Seminar N° 81 Fractional non-symmetric Fick's law in heterogeneous media

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Abstract. *Field data, obtained in heterogeneous aquifers, show behaviors not compatible with Fick's law, which holds for homogeneous media, and states that the flux of a passive tracer is proportional to the gradient of the concentration. Depending on the medium, deviations may be due to trapping sites or to preferential paths. Here, we focus on the second possibility, which corresponds to concentration profiles showing heavy tails and skewness.*

For those situations, small scale models are random walks with Lévy laws for the jump-length distribution. On the large scale, they correspond to partial differential equations involving fractional derivatives for the evolution of the concentration. The solutions to those equations were shown to satisfy fractional variants of Fick's law, with flux proportional to a fractional derivative of the concentration, in infinite domains.

Here, we use a natural and physical definition of fractional derivatives, which allows us to extend the result to domains limited by boundary-conditions, more especially when the jump-length distribution is non-symmetric.

Keywords*. Fractional Fick's Law, heterogeneous media, fractional derivatives, random walks, Lévy flights .*

1. Introduction

Many field data indicate that mass spreading in underground porous media may depart from Fick's law. This may be due to memory effects, which we disregard here, focusing on situations dominated by heavy tails, not compatible with the notion of a mean square displacement (Benson *et al* (2000, 2001), Cushman *et al* (2000), Zhang *et al* (2005), and Deng *et al* (2006)). Lévy Flights serve as a small scale model for such situations. They are random walks, more general than Brownian motion since trajectories result from successive increments, distributed by stable Lévy laws. On the macroscopic scale, the corresponding density of particles satisfies a Space-Fractional Dispersion Equation (Scalas *and al* (2004)) , and in infinite media both points of view are equivalent.

In semi-infinite media, due to non-locality of fractional derivatives, it may be necessary to modify such equations (Krepysheva et al (2006a & 2006b)), depending on the physical properties of the boundary, limiting the domain. For symmetric Lévy flights, the result was obtained by considering the even extension of the density of spreading particles. For skewed random walks the method becomes uncomfortable.

In fact, addressing directly the flux of particles performing Lévy Flights is possible, in bounded or unbounded media. We will see that the flux is given by a fractional variant of Fick's law, which agrees with previous results for infinite media, and adapts to possibly skewed Lévy flights in domains, limited by reflective or absorbing barriers. The result is based upon a novel expression for the left inverse of Riemann-Liouville's fractional integrals (Samko *et al* (1993)).

2. The flux of particles performing Lévy Flights

Independent walkers are said to perform Lévy flights when each of them makes successive instantaneous independent jumps whose amplitudes are distributed according to the density $\varphi_l(x) = \frac{1}{l} L_\alpha^\theta(\frac{x}{l})$, which accounts for the possibility, for dissolved particles, to travel very fast very far, as in media with hidden preferential paths. Here L_α^θ denotes the density of a normalized Lévy law of stability index α and skewness parameter θ (Gnedenko *et al* (1968)). Waiting times between jumps are independent too, and distributed according to the density $\psi_\tau(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}$. Here, τ is the mean waiting time, and l is a length scale.

Let $P(x,t)$ be the density of the probability of finding a walker in $[x,x+dx]$ at time t. The flux through abscissa x is the balance of particles crossing x to the right or to the left during $[t,t+dt[$, divided by dt, and the probability of making one jump during infinitesimal time interval $[t,t+dt[$ is $\frac{dt}{\tau}$. Hence, in an infinite medium, the flux is

$$\tau^{-1}[\int_0^\infty P(x-y,t)F^+_{\alpha,\theta}(\frac{y}{l})dy-\int_0^\infty P(x+y,t)F^-_{\alpha,\theta}(\frac{-y}{l})dy], \tag{1}$$

with $F^+_{\alpha,\theta}(y/l)$ being the probability $\int_y^{+\infty}\frac{1}{l}L_{\alpha,\theta}(z/l)dz=\int_{y/l}^{+\infty}L_{\alpha,\theta}(z)dz$ for a jump to be to the right while having an amplitude of more than y. Similarly, $F^-_{\alpha,\theta}(-y/l)$ is the probability $\int_{-\infty}^{-y}\frac{1}{l}L_{\alpha,\theta}(z/l)dz$ for a jump to have a modulus of more than y, but to the left. Since there is no confusion, we will skip t in $P(.,t)$ from now on, except in final results at the end of Section 4.

Expression (1) giving the flux on the small scale may be modified more or less deeply by the presence of a boundary at, say $x=0$, depending on whether it is allowed to release particles or not.

To see this, consider an absorbing boundary, such that walkers are killed when hitting the wall. Then, (1) holds with HP in place of P on the right-hand side. Here H denotes Heavisde's function. Oppositely, imagine a purely reflecting wall such that each particle hitting the barrier flies the length of the jump, which had been assigned to it before the shock, just remaining on the same side ($x>0$) see Krepysheva *et al* (2006a). Then, when counting particles crossing x to the left or to the right, we have to take into account that jumps, directed to the left and starting from $x+y$ ($y>0$), arrive at the right of x hence do not enter the balance if the amplitude is larger than $2x+y$. Jumps directed to theleft and starting from $x-y$, with $0<y<x$, may cross x to the right if the amplitude is of more than $2x-y$. In this case the flux is

$$\int_0^x \frac{P(x-y)}{\tau}[F^+_{\alpha,\theta}(\frac{y}{l})+F^-_{\alpha,\theta}(\frac{y-2x}{l})]dy-\int_0^\infty \frac{P(x+y)}{\tau}[F^+_{\alpha,\theta}(\frac{-y}{l})-F^-_{\alpha,\theta}(\frac{-2x-y}{l})]dy.$$

Setting $P^*(x)=P(x)$ for $x>0$ and $P^*(x)=P(-x)$ for $x<0$ (even extension of P) we obtain that this expression is

$$\int_0^\infty \frac{P^*(x-y)}{\tau}F^+_{\alpha,\theta}(\frac{y}{l})dy-\int_0^\infty \frac{P(x+y)}{\tau}F^-_{\alpha,\theta}(\frac{-y}{l})dy-\int_x^\infty \frac{P^*(x-y)}{\tau}(F^+_{\alpha,\theta}(\frac{y}{l})-F^-_{\alpha,\theta}(\frac{-y}{l}))dy \tag{2}$$

Since fractional dispersion equations were shown to hold in the diffusive limit ($l\to 0$) provided the scaling $l^\alpha/\tau=K$ (Scalas *et al* (2004)) holds, we address (1) and (2) in this context.

3. Fractional tools

Some mathematical tools make it possible to see that the limits of (1) and (2) combine several kinds of fractional derivatives of the density P.

3.1. A novel expression for for Riemann-Liouville's derivatives

Left and right-sided Riemann-Liouville integrals of the order of α', which we use here are those of Samko *et al* (1993) associated with semi-infinite intervals according to

$$I_+^{\alpha'}\varphi(x)=\frac{1}{\Gamma(\alpha')}\int_{-\infty}^x (x-y)^{\alpha'-1}\varphi(y)dy, I_-^{\alpha'}\varphi(x)=\frac{1}{\Gamma(\alpha')}\int_x^{+\infty}(y-x)^{\alpha'-1}\varphi(y)dy. \tag{3}$$

The left-sided Riemann-Liouville derivative of order α' is $D_+^{\alpha'}\varphi(x)=(\frac{d}{dx})^{[\alpha']+1}I_+^{1-\{\alpha'\}}=$

$$(\frac{d}{dx})^{[\alpha']+1}\frac{1}{\Gamma([\alpha']+1-\alpha')}\int_{-\infty}^{x}(x-y)^{-\{\alpha'\}}\varphi(y)dy, \tag{4}$$

where $[.]$ denotes integer part, while $\{.\}$ is defined by $\alpha'=[\alpha']+\{\alpha'\}$. The right-sided Riemann-Liouville derivative is

$$D_{-}^{\alpha'}\varphi(x)=(-\tfrac{d}{dx})^{[\alpha']+1}I_{-}^{1-\{\alpha'\}}=(-\frac{d}{dx})^{[\alpha']+1}\frac{1}{\Gamma([\alpha']+1-\alpha')}\int_{x}^{+\infty}(y-x)^{-\{\alpha'\}}\varphi(y)dy. \tag{5}$$

For a large set of functions, they coincide with Marchaud's derivatives which are left inverses to fractional integrals $I_{\pm}^{\alpha'}$. We will not give here the proof of a new result by Néel and Abdennadher (2007), stating that the limit, when l tends to zero, of $l^{-\alpha}\int_0^{\infty}f(x\pm y)F(\frac{\pm y}{l})dy$ is the left inverse of $I_{\pm}^{\alpha-1}$, hence a Marchaud's derivative of order $\alpha-1$.

The claimed result holds provided F, integrable over $[0,+\infty[$, satisfies $\int_0^{\infty}F(y)dy=0$ while, moreover, it is of the form of $F_1(y)+Cy^{-\alpha}$, with F_1 and $F_1(y)y^{\alpha-1}$ being integrable near infinity.

Expressions of the form of $l^{-\alpha}\int_0^{\infty}f(x\pm y)F_{\alpha,\theta}^{\mp}(\frac{\mp y}{l})dy$ are present on the right-hand sides of (1)-(2). Nevertheless, cumulated probabilities $F_{\alpha,\theta}^{-}(-.)$ and $F_{\alpha,\theta}^{+}$ satisfy $H_2(\alpha)$, but of course not H_1. Hence, we will take a function $f_{\alpha,\theta}$, supported in $[0,1]$ (e.g.) and satisfying $\int_0^{+\infty}f_{\alpha,\theta}(y)dy=J_{\alpha,\theta}=\int_0^{+\infty}F_{\alpha,\theta}^{-}(-y)dy$. Due to (15), proved in Appendix B, $J_{\alpha,\theta}$ is also equal to $\int_0^{+\infty}F_{\alpha,\theta}^{+}(y)dy$, so that setting $\tilde{F}_{\alpha,\theta}^{\pm}(y)=F_{\alpha,\theta}^{\pm}(y)-f_{\alpha,\theta}(y)$ yields functions $\tilde{F}_{\alpha,\theta}^{\pm}$ matching the conditions for F. With these notations, since $\tau^{-1}=Kl^{-\alpha}$ holds, the second integrals on the right-hand side of (1), $Kl^{-\alpha}\int_0^{\infty}P(x+y)F_{\alpha,\theta}^{-}(\frac{-y}{l})dy$, is equal to

$$Kl^{1-\alpha}[P(x)J_{\alpha,\theta}+\int_0^{\infty}P(x+y)\tilde{F}_{\alpha,\theta}^{-}(\frac{-y}{l})dy+\int_0^{\infty}(P(x+y)-P(x))f_{\alpha,\theta}(\frac{y}{l})dy]. \tag{6}$$

Similarly, the first expression in (1), $Kl^{-\alpha}\int_0^{\infty}P(x-y)F_{\alpha,\theta}^{+}(\frac{y}{l})dy$, is equal to

$$Kl^{1-\alpha}[P(x)J_{\alpha,\theta}+\int_0^{\infty}P(x-y)\tilde{F}_{\alpha,\theta}^{+}(\frac{y}{l})dy+\int_0^{\infty}(P(x-y)-P(x))f_{\alpha,\theta}(\frac{y}{l})dy]. \tag{7}$$

The $l^{-\alpha}P(x,t)J_{\alpha,\theta}$ in (1) or (2) cancel each other when we take the difference between (6) and (7). Moreover, we will see that appropriately choosing $f_{\alpha,\theta}$ yields that the limits of $l^{-\alpha}\int_0^{\infty}(P(x\pm y)-P(x))f_{\alpha,\theta}(\frac{y}{l})dy$ are Kolwankar and Gangal's local fractional derivatives, which are less currently used than the ones of Riemann, Liouville and Marchaud, and will be recalled in next Subsection.

3.2. Kolwankar and Gangal's local fractional derivatives

The notion of a local fractional derivative was introduced by Kolwankar and Gangal (1996) in view of building a tool, designed for the study of continuous but nowhere differentiable functions frequently occuring in Nature and Economics. Those fractional derivatives vanish for derivable functions, and hence can become "invisible".

For q between 0 and 1, the right and left-sided Kolwankar and Gangal's fractional derivatives $D_{\pm}^{KG,q}f(x)$ of order q of function f, computed at x, are obtained from Riemann-Liouville derivatives by letting the range of integration tend to zero. To be more precise, it is the limit of $\frac{d}{dh}\frac{1}{\Gamma(1-q)}\int_x^{x+h}(1-t)^{-q}\frac{f(y)-f(x)}{(x+h-y)^{1-q}}dt$ when h tends to zero. When it does exist, it is equal to the limit of $\frac{h^{-q}}{\Gamma(1-q)}\int_0^1(1-t)^{-q}(f(x+th)-f(x))dt$, due to L'Hôpital's rule.

For $\alpha<2$, choosing $f_{\alpha,\theta}(t)=J_{\alpha,\theta}(2-\alpha)(1-t)^{1-\alpha}\chi_{[0,1]}$ yields that, when P has a local derivative (w.r.t. space) of order $\alpha-1$ at point $x+$, $l^{-\alpha}\int_0^{+\infty}f_{\alpha,\theta}(y/l)(P(x+y)-P(x,t))dy$, equal to $J_{\alpha,\theta}l^{1-\alpha}(2-\alpha)\int_0^1(1-s)^{1-\alpha}(P(x+ls)-P(x))ds$, has a limit when l tends to zero. And the limit is $J_{\alpha,\theta}\Gamma(3-\alpha)D_+^{KG,\alpha-1}P(x)$. At the left, $l^{-\alpha}\int_0^{+\infty}f_{\alpha,\theta}(y/l)(P(x-y)-P(x))dy$ tends to $-J_{\alpha,\theta}\Gamma(3-\alpha)D_-^{KG,\alpha-1}P(x)$. Moreover, when P is derivable at x, $D_\pm^{KG,\alpha-1}P(x)$ is equal to zero.

Both kinds of fractional derivatives, local and non-local, appear in the limits when l tends to zero, of expressions (1) and (2), hence in the macroscopic flux.

4. Fractional Fick's law and dispersion equation

4.1. Limits of the right-hand sides of (5) and (6)

Let α be strictly between 1 and 2. When P has Marchaud's and Kolwankar-Gangal's derivatives of order $\alpha-1$, which are integrable, bounded and continuous, terms on the right-hand sides of (7) and (6) have limits when l tends to zero. The limit is $\lambda_+D_+^{\alpha-1}P(x)$ for $l^{-\alpha}\int_0^{\infty}P(x+y)\tilde{F}_{\alpha,\theta}^-(\frac{-y}{l})dy$ in (6) and $\lambda_-D_-^{\alpha-1}P(x)$ for $l^{-\alpha}\int_0^{\infty}P(x-y)\tilde{F}_{\alpha,\theta}^+(\frac{y}{l})dy$ in (7), with

$$\lambda_+=\int_0^{+\infty}I_+^{\alpha-1}H\,\tilde{F}_{\alpha,\theta}^+(y)dy,\quad \lambda_-=\int_0^{+\infty}I_+^{\alpha-1}H\,\tilde{F}_{\alpha,\theta}^-(-y)dy. \tag{8}$$

For integrals $Kl^{-\alpha}\int_0^{+\infty}f_{\alpha,\theta}(y/l)(P(x\pm y)-f(x))dy$, the limit is $\mp KJ_{\alpha,\theta}D_\pm^{KG,\alpha-1}P(x)$ which is zero on intervals where P is derivable. For parameter $J_{\alpha,\theta}$, we have the exact expression (15).

To compute $\lambda_\pm$, let us compare $D_\pm^{\alpha-1}f$ against the limit of $l^{-\alpha}\int_0^{\infty}(f(x\pm y)-f(x))F_{\alpha,\theta}^{\mp}(\frac{\mp y}{l})dy$ for some particular function f. For instance, we take $f=\chi_{[1,2[}$. For x in $]1,2[$, the local derivative exists and is equal to zero, while we have $l^{-\alpha}\int_0^{+\infty}(f(x+y)-f(x))F_{\alpha,\theta}^-(-y/l)dy=l^{-\alpha}\int_{2-x}^{+\infty}F_{\alpha,\theta}^-(-y/l)dy\ =C_{\alpha,-\theta}\frac{(2-x)^{1-\alpha}}{\alpha-1}+O(l)$ for $1<\alpha<2$, with $C_{\alpha,-\theta}$ being defined by (13). We also have $D_-^{\alpha-1}\chi_{[1,2[}(x)=\frac{-1}{\Gamma(1-\alpha)}\int_{2-x}^{+\infty}y^{-\alpha}dy=\frac{(2-x)^{1-\alpha}}{\Gamma(2-\alpha)}$. This implies $\lambda_-=-\frac{\Gamma(2-\alpha)}{\alpha-1}C_{\alpha,-\theta}=\frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}$, and similarly $\lambda_+=\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}$.

4.2. Fractional Fick's law

Hence, the limit of (1), which is the flux $Q(x)P$ through x (in an infinite medium) on the macroscopic scale is

$$Q(x)P=K\left(\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}D_+^{\alpha-1}P(x)-\frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}D_-^{\alpha-1}P(x)\right) \tag{9}$$

$$-KJ_{\alpha,\theta}\Gamma(3-\alpha)[D_+^{KG,\alpha-1}P(x)+D_-^{KG,\alpha-1}P(x)].$$

This is a fractional variant of Fick's law, which Paradisi *et al* (2001) obtained for derivable functions, of course satisfying $D_{\pm}^{KG,\alpha-1}P=0$ identically.
In a semi-infinite domain limited by a reflective barrier, according to (2) the flux is given by the same expression, with P^* instead of P, but we have to add $-Kl^{-\alpha}\int_0^{+\infty}((1-H)P^*)(x-y)(F_{\alpha,\theta}^{-}(-y/l)-F_{\alpha,\theta}^{+}(y/l))dy$, which tends to $(\lambda^- - \lambda^+)D_+^{\alpha-1}((1-H)P^*)(x)$. Hence, for $x>0$ the macroscopic flux is

$$Q(x)P = K[\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}(D_+^{\alpha-1}P)(x)-\frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}(D_-^{\alpha-1}P^*)(x)] \qquad (10)$$

$$+K\frac{\sin\frac{\pi}{2}(\alpha+\theta)-\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}D_+^{\alpha-1}((1-H)P^*)(x)-KJ_{\alpha,\theta}\Gamma(3-\alpha)[D_+^{KG,\alpha-1}P(x,t)+D_-^{KG,\alpha-1}P^*(x)]$$

Case $\alpha=2$ had been left apart in Subsection 3.2. To retrieve Fick's law, take $A(l)=l^{-1/2}$. Parameter θ is equal to zero, and it is enough to consider the case of an infinite medium, since due to $\theta=0$ (2) is exactly of the form of (1), with P^* instead of P. We have

$$l^{-2}\int_0^{+\infty}F_{2,0}^{\pm}(\tfrac{y}{l})(P(x+y)-P(x))dy=\int_0^{A(l)}F_{2,0}^{\pm}(y)\frac{P(x+ly)-P(x)}{ly}dy+l^{-1}\int_{A(l)}^{+\infty}F_{2,0}^{\pm}(y)(P(x+ly)-P(x))dy.$$

When P is differentiable at point x, $\int_0^{A(l)}F_{2,0}^{\pm}(y)\frac{P(x+ly)-P(x)}{ly}dy$ tends to the usual derivative $\partial_x P$, times $\mp\int_0^{+\infty}F_{2,0}^{\pm}(y)ydy$, itself equal to $\mp1/2$ due to $F_{2,0}^{\pm}(x)=\int_x^{+\infty}\frac{1}{2\sqrt{\pi}}e^{-y^2/4}dy$. And $l^{-1}\int_{A(l)}^{+\infty}F_{2,0}^{\pm}(y)(P(x+ly)-P(x))dy$ is less than $l^{-2}F_{2,0}^{\pm}(A(l))\int_{A(l)}^{+\infty}\left|P(x+y)-P(x)\right|dy$, which tends to 0 when P is integrable and fixed. Similar results are obtained at the left of x, hence for $\alpha=2$, in the limit "l tends to zero" operator flux tends to $-K\partial_x P$, which is classical Fick's law.
The more general fractional version implies a space-fractional variant of the classical diffusion equation.

4.3 Space-fractional diffusion equation

When the density of particles and the macroscopic flux are derivable, mass conservation without sources implies $\partial_t P=-\partial_x Q$. Moreover, we have $\partial_x D_{\pm}^{\alpha-1}=\pm D_{\pm}^{\alpha}$, and local Kolwankar-Gangal derivatives with order of less than 1 are identically zero. Hence, in an infinite medium, (9) implies that P evolves according to

$$\partial_t P(x,t)=-K[\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}D_+^{\alpha}P(x,t)+\frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}D_-^{\alpha}P(x,t)]. \qquad (11)$$

This Fractional Dispersion Equation had been obtained by Gorenflo *et al* (2002) via Fourier's analysis, from the Generalized Master Equation for the density of particles performing possibly skewed Lévy flights.
In a medium, limited by a reflective barrier, (10) implies

$$\partial_t P(x,t)=-K[\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}D_+^{\alpha}P^*(x,t)+\frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}D_-^{\alpha}P(x,t)] \qquad (12)$$

$$-K\frac{\sin\frac{\pi}{2}(\alpha-\theta)-\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}D_+^{\alpha}((1-H)P^*)(x,t).$$

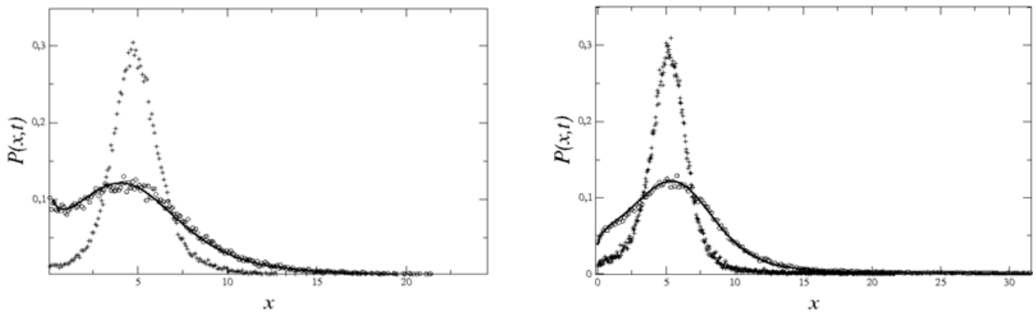

Figure 1 Numerical solution of (12), compared with Monte Carlo simulation of skewed Lévy flights with a reflective barrier, for $\alpha = 1.5$, with $\theta = 0.2$ at the left and $\theta = -0.2$ at the right. Full line represents the numerical solution to (12) with $K = 1$ at instant $t = 4$, and circles stand for random walk histograms. Histograms at time $t = 1$ correspond to symbols "plus".

For symmetric random walks (with $\theta = 0$) we retrieve a result of Krepysheva *et al* (2006a, 2006b). That (12) represents the evolution of the concentration of walkers had been checked in Krepysheva *et al* (2006a) by comparing the discretized solution of the partial differential equation against Monte Carlo simulations of symmetric Lévy flights.

4.4 Numerical illustration of (12)

In order to solve (12), derivatives of order α were discretized according to a shifted Grünwald-Letnikov scheme (Gorenflo *et al* (2002)), and we have set $K = 1$, as in Krepysheva *et al* (2006a). The issue was compared with histograms of Lévy flights corresponding to smallvalues of τ and l satisfying $\tau = l^{\alpha}$. In order to keep coherent data, we had to take into account that Dirac impulses are easy to implement in Monte Carlo simulations, but not in the discretized partial differential equation (12). Hence numerical simulations of (12) were started at time $t = 1$ from corresponding histograms, represented by symbols "plus" on Figures 1 and 2. Random walks were started at $t = 0$ from Dirac impulses applied at $x = 5$.

We observe that already at instant $t = 1$, the maximum of the density of particles has moved from initial impulse's location $x = 5$, to the left for $\theta = 0.2$, and to the right for $\theta = -0.2$. The trend is confirmed at larger values of t, but for positive valued θ, the peak of the distribution of particles is perturbed by the wall (at $x = 0$).

5. Conclusion

For a cloud of particles, performing one-dimensional Lévy flights with time and length scales τ and l satisfying $\tau = Kl^{\alpha}$, the mass flux through abscissa x is a difference between convolutions involving the density of walkers, in infinite media. In the limit when l tends to zero, the convolutions of this form tend to combinations of non-local and local fractional derivatives of order $\alpha - 1$. That distribution functions of stable Lévy laws on the left and on the right have equalintegrals over $]-\infty, 0]$ and $[0, +\infty[$, even when they are skewed was essential for that. Hence a fractional generalization of Fick's law was derived, without passing through any partial differential equation for the time evolution of the concentration. That it adapts to domains, limited by boundaries, was shown for two examples, trivial and less trivial.

The thus obtained fractional Fick's law, when recast into mass conservation principle, yields a fractional dispersion, equation, provided local derivatives are zero, which holds when the concentration is derivable (w.r.t. space).

6. Appendixes

6.1. Densities of alpha stable Lévy laws

The random variable X, with law F, is said to be stable if, for any sequence of independent random variables X_i distributed like X, there exists a sequence c_n of positive numbers such that $\frac{X_1 + \ldots X_n}{c_n}$ be distributed according to F itself for any positive integer n (Feller (1970) and Gnedenko *et al* (1968)). Then, c_n is a power of n, and the inverse α of the exponent belongs to $]0, 2]$ and serves as a label for the law: it is called the stability exponent of the

α stable law. The value $\alpha = 2$ corresponds to normal law, which is symmetric. For $\alpha \in]0,2[$, stable laws may be symmetric or skewed.

Stable laws play an important role in Nature because they are attractors, again in the context of the addition of many independent random variables X_n , distributed according to law F . Probability law G is an attractor for F if there exists sequences A_n and B_n , with $B_n > 0$, such that the law of $\frac{X_1+...+X_n}{B_n} - A_n$ tends to G when n tends to ∞ (Feller (1970)). When α belongs to $]0,2[$, α stable laws are attractors for probability laws whose density behaves asymptotically as $x^{-\alpha-1}$. Normal law ($\alpha = 2$) is an attractor for laws whose asymptotics is $x^{-\gamma-1}$ with $\gamma \geq 2$ (Feller (1970) and Gnedenko *et al* (1968)).

In general, the density of a stable law cannot be given in closed form. But, up to translations and dilatations, the Fourier transform is $e^{-k^{\alpha} e^{(i\, sign(k)\ \pi\theta/2)}}$. Hence, the corresponding density L_α^θ satisfies $L_\alpha^\theta(-x) = L_\alpha^{-\theta}(x)$, is labeled by the stability exponent α , and the skewness parameter θ, which belongs to $[\alpha-2, 2-\alpha]$. Moreover, for α stricly between 1 and 2 with $\alpha - 2 < \theta \leq 2-\alpha$, provided $x > A > 0$ holds with A large enough, we have $L_\alpha^\theta(x) = \frac{1}{\pi x}\Sigma_{n=1}^{+\infty}(-x^{-\alpha})^n \frac{\Gamma(1+n\alpha)}{n!}\sin\frac{n\pi}{2}(\theta-\alpha)$. We will denote by

$$C_\alpha^\theta = \frac{-1}{\pi}\Gamma(1+\alpha)\sin\frac{\pi}{2}(\theta-\alpha) \tag{13}$$

the coefficient of the leading term in this expansion.

6.2 Integrals of cumulated alpha stable Lévy laws

In Subsection 3.1, we use the fact that $\int_0^{+\infty} F_{\alpha,\theta}^{+}(y)dy$ and $\int_0^{+\infty} F_{\alpha,\theta}^{-}(-y)dy$ are equal. To prove the claim, notice that $F_{\alpha,\theta}^{-}(-x) = \int_{-\infty}^{-x} L_\alpha^\theta(y)dy = \int_x^{+\infty} L_\alpha^{-\theta}(-y)dy = F_{\alpha,-\theta}^{+}(x)$. Then, we will uses Mellin's transform, defined by $M\omega(z) = \int_0^{+\infty} t^{z-1}\omega(t)dt$ for function ω . With $z=1$ we see that $\int_0^{+\infty} F_{\alpha,\theta}^{+}(y)dy = MF_{\alpha,\theta}^{+}(1)$, while we have $F_{\alpha,\theta}^{+}(x) = I_{-}^{1}L_\alpha^{-\theta}(x)$, hence $\int_0^{+\infty} F_{\alpha,\theta}^{+}(y)dy = (MI_{-}^{1}L_\alpha^{-\theta})(1)$.

For $\alpha > \mathrm{Re}(z) > 0$ and sufficiently good-behaved functions in neighbourhoods of ∞ , such as L_α^θ , we have

$$(MI_{-}^{1}\omega)(z) = \frac{\Gamma(z)}{\Gamma(z+1)}(M\omega)(z+1),$$

According to Rubin (1996, see at page 44). From this, due to $F_{\alpha,\theta}^{+}(x) = \int_x^{+\infty} L_\alpha^\theta(y)dy = I_{-}^{1}L_\alpha^\theta(x)$, we deduce

$$(MF_{\alpha,\theta}^{d})(z) = \frac{\Gamma(z)}{\Gamma(z+1)}(ML_\alpha^\theta)(z+1).$$

The Mellin transform ML_α^θ is given in Schneider (1986) for $0 < Re(z) < 1$

$$(ML_\alpha^\theta)(z) = \frac{1}{\alpha}\frac{\Gamma(z)\Gamma((1-z)\alpha^{-1})}{\Gamma((1-z)\frac{\alpha-\theta}{2\alpha})\Gamma(1-(1-z)\frac{\alpha-\theta}{2\alpha})}.$$

This implies

$$(ML_\alpha^\theta)(z) = \frac{1}{\pi\alpha}\Gamma(z)\Gamma(\frac{1-z}{\alpha})\sin((1-z)\pi\frac{\alpha-\theta}{2\alpha}) \tag{14}$$

due to complements formula for Gamma functions (Abramowitz and Stegun (1965)). Nevertheless, $ML_\alpha^\theta(z)$, as a function of z, is holomorphic for $0 < Re(z) < \alpha + 1$, due to the behavior of $L_\alpha^\theta(x)$ for large real values of x. On the right-hand side of (14), $\Gamma(z)\Gamma((1-z)\alpha^{-1})$ is holomorphic also except at poles of $\Gamma((1-z)\alpha^{-1})$. Then, analytic continuation extends (14) to $\{z \in C/0 < Re(z) < \alpha + 1\} - \{1\}$.
From this we deduce

$$\int_0^{+\infty} F_{\alpha,\theta}^+(y)dy = (MF_{\alpha,\theta}^+)(2) \quad = \frac{\Gamma(2)\Gamma(-1/\alpha)}{\alpha\pi}\sin\pi\frac{\theta-\alpha}{2\alpha} = -\frac{\Gamma(-1/\alpha)}{\alpha\pi}\cos\pi\frac{\theta}{2\alpha} \tag{15}$$

is an even function of θ, hence the claimed result.

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