consider the weak formulations of generalized Minty vector variational-like inequality problems and generalized Stampacchia vector variational-like inequality problems in a very general setting and establish the existence results for their solutions. The result of this paper either generalize or different from those appeared in [1,3,5,7,8,11,15,16,19].

2 Preliminaries

Throughout this section, unless otherwise specified, we assume that K is a nonempty subset of $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ is a given map. The interior of K is denoted by int K.

Let $f = (f_1, \dots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function. We consider the following *vector optimization problem*:

(VOP) Minimize $f(x) = (f_1(x), \dots, f_\ell(x))$ subject to $x \in K$.

A point $\bar{x} \in K$ is said to be an *efficient* (or *Pareto*) *solution* of (VOP) if

$$f(y) - f(\bar{x}) = (f_1(y) - f_1(\bar{x}), \dots, f_\ell(y) - f_\ell(\bar{x})) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K,$$

where $\mathbb{R}^\ell_+$ is the nonnegative orthant of $\mathbb{R}^\ell$ and $\mathbf{0}$ is the zero vector of $\mathbb{R}^\ell$.

Definition 1 [4] Let $g : K \to \mathbb{R}$ be locally Lipschitz at a given point $x \in K$. The *Clarke's generalized directional derivative* of g at $x \in K$ in the direction of a vector $v \in K$, denoted by $g^\circ(x; v)$, is defined by

$$g^\circ(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{g(y + tv) - g(y)}{t}.$$

Definition 2 [4] Let $g : K \to \mathbb{R}$ be locally Lipschitz at a given point $x \in K$. The *Clarke's generalized subdifferential* of g at $x \in K$, denoted by $\partial^c g(x)$, is defined by

$$\partial^c g(x) = \left\{\xi \in \mathbb{R}^n : g^\circ(x; v) \geq \langle \xi, v \rangle \text{ for all } v \in \mathbb{R}^n\right\},$$

where $\langle ., . \rangle$ denotes the scalar product in $\mathbb{R}^n$.

We note that $\partial^c g(x)$ is a nonempty, convex and compact subset of $\mathbb{R}^n$ if g is locally Lipschitz on K.

Theorem 1 (Lebourg Mean Value Theorem) [4] *Let x and y be points in $K \subseteq \mathbb{R}^n$ and suppose that $g : K \to \mathbb{R}$ is locally Lispschitz on an open set containing the line segment $[x, y]$. Then there exists a point $u \in (x, y)$ such that*

$$g(x) - g(y) \in \left\langle \partial^c g(u), x - y \right\rangle,$$

where (x, y) denotes the line segment joining x and y excluding the end points x and y.

A mapping $\eta : K \times K \to \mathbb{R}^n$ is said to be *skew* if for all $x, y \in K$,

$$\eta(y, x) + \eta(x, y) = 0.$$

Definition 3 Let x be an arbitrary point of K. The set K is said to be *invex at x* w.r.t. η if for all $y \in K$,

$$x + t\eta(y, x) \in K, \quad \text{for all } t \in [0, 1].$$

K is said to be *invex* w.r.t. η if K is invex at every point $x \in K$ w.r.t. η.

Condition C. Let $K \subseteq \mathbb{R}^n$ be an invex set w..r.t. $\eta : K \times K \to \mathbb{R}^n$. Then, for all $x, y \in K$ and all $t \in [0, 1]$,

(a) $\eta(x, x + t\eta(y, x)) = -t\eta(y, x)$
(b) $\eta(y, x + t\eta(y, x)) = (1 - t)\eta(y, x)$.

Obviously, the map $\eta(y, x) = y - x$ satisfies Condition C. The examples of the map η that satisfies Condition C are given in [17,18].

Gang and Liu [7] considered the following Condition C*.

Condition C.* Let $K \subseteq \mathbb{R}^n$ be an invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$. We say that the mapping $\eta : K \times K \to \mathbb{R}^n$ satisfies the Condition C* if for any $x, y \in X$ and for all $t \in [0, 1]$

$$\eta(x, x + t\eta(y, x)) = -\alpha(t)\eta(y, x)$$

and

$$\eta(y, x + t\eta(y, x)) = \beta(t)\eta(y, x),$$

where $\alpha(t) > 0,\ \beta(t) > 0$ for all $t \in (0, 1)$.

Remark 1 We note that if η satisfies the Condition C, then it satisfies the Condition C*. However, the converse is not true in general.

Example 1 [7] Let $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\eta(x, y) := \begin{cases} x - y & \text{if } x \geq 0,\ y \geq 0, \\ \frac{1}{2}(x - y) & \text{if } x \leq 0,\ y \leq 0, \\ \frac{1}{3}(x - y) & \text{if } x > 0,\ y < 0, \\ \frac{1}{3}(x - y) & \text{if } x < 0,\ y > 0. \end{cases}$$

It is easy to check that η satisfies Condition C* and is skew, but it does not satisfy Conditions C.

Definition 4 Let $g : K \to \mathbb{R}$ be locally Lipschitz at a given point $x \in K$. Then g is said to be

(a) *invex* w.r.t. η on K if for all $x, y \in K$ and all $\xi \in \partial^c g(x)$,

$$\langle \xi, \eta(y, x) \rangle \leq g(y) - g(x);$$

(b) *pseudoinvex* w.r.t. η on K if for all $x, y \in K$ and all $\xi \in \partial^c g(x)$,

$$\langle \xi, \eta(y, x) \rangle \geq 0 \quad \text{implies} \quad g(y) \geq g(x);$$

(c) *strictly pseudoinvex* w.r.t. η on K if for all $x, y \in K$ with $x \neq y$ and all $\xi \in \partial^c g(x)$,

$$\langle \xi, \eta(y, x) \rangle \geq 0 \quad \text{implies} \quad g(y) > g(x);$$

(d) *quasiinvex* w.r.t. η on K if for all $x, y \in K$ and all $\xi \in \partial^c g(x)$,

$$g(y) \leq g(x) \quad \text{implies} \quad \langle \xi, \eta(y, x) \rangle \leq 0.$$

Definition 5 Let $K \subseteq \mathbb{R}^n$ be an invex set w.r.t. η. A function $g : K \to \mathbb{R}$ is said to be

(a) *preinvex* w.r.t. η if

$$g(x + t\eta(y, x)) \leq tg(y) + (1 - t)g(x), \quad \text{for all } x, y \in K \text{ and all } t \in [0, 1];$$

(b) *prequasiinvex* w.r.t. η on K if for all $x, y \in K$, $0 \leq t \leq 1$,

$$g(x + t\eta(y, x)) \leq \max\{g(x), g(y)\};$$

(c) *semi-strictly prequasiinvex* w.r.t. η on K if for all $x, y \in K$, $0 < t < 1$ with $g(x) \neq g(y)$,

$$g(x + t\eta(y, x)) < \max\{g(x), g(y)\}.$$

Theorem 2 [15, Theorem 3.1] *If $g : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is pseudoinvex w.r.t. η, then g is a semi-strictly prequasiinvex function w.r.t. the same η.*

Theorem 3 [15, Theorem 3.2] *Let $g : K \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function and let $\eta : K \times K \to \mathbb{R}^n$ be satisfy Condition C.*

(a) *If g is quasiinvex w.r.t. η, then g is prequasiinvex w.r.t. the same η.*
(b) *If g is prequasiinvex w.r.t. η and the mapping $x \mapsto \eta(y, x)$ is continuous, then g is quasiinvex w.r.t. the same η.*

Theorem 4 [16, Theorem 2.2] *If $g : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous function and semi-strictly prequasiinvex w.r.t. η on K, then g is prequasiinvex w.r.t the same η on K.*

Remark 2 When $g : K \to \mathbb{R}$ is pseudoinvex w.r.t. η, then by Theorem 2, g is semi-strictly prequasiinvex w.r.t. the same η, and hence, Theorem 4 implies that g is prequasiinvex w.r.t. η if it is lower semicontinuous.

Theorem 5 [10, Theorem 4.1] *If $g : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz and quasiinvex w.r.t. η on K. Then $\partial^c g$ is quasimonotone w.r.t. $\eta : K \times K \to \mathbb{R}^n$, that is, for all $x, y \in K$ and all $\xi \in \partial^c g(x)$, $\zeta \in \partial^c g(y)$, we have*

$$\langle \xi, \eta(y, x) \rangle > 0 \quad \textit{implies} \quad \langle \zeta, \eta(x, y) \rangle \leq 0.$$

Let K be a nonempty convex subset of a vector space X. A mapping $F : K \to 2^X$ is said to be a *KKM mapping* if for each nonempty finite subset A of K, $\text{conv}A \subset F(A)$, where $\text{conv}A$ denotes the convex hull of A, and $F(A) = \bigcup\{F(x) : x \in A\}$.

The following form of Fan-KKM lemma is appeared in [6].

Lemma 1 *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $\Gamma,\ \hat{\Gamma} : K \to 2^K$ be two set-valued maps such that the following conditions hold:*

(A1) *For all $x \in K$, $\hat{\Gamma}(x) \subseteq \Gamma(x)$;*
(A2) *Gam$\hat{\ }$ma is a KKM map;*
(A3) *For all $x \in K$, $\Gamma(x)$ is closed,*
(A4) *there is a nonempty compact convex set $B \subseteq K$ such that $cl_K\left(\bigcap_{x\in B}\Gamma(x)\right)$ is compact.*

Then $\bigcap\limits_{x\in K}\Gamma(x) \neq \emptyset$.

3 Generalized Minty vector variational-like inequalities and vector optimization

All the results of this section can be extended to Banach space setting. However, for the sake of simplicity, we consider finite dimensional space $\mathbb{R}^n$.

Let K be a nonempty subset of $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ be a given map. We denote by $\mathbf{0}$ the zero element of $\mathbb{R}^\ell$. Let $f = (f_1, \ldots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function such that each f_i is locally Lipschitz on K. We consider the following *generalized Minty vector variational-like inequality problems*:

(GGMVVLIP) (GMVVLIP) Find $\bar{x} \in K$ such that for all $x \in K$, there exists $\zeta_i \in \partial^c f_i(x)$, $i \in \mathscr{I} = \{1, \ldots, \ell\}$, satisfying

$$\langle \zeta, \eta(x, \bar{x})\rangle_\ell = (\langle \zeta_1, \eta(x, \bar{x})\rangle, \ldots, \langle \zeta_\ell, \eta(x, \bar{x})\rangle) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}.$$

(GGMVVLIIP) (GGMVVLIP) Find $\bar{x} \in K$ such that for all $x \in K$ and all $\zeta_i \in \partial^c f_i(x)$, $i \in \mathscr{I} = \{1, \ldots, \ell\}$,

$$\langle \zeta, \eta(x, \bar{x})\rangle_\ell = (\langle \zeta_1, \eta(x, \bar{x})\rangle, \ldots, \langle \zeta_\ell, \eta(x, \bar{x})\rangle) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}.$$

(GGMVVLIIIP) (GWMVVLIP) Find $\bar{x} \in K$ such that for all $x \in K$, there exists $\zeta_i \in \partial^c f_i(x)$, $i \in \mathscr{I} = \{1, \ldots, \ell\}$, satisfying

$$\langle \zeta, \eta(x, \bar{x})\rangle_\ell = (\langle \zeta_1, \eta(x, \bar{x})\rangle, \ldots, \langle \zeta_\ell, \eta(x, \bar{x})\rangle) \notin -\text{int}\,\mathbb{R}^\ell_+.$$

(GGMVVLIP) and (GWMVVLIP) are considered and studied in [1] with further applications to (VOP). The relationship between a solution of (GMVVLIP) and an efficient solution of (VOP) is established in [1] under the condition that each f_i is preinvex. The existence of solutions of (GGMVVLIP) is studied in [2]. When $\eta(y, x) = y - x$, then (GGMVVLIP) reduces to the generalized Minty vector variational inequality problem considered and studied in [12]. Of course, (GGMVVLIP) is more general than (GMVVLIP) as every solution of (GGMVVLIP) is a solution of (GMVVLIP).

Theorem 6 *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that η is skew and satisfies Condition C. For each $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$, let f_i be pseudoinvex w.r.t. η, locally Lipschitz on K and let the mapping $y \mapsto \eta(x, y)$ be continuous.*

(a) *If $\bar{x} \in K$ is a solution of (GGMVVLIP), then it is an efficient solution of (VOP).*
(b) *If $\bar{x} \in K$ is an efficient solution of (VOP), then it is a solution of (GMVVLIP).*

Proof (a) Let $\bar{x}$ be a solution of (GGMVVLIP) but not an efficient solution of (VOP). Then there exists $x_0 \in K$ such that

$$f(\bar{x}) - f(x_0) = (f_1(\bar{x}) - f_1(x_0), \ldots, f_\ell(\bar{x}) - f_\ell(x_0)) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \tag{1}$$

Since K is invex w.r.t. η, we have $x(t) := \bar{x} + t\eta(x_0, \bar{x}) \in K$ for all $t \in [0, 1]$. Since each f_i is pseudoinvex w.r.t. η on K, it follows from Theorems 2 and 4 and Remark 2 that each f_i is both prequasiinvex and semi-strictly prequasiinvex w.r.t. the same η. Then by using prequasiinvexity, semi-strict prequasiinvexity and (1), we have

$$f(\bar{x}) - f(x(t)) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}, \quad \text{for all } t \in (0, 1),$$

that is,

$$f(x(0)) - f(x(t)) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}, \quad \text{for all } t \in (0, 1). \tag{2}$$

By Mean Value Theorem 1, there exist $t_i \in (0, 1)$ and $\xi_i \in \partial^c f_i(x(t_i))$ for all $i \in \mathscr{I}$ such that

$$f_i(x(0)) - f_i(x(t)) = \langle \xi_i, -t\eta(x_0, \bar{x})\rangle, \quad \text{for all } i \in \mathscr{I}.$$

By using (2), we obtain

$$\langle \xi_i, \eta(x_0, \bar{x})\rangle \le 0, \quad \text{for all } i \in \mathscr{I}, \tag{3}$$

and one of which becomes strict inequality. From Condition C (a) and skewness of η, we have

$$\eta(x(t_i), \bar{x}) = t_i\eta(x_0, \bar{x}), \quad \text{for all } i \in \mathscr{I}$$

and hence

$$\langle \xi_i, \eta(x(t_i), \bar{x})\rangle \le 0, \quad \text{for all } i \in \mathscr{I}$$

and one of which becomes a strict inequality. Therefore,

$$(\langle \xi_1, \eta(x(t_1), \bar{x})\rangle, \ldots, \langle \xi_\ell, \eta(x(t_\ell), \bar{x})\rangle) \in -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \tag{4}$$

Suppose that $t_1, t_2, \ldots, t_\ell$ are all equal. Then it follows from (4) that $\bar{x} \in K$ is not a solution of (GGMVVLIP), a contradiction of our supposition.

Consider the case where $t_1, t_2, \ldots, t_\ell$ are all not equal.

Case 1 (i). If $t_1 > t_2$, and in (4) the inequality is strict for $k = 1$, then from Condition C, we have

$$\langle \xi_1, \eta(x(t_2), x(t_1))\rangle = \frac{t_2 - t_1}{t_1}\langle \xi_1, \eta(x(t_1), \bar{x})\rangle > 0.$$

By Theorem 5, it follows that $\partial^c f_1(x)$ is quasiomonotone. Thus by virtue of pseudoinvexity of f_1, we have for all $\zeta_1 \in \partial^c f_1(x(t_2))$,

$$\langle \zeta_1, \eta(x(t_1), x(t_2))\rangle \le 0.$$

From Condition C, we deduce that

$$\langle \zeta_1, \eta(x(t_2), \bar{x})\rangle = \frac{t_2}{t_1 - t_2}\langle \zeta_1, \eta(x(t_1), x(t_2))\rangle \le 0. \tag{5}$$

Therefore, from (4) and (5), for all $\zeta_1 \in \partial^c f_1(x(t_2))$ and $\xi_2 \in \partial^c f_2(x(t_2))$, we obtain

$$\langle \zeta_1, \eta(x(t_2), \bar{x})\rangle \leq 0 \quad \text{and} \quad \langle \xi_2, \eta(x(t_2), \bar{x})\rangle \leq 0. \tag{6}$$

Case 1 (ii). If $t_1 < t_2$ and in (4) the inequality is strict for $k = 1$. From Condition C, we have

$$\langle \xi_2, \eta(x(t_1), x(t_2))\rangle = \frac{t_1 - t_2}{t_2} \langle \xi_2, \eta(x(t_2), \bar{x})\rangle \geq 0.$$

The pseudoinvexity of f_2 implies that $f_2(x(t_1)) \geq f_2(x(t_2))$. Since f_2 is prequasiinvex, by Theorem 3 (b), f_2 is quasiinvex. Therefore, for any $\xi_1{}' \in \partial^c f_2(x(t_1))$, we have

$$\left\langle \xi_1{}', \eta(x(t_2), x(t_1))\right\rangle \leq 0.$$

Thus from (4) and the assumption that strict inequality holds in (4) for $k = 1$, we have for all $\xi_1 \in \partial^c f_1(x(t_1))$ satisfies $\langle \xi_1, \eta(x(t_1), \bar{x})\rangle < 0$. Therefore, for all $\xi_1 \in \partial^c f_1(x(t_1))$ and $\xi_1{}' \in \partial^c f_2(x(t_1))$, we have

$$\langle \xi_1, \eta(x(t_1), \bar{x})\rangle < 0 \quad \text{and} \quad \left\langle \xi_1{}', \eta(x(t_1), \bar{x})\right\rangle \leq 0.$$

The above inequalities contradict the fact that $\bar{x}$ is a solution for (GGMVVLIP).

Case 2 (i). If $t_1 < t_2$ and in (4) the inequality is strict for $k = 2$, then from Condition C, we have

$$\langle \xi_2, \eta(x(t_1), x(t_2))\rangle = \frac{t_2 - t_1}{t_2} \langle \xi_2, \eta(x(t_2), \bar{x})\rangle > 0.$$

By Theorem 5, it follows that $\partial^c f_2(x)$ is quasimonotone. Thus by virtue of pseudoinvexity of f_2, for all $\xi_1{}' \in \partial^c f_2(x(t_1))$, we have

$$\left\langle \xi_1{}', \eta(x(t_2), x(t_1))\right\rangle \leq 0.$$

From Condition C, we deduce that

$$\left\langle \xi_1{}', \eta(x(t_1), \bar{x})\right\rangle = \frac{t_1}{t_2 - t_1} \left\langle \xi_1{}', \eta(x(t_1), \bar{x})\right\rangle \leq 0. \tag{7}$$

Therefore, from (4) and (7), for all $\xi_1 \in \partial^c f_1(x(t_1))$ and $\xi_1{}' \in \partial^c f_2(x(t_1))$ we obtain

$$\langle \xi_1, \eta(x(t_1), \bar{x})\rangle \leq 0 \quad \text{and} \quad \left\langle \xi_1{}', \eta(x(t_1), \bar{x})\right\rangle \leq 0. \tag{8}$$

Case 2 (ii). It $t_1 > t_2$ and in (4) the inequality is strict for $k = 2$, by a similar method as in *Case 1 (i)*, we deduce a contradiction.
Hence for $t_1 \neq t_2$, let $t_0 = \min\{t_1, t_2\}$. Then from (6) and (8), for $\gamma_i \in \partial^c f_i(x(t_0))$, $i = 1, 2$ we have

$$\langle \gamma_i, \eta(x(t_0), \bar{x})\rangle \leq 0, \quad \text{for } i = 1, 2.$$

By continuing this process, we can find $t^* \in (0, 1)$ such that for $\tau_i \in \partial^c f_i(x(t^*))$, $i = 1, 2, \ldots, n$

$$\langle \tau_i, \eta(x(t^*), \bar{x})\rangle \leq 0.$$

This contradicts the fact that $\bar{x} \in K$ is a solution of (GGMVVLIP).

(b) Let $\bar{x}$ be an efficient solution of (VOP) but not a solution of (GMVVLIP). Then there exists $x_0 \in K$ such that

$$\langle \zeta, \eta(x_0, \bar{x}) \rangle_\ell = (\langle \zeta_1, \eta(x_0, \bar{x}) \rangle, \ldots, \langle \zeta_\ell, \eta(x_0, \bar{x}) \rangle) \in -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$$

for all $\zeta_i \in \partial^c f(x_0)$, $i = 1, 2, \ldots, \ell$. Since η is skew, we have

$$\langle \zeta, \eta(\bar{x}, x_0) \rangle_\ell = (\langle \zeta_1, \eta(\bar{x}, x_0) \rangle, \ldots, \langle \zeta_\ell, \eta(\bar{x}, x_0) \rangle) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$$

for all $\zeta_i \in \partial^c f(x_0)$, $i = 1, 2, \ldots, \ell$. From the pseudoinvexity of each f_i, it follows that

$$f(\bar{x}) - f(x_0) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\},$$

contradicting the fact that $\bar{x}$ is an efficient solution of (VOP). □

Remark 3 Theorem 6 generalizes the half part of Theorem 3.1 in [1] in the following way:

(a) In Theorem 6, we established that every efficient solution of (VOP) is a solution of (GMVVLIP) for pseudoinvex functions while it is proved in [1] for invex functions with some extra conditions.
(b) In the proof of part (b) of Theorem 6, we only assumed that K is an invex set while it is invex with some other condition in [1].

It is worth to mention that we used a mean value theorem for invex functions to establish Theorem 3.1 in [1], however, in the proof of Theorem 6, we used simple mean value theorem for Clarke's generalized subdifferentials. Therefore, the proof of Theorem 6 and Theorem 3.1 in [1] are different.

Now we consider the perturbed form of generalized weak Stampacchia vector variational-like inequality problem (PGWSVVLIP): find $\bar{x} \in K$ for which there exists $t_0 \in (0, 1)$ such that

$$\left\langle \partial^c f(\bar{x} + t\eta(x, \bar{x})), \eta(x, \bar{x}) \right\rangle \nsubseteq -\text{int}\, \mathbb{R}^\ell_+, \quad \text{for all } x \in K \text{ and all } t \in (0, t_0].$$

It is equivalent to find $\bar{x} \in K$ for which there exists $t_0 \in (0, 1)$ such that for all $x \in K$ and all $t \in (0, t_0]$, there exists $\xi_i \in \partial^c f_i(\bar{x} + t\eta(x, \bar{x}))$, $i \in \mathscr{I}$, satisfying

$$\langle \xi, \eta(x, \bar{x}) \rangle_\ell = (\langle \xi_1, \eta(x, \bar{x}) \rangle, \ldots, \langle \xi_\ell, \eta(x, \bar{x}) \rangle) \notin -\text{int}\, \mathbb{R}^\ell_+.$$

The following result provides the relationship between a solution of (PGWSVVLIP) and (GWMVVLIP).

Theorem 7 *Let K be an invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that η is skew and satisfies Condition C^*. Let $\partial^c f$ be strictly $\mathbb{R}^\ell_+$-quasimonotone w.r.t. η, that is,*

$$\left\langle \partial^c f(x), \eta(y, x) \right\rangle \subseteq \mathit{int}\, \mathbb{R}^\ell_+ \quad \mathit{implies} \quad \left\langle \partial^c f(y), \eta(x, y) \right\rangle \subseteq -\mathit{int}\, \mathbb{R}^\ell_+,$$

for all $x, y \in K$. Then $\bar{x} \in K$ is a solution of (PGWSVVLIP) if and only if it is a solution of (GWMVVLIP).

Proof Let $\bar{x}$ be a solution of (PGWSVVLIP). Then there exists $t_0 \in (0, 1)$ such that

$$\left\langle \partial^c f(\bar{x} + t\eta(x, \bar{x})), \eta(x, \bar{x}) \right\rangle \nsubseteq -\text{int}\, \mathbb{R}^\ell_+, \tag{9}$$

for all $x \in K$ and all $t \in (0, t_0]$. By the Condition C^*, we have

$$\eta(x, \bar{x} + t\eta(x, \bar{x})) = \beta(t)\eta(x, \bar{x}), \tag{10}$$

where $\beta(t) > 0$ for all $t \in (0, 1)$. It follows from (9) that

$$\left\langle \partial^c f(\bar{x} + t\eta(x, \bar{x})), \eta(x, \bar{x} + t\eta(x, \bar{x}))\right\rangle \nsubseteq -\text{int}\,\mathbb{R}^\ell_+,$$

for all $x \in K$ and all $t \in (0, t_0]$. By strictly $\mathbb{R}^\ell_+$-quasimonotonicity of $\partial^c f$, we have

$$\left\langle \partial^c f(x), \eta(\bar{x} + t\eta(x, \bar{x}), x)\right\rangle \nsubseteq \text{int}\,\mathbb{R}^\ell_+.$$

By (10) and skewness of η, we obtain

$$\left\langle \partial^c f(x), \eta(x, \bar{x})\right\rangle \nsubseteq -\text{int}\,\mathbb{R}^\ell_+,$$

that is, for all $x \in K$, there exists $\zeta_i \in \partial^c f(x),\ i \in \mathscr{I}$ satisfying

$$\langle \zeta, \eta(x, \bar{x})\rangle_\ell = (\langle \zeta_1, \eta(x, \bar{x})\rangle, \ldots, \langle \zeta_\ell, \eta(x, \bar{x})\rangle) \notin -\text{int}\,\mathbb{R}^\ell_+.$$

Hence $\bar{x} \in K$ is a solution of (GWMVVLIP).

Conversely, let $\bar{x}$ be a solution of (GWMVVLIP). Then

$$\left\langle \partial^c f(\bar{x} + t\eta(x, \bar{x})), \eta(\bar{x}, \bar{x} + t\eta(x, \bar{x}))\right\rangle \nsubseteq \text{int}\,\mathbb{R}^\ell_+, \tag{11}$$

for all $x \in K$ and all $t \in (0, t_0]$. By Condition C*, we have

$$\eta(\bar{x}, \bar{x} + t\eta(x, \bar{x})) = -\alpha(t)\eta(x, \bar{x}),$$

where $\alpha(t) > 0$ for all $t \in (0, 1)$. It follows from (11) that

$$\left\langle \partial^c f(\bar{x} + t\eta(x, \bar{x})), \eta(x, \bar{x}))\right\rangle \nsubseteq -\text{int}\,\mathbb{R}^\ell_+, \quad \text{for all } x \in K \text{ and all } t \in (0, t_0].$$

Thus, $\bar{x}$ is a solution of (PGWSVVLIP). □

Remark 4 Theorem 7 generalizes and extends Proposition 2 in [8] and Theorem 3.2 in [19] for nondifferentiable and pseudoinvex functions.

4 Generalized vector variational-like inequalities and existence results

In this section, we present different kinds of generalized vector variational-like inequality problems (in short, GVVLIP) and prove the existence of their solutions. The GVVLIP considered in the previous section are the particular forms of the GVVLIP considered in this section.

For any two Hausdorff topological vector spaces X and Y, let $L(X, Y)$ denote the family of all continuous linear operators from X into Y. The zero element of the vector space Y is denoted by $\mathbf{0}$. When Y is the set $\mathbb{R}$ of real numbers, $L(X, Y)$ denotes the usual dual space X^* of X. For any $x \in X$ and $u \in L(X, Y)$, we denote by $\langle u, x\rangle$ the evaluation of u at x. Throughout this section, unless otherwise specified, we assume that K is a nonempty convex subset of X, $T : K \to 2^{L(X,Y)}$ is a set-valued map, $\eta : K \times K \to X$ is a map and $\{C(x) : x \in K\}$ is a family of closed convex and pointed cones in Y with $\text{int}\,C(x) \neq \emptyset$ for all $x \in K$. Note that $\mathbf{0} \notin \text{int}\,C(x)$ for all $x \in K$.

Let σ be the family of all bounded subsets of X whose union is total in X, that is, the linear hull of $\cup\{S : S \in \sigma\}$ is dense in X. Let $\mathfrak{B}$ be a neighborhood base of $\mathbf{0}$ in Y. When S runs through σ, V through $\mathfrak{B}$, the family

$$M(S, V) = \left\{u \in L(X, Y) : \bigcup_{x \in S} \langle u, x\rangle \subset V\right\}$$

is a neighborhood base of $\mathbf{0}$ in $L(X, Y)$ for a unique translation-invariant topology, called the topology of uniform convergence on the sets $S \in \sigma$, or, briefly the σ-topology. Throughout this section, we suppose that the space $L(X, Y)$ is equipped with the σ-topology.

We set $\langle T(\bar{x}), \eta(\bar{x}, x)\rangle = \bigcup_{u \in T(\bar{x})} \langle u, \eta(\bar{x}, x)\rangle$.

We consider the following forms of generalized Minty and generalized Stampacchia vector variational-like inequality problems:

Generalized Minty Vector Variational-Like Inequality Problem (GMVVLIP): Find $\bar{x} \in K$ such that

$$\langle T(x), \eta(x, \bar{x})\rangle \nsubseteq -C(\bar{x}) \setminus \{\mathbf{0}\}, \quad \text{for all } x \in K.$$

It is equivalent to find $\bar{x} \in K$ such that for all $x \in K$, there exists $u \in T(x)$ satisfying

$$\langle u, \eta(x, \bar{x})\rangle \notin -C(\bar{x}) \setminus \{\mathbf{0}\}.$$

Generalized Weak Minty Vector Variational-Like Inequality Problem (GWMVVLIP): Find $\bar{x} \in K$ such that

$$\langle T(x), \eta(x, \bar{x})\rangle \nsubseteq -\text{int}\, C(\bar{x}), \quad \text{for all } x \in K.$$

It is equivalent to find $\bar{x} \in K$ such that for all $x \in K$, there exists $u \in T(x)$ satisfying

$$\langle u, \eta(x, \bar{x})\rangle \notin -\text{int}\, C(\bar{x}).$$

Generalized Stampacchia Vector Variational-Like Inequality Problem (GSVVLIP): Find $\bar{x} \in K$ such that

$$\langle T(\bar{x}), \eta(x, \bar{x})\rangle \nsubseteq -C(\bar{x}) \setminus \{\mathbf{0}\}, \quad \text{for all } x \in K.$$

It is equivalent to find $\bar{x} \in K$ such that for all $x \in K$, there exists $\bar{u} \in T(\bar{x})$ satisfying

$$\langle \bar{u}, \eta(x, \bar{x})\rangle \notin -C(\bar{x}) \setminus \{\mathbf{0}\}.$$

Generalized Weak Stampacchia Vector Variational-Like Inequality Problem (GWSVVLIP): Find $\bar{x} \in K$ such that

$$\langle T(\bar{x}), \eta(x, \bar{x})\rangle \nsubseteq -\text{int}\, C(\bar{x}), \quad \text{for all } x \in K.$$

It is equivalent to find $\bar{x} \in K$ such that for all $x \in K$, there exists $\bar{u} \in T(\bar{x})$ satisfying

$$\langle \bar{u}, \eta(x, \bar{x})\rangle \notin -\text{int}\, C(\bar{x}).$$

It can be easily seen that every solution of (GMVVLIP) (respectively, (GSVVLIP)) is a solution of (GWMVVLIP) (respectively, (GWSVVLIP)).

If we consider $X = \mathbb{R}^n$, $Y = \mathbb{R}^\ell$, $C(x) = \mathbb{R}^\ell_+$ and $T(x) = \partial^c f(x)$, then (GMVVIP) and (GWMVVLIP) reduce to the ones considered in the previous section.

To study the existence of solutions of above mentioned problems, we define the following concepts of monotonicities.

Definition 6 For any $y \in K$, a set-valued map $T : K \to 2^{L(X,Y)}$ is said to be

(a) $C(y)$-*pseudomonotone* w.r.t. η on K if for all $x \in K$

$$\langle T(x), \eta(y, x)\rangle \subseteq C(y) \setminus \{\mathbf{0}\} \quad \text{implies} \quad \langle T(y), \eta(x, y)\rangle \subseteq -C(y) \setminus \{\mathbf{0}\};$$

(b) $C(y)$-*quasimonotone* w.r.t. η on K if for all $x \in K$

$$\langle T(x), \eta(y,x)\rangle \subseteq \text{int}\, C(y) \quad \text{implies} \quad \langle T(y), \eta(x,y)\rangle \subseteq -C(y) \setminus \{\mathbf{0}\};$$

(c) $C(y)$-*strictly quasimonotone* w.r.t. η on K if for all $x \in K$

$$\langle T(x), \eta(y,x)\rangle \subseteq \text{int}\, C(y) \quad \text{implies} \quad \langle T(y), \eta(x,y)\rangle \subseteq -\text{int}\, C(y);$$

(d) $C(y)$-*properly quasimonotone* w.r.t. η on K if for all $\{x_1, x_2, \ldots, x_n\} \subseteq K$ and for all $y \in \text{conv}\{x_1, x_2, \ldots, x_n\}$, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$\langle T(y), \eta(x_i, y)\rangle \subseteq C(y) \setminus \{\mathbf{0}\};$$

(e) $C(y)$-*weakly properly quasimonotone* w.r.t. η on K if for all $\{x_1, x_2, \ldots, x_n\} \subseteq K$ and for all $y \in \text{conv}\{x_1, x_2, \ldots, x_n\}$, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$\langle T(y), \eta(x_i, y)\rangle \nsubseteq -\text{int}\, C(y).$$

We obtain the following existence result for solutions of (GSVVLIP) under $C(y)$-proper quasimonotonicity and for solutions of (GMVVLIP) under $C(y)$-pseudomonotonicity.

Theorem 8 *For any $y \in K$, let the set-valued map $T : K \to 2^{L(X,Y)}$ be $C(y)$-properly quasimonotone w.r.t. $\eta : K \times K \to X$. Assume that the following conditions are satisfied.*

(i) *The set-valued map $\Gamma : K \to 2^K$ defined by*

$$\Gamma(x) = \left\{ y \in K : \langle T(y), \eta(x,y)\rangle \nsubseteq -C(y) \setminus \{\mathbf{0}\} \right\}$$

is closed valued.

(ii) *There exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that for each $y \in K \setminus M$, there exists $x \in B$ such that $y \notin \Gamma(x)$.*

Then (GSVVLIP) has a solution. Furthermore, if T is $C(y)$-pseudomonotone w.r.t. η and η is skew, then (GMVVLIP) also has a solution.

Proof We claim that Γ is a KKM mapping on K. Suppose Γ is not a KKM map, then there exists $\{x_1, x_2, \ldots, x_n\} \subset K$, $t_i \geq 0$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n t_i = 1$ such that $y = \sum_{i=1}^n t_i x_i \notin \bigcup_{i=1}^n \Gamma(x_i)$. Thus for any $i = 1, 2, \ldots, n$

$$\langle T(y), \eta(x_i, y)\rangle \subseteq -C(y) \setminus \{\mathbf{0}\},$$

which contradicts the $C(y)$-proper quasimonotonicity of T. Hence, Γ is a KKM mapping.

By condition (ii), $\Gamma(x)$ is a closed subset of a compact set and hence compact. Then by Lemma 1

$$\bigcap_{x \in K} \Gamma(x) \neq \emptyset,$$

that is, there exists $\bar{x} \in K$ such that

$$\langle T(\bar{x}), \eta(x, \bar{x})\rangle \nsubseteq -C(\bar{x}) \setminus \{\mathbf{0}\}, \quad \text{for all } x \in K.$$

Hence (GSVVLIP) has a solution.

Further, suppose that $\bar{x}$ is not a solution of (GMVVLIP). Then there exists $x \in K$ such that

$$\langle T(x), \eta(x, \bar{x})\rangle \subseteq -C(\bar{x}) \setminus \{\mathbf{0}\}.$$

Since η is skew, we have

$$\langle T(x), \eta(\bar{x}, x)\rangle \subseteq C(\bar{x}) \setminus \{\mathbf{0}\}.$$

By $C(x)$-pseudomonotonicity of T, we have

$$\langle T(\bar{x}), \eta(x, \bar{x})\rangle \subseteq -C(\bar{x}) \setminus \{\mathbf{0}\},$$

and thus, $\bar{x} \in K$ is not a solution of (GSVVLIP), a contradiction. □

Remark 5 Theorem 8 generalizes Theorems 2.1 and 2.3 in [21] to an arbitrary topological vector space. In fact, if η is affine in the first argument and $\eta(x, x) = 0$ for all $x \in K$, then trivially T is C-properly quasimonotone w.r.t. η. Therefore, condition 4 in Theorem 2.3 in [21] is superfluous, since it can be easily deduced from condition 3 in this theorem. Therefore, Theorem 8 also refines Theorem 2.3 in [21].

The following example shows that the $C(x)$-properly quasimonotonicity w.r.t. η does not imply the affineness of η in the first argument.

Example 2 Let $X = Y = \mathbb{R}$, $K = [-2, 2)$, $C(x) = \mathbb{R}_+ \setminus \{\mathbf{0}\}$ for all $x \in K$ and let $T : K \to 2^{\mathbb{R}}$ be defined as

$$T(x) := \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x > 0, \\ (-\infty, +\infty) & \text{if } x = 0, \\ \frac{-1}{2\sqrt{-x}} & \text{if } x < 0. \end{cases}$$

Assume that $\eta : X \times X \to X$ is a mapping defined by

$$\eta(x, y) := \begin{cases} x - y & \text{if } \ x \geq 0, y \geq 0 \ \text{ or } \ x \leq 0, y \leq 0, \\ -y & \text{if } \ x \leq 0, y > 0 \ \text{ or } \ x > 0, y \leq 0. \end{cases}$$

Then, it is easy to see that T is $C(x)$-properly quasimonotone w.r.t. η, but η is not affine in the first argument.

When X and Y are normed spaces, we establish the following existence result for a solution of (GWMVVLIP).

Theorem 9 *For all $y \in K$, let $T : K \to 2^{L(X,Y)}$ be compact-valued, $C(y)$-properly quasimonotone and $C(y)$-strictly quasimonotone. Assume that the following conditions are satisfied.*

(i) *The set-valued mapping $W : K \to 2^Y$ defined by $W(x) = Y \setminus \{int\, C(x)\}$ is closed.*
(ii) *η is continuous in the first argument.*
(iii) *There exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there exists $y \in B$ such that*

$$y \notin \Gamma(x) := \{y \in K : \langle T(y), \eta(x, y)\rangle \nsubseteq -intC(y)\}$$

Then (GMWVVLIP) has a solution.

Proof By the same argument as the first part of the proof of Theorem 8, Γ is a KKM mapping.

We claim that the set-valued mapping $\hat{\Gamma}$ defined by

$$\hat{\Gamma}(x) := \{y \in K : \langle T(x), \eta(y, x)\rangle \nsubseteq \text{int}\, C(y)\} \quad \text{for all } x \in K,$$

is closed valued.

Let $\{y_n\}$ be a sequence in $\hat{\Gamma}(x)$ converges to $y \in K$. Then

$$\langle T(x), \eta(y_n, x)\rangle \nsubseteq \text{int}\, C(y_n),$$

and therefore, there exists $u_n \in T(x)$ satisfying

$$z_n = \langle u_n, \eta(y_n, x)\rangle \notin \text{int}\, C(y_n).$$

Then $z_n \in W(y_n)$, and hence, $(y_n, z_n) \in \text{Graph}(W)$. Since $T(x)$ is compact, $\{u_n\}$ has a convergent subsequence in $T(x)$. Let $\{u_{n_k}\}$ be a subsequence of $\{u_n\}$ that converges to $u_0 \in T(x)$. By continuity of η, $\{\eta(y_{n_k}, x)\}$ is a convergent sequence. Hence there exists k_0 such that the set $\{\eta(y_{n_k}, x) : k \geq k_0\}$ is norm bounded. Therefore,

$$z_0 = \lim_{k \geq k_0} z_{n_k} = \langle u_0, \eta(y, x)\rangle.$$

Since Graph(W) is closed, then $(y, z_0) \in \text{Graph}(W)$, and hence,

$$\langle u_0, \eta(y, x)\rangle \nsubseteq \text{int}\, C(y).$$

Thus, $y \in \hat{\Gamma}(x)$.

Since T is $C(x)$-strictly quasimonotone, we have $\Gamma(x) \subseteq \hat{\Gamma}(x)$ for all $x \in K$. Therefore, $\hat{\Gamma}$ is also a KKM mapping. By Lemma 1,

$$\bigcap_{x \in K} \hat{\Gamma}(x) \neq \emptyset.$$

Therefore, there exists $\bar{x} \in K$ such that

$$\langle T(x), \eta(\bar{x}, x)\rangle \nsubseteq \text{int}\, C(\bar{x}), \quad \text{for all } x \in K,$$

Hence (GWMVVLIP) has a solution. □

Remark 6 When K is compact, then the condition (iii) of Theorem 9 is trivially satisfied.

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