

COMPOSITIONS OF EQUI-DIMENSIONAL FOLD MAPS

YOSHIHIRO HIRATO AND MASAMICHI TAKASE

ABSTRACT. According to Ando's theorem, the oriented bordism group of fold maps of n -manifolds into n -space is isomorphic to the stable n -stem. Among such fold maps we define two geometric operations corresponding to the composition and to the Toda bracket in the stable stem through Ando's isomorphism. By using the operations we explicitly construct several fold maps with favourable properties, including a fold map which represents the generator of the stable 6-stem.

1. INTRODUCTION

A fold map, which is a smooth map between smooth manifolds with only fold singularities, can be considered as a simple extension of an immersion and also as a high-dimensional analogue of a Morse function. Many studies on fold maps have indicated that they are closely related to the geometry of manifolds (e.g. see [3, 4, 9, 16]). In this note we study equi-dimensional fold maps, rather in the light of their relation to algebraic topology.

A smooth map $f: N^n \rightarrow \mathbf{R}^n$ from an n -dimensional closed oriented manifold N^n into the n -dimensional Euclidean space is said to be a *fold map* if each of its singular points has the local form $f(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, \pm x_n^2)$ with respect to suitable local coordinate systems in N^n and in \mathbf{R}^n . For two such fold maps $f_i: N_i \rightarrow \mathbf{R}^n$ ($i = 0, 1$), we say that they are *oriented bordant* if there exists a fold map F to $\mathbf{R}^n \times [0, 1]$ from an oriented cobordism (as a manifold) W^{n+1} between N_0 and N_1 , such that $F|_{N_0 \times [0, \epsilon)} = f_0 \times \text{Id}_{[0, \epsilon)}$ and that $F|_{N_1 \times (1 - \epsilon, 1]} = f_1 \times \text{Id}_{(-\epsilon, 0]}$ (with ϵ being a small positive real number). This gives an equivalence relation among all fold maps from closed oriented n -manifolds into \mathbf{R}^n and the quotient space form an abelian group called the *oriented fold bordism group* in a usual manner, which we denote by $\text{SFold}(n, 0)$. Note that for more general singular maps the notion bordism group has been introduced [11, 18] and intensively studied in recent years (e.g. [13, 17, 19]).

2010 *Mathematics Subject Classification*. Primary: 57R45; Secondary: 55Q45, 57R35, 55Q35.

Key words and phrases. fold map, Toda bracket, composition, stable homotopy group, bordism.

The fold bordism groups have been studied by many authors (e.g. see [3, 4, 5, 6, 7, 8, 14]). In particular, Ando [3, 5, 6] has proven that $\text{SFold}(n, 0)$ is isomorphic to the stable homotopy group π_n^S of spheres. Under Ando's isomorphism, we introduce two geometric operations for (bordism classes of) fold maps corresponding to the composition and to the Toda bracket in the stable homotopy groups of spheres (in §3). In fact, Koschorke [10] formulated the similar operations for the immersion bordism group $\text{SI}(n, 1)$ of immersions of n -manifolds into \mathbf{R}^{n+1} , which is also isomorphic to π_n^S [22]. Therefore in practice, we first establish in a geometric manner an isomorphism between $\text{SFold}(n, 0)$ and $\text{SI}(n, 1)$ in §2, and then just interpret Koschorke's composition and Toda bracket through the isomorphism. This attempt is natural and useful since for codimension one immersions a kind of the Pontrjagin-Thom construction gives good understandings of geometric counterparts to many algebraic operations in the stable stems (see e.g. [1, 2] for recent papers). We detail many low-dimensional examples. As an application, we describe a construction of a fold map $S^3 \times S^3 \rightarrow \mathbf{R}^6$ which represents the generator of the stable 6-stem $\pi_6^S \approx \mathbf{Z}/2\mathbf{Z} = \mathbf{Z}_2$ in §4.

2. AN ISOMORPHISM BETWEEN FOLD AND IMMERSION BORDISM GROUPS

Wells [22] studied the bordism groups of immersions and reduced the problem to the study of embeddings with appropriate vector fields, by lifting immersions into a higher dimensional space. In particular, since a codimension one immersion $f: N^n \looparrowright \mathbf{R}^{n+1}$ of an oriented n -manifold N^n naturally has the homotopically unique normal framing, by suspending and slightly perturbing it in a euclidian space of enough high dimension, we can obtain a normally framed embedding. The isomorphism $\text{SI}(n+1) \approx \pi_n^S$ is given by applying the usual Pontrjagin-Thom construction for the resultant normally framed embedding.

2.1. The isomorphism \mathfrak{m} . In this section, we construct a natural isomorphism between the oriented fold bordism group $\text{SFold}(n, 0)$ and the oriented immersion bordism group $\text{SI}(n, 1)$, each of which we already know is isomorphic to the stable homotopy group π_n^S of spheres.

Let $f: N^n \rightarrow \mathbf{R}^n$ be a fold map from an oriented n -manifold N^n to \mathbf{R}^n ($n \geq 1$). Then the fold set $S(f)$ of f is an $(n-1)$ -dimensional orientable submanifold of N^n and the restriction $f|_{S(f)}$ is an immersion in \mathbf{R}^n with trivial normal line bundle (e.g. see [15, Lemma 2.2]). For each component S_i of the fold set $S(f)$, we can take a tubular neighbourhood $S_i \times \mathbf{R} \subset N^n$ so that f immerses $S_i \times [0, \infty)$ orientation-preservingly into \mathbf{R}^n and that

f immerses $S_i \times (-\infty, 0]$ orientation-reversingly into \mathbf{R}^n . This determines the orientation of the normal bundle of $S_i \subset N^n$, that further induces the orientation of S_i from the given orientation of N^n . Thus, $S(f)$ becomes an oriented $(n - 1)$ -manifold.

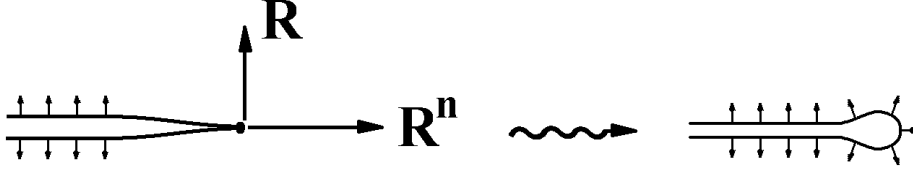


FIGURE 1. Desingularisation of folds

Let $j: \mathbf{R}^n \hookrightarrow \mathbf{R}^{n+1}$ be the inclusion and consider the composition $j \circ f: N^n \rightarrow \mathbf{R}^{n+1}$. Then, $j \circ f$ is an immersion on $N^n \setminus S(f)$ and hence we can take a normal vector field ν on $N \setminus S(f)$ with respect to the orientations of N^n and of \mathbf{R}^{n+1} . Then, the situation inside each fibre D_p^2 (at $p \in S(f)$) of the 2-dimensional normal disk bundle of $(j \circ f)(S(f))$ in \mathbf{R}^{n+1} , which is the trivial bundle, is as the left figure in Figure 1 (the two curves in left figure, depicted slightly away from each other, are in reality on \mathbf{R}^n , and the arrows attached to them indicate the normal vector field ν). Therefore, we can “desingularise” $j \circ f$ by modifying it inside each D_p^2 as in Figure 1. Since the normal bundle of $(j \circ f)(S(f))$ in \mathbf{R}^{n+1} is trivial, this process can be done globally on each component of $(j \circ f)(S(f))$. Thus we obtain an immersion of N^n in \mathbf{R}^{n+1} , which we denote by \bar{f} . Furthermore, this gives rise to a homomorphism between the bordism groups,

$$\mathbf{m}: \text{SFold}(n, 0) \rightarrow \text{SI}(n, 1), [f] \mapsto [\bar{f}]$$

since we can perform the same operation for a fold bordism between two bordant fold maps, so that we can obtain an immersion bordism between the corresponding immersions.

Example 2.1. The fold map $S^1 \rightarrow \mathbf{R}^1$ shown in Figure 2 (which we call the \bowtie -fold map) generates $\text{SFold}(1, 0) \approx \pi_1^S \approx \mathbf{Z}_2$. This is easily seen from [6, Theorem 1.3] and Figure 3, which depicts a stable map from D^2 to the half plane \mathbf{R}_+^2 with one cusp point extending the fold map.

Then, Figure 4 describes the image of the \bowtie -fold map shown in Figure 2 under the homomorphism $\mathbf{m}: \text{SFold}(1, 0) \rightarrow \text{SI}(1, 1)$. In fact, the immersion with one crossing represents the generator of $\text{SI}(1, 1)$, since for a self-transverse immersion $S^1 \looparrowright \mathbf{R}^2$ the number modulo 2 of its double points gives the isomorphism $\text{SI}(1, 1) \rightarrow \mathbf{Z}_2$. Thus, we see that $\mathbf{m}: \text{SFold}(1, 0) \rightarrow \text{SI}(1, 1)$ is an isomorphism.

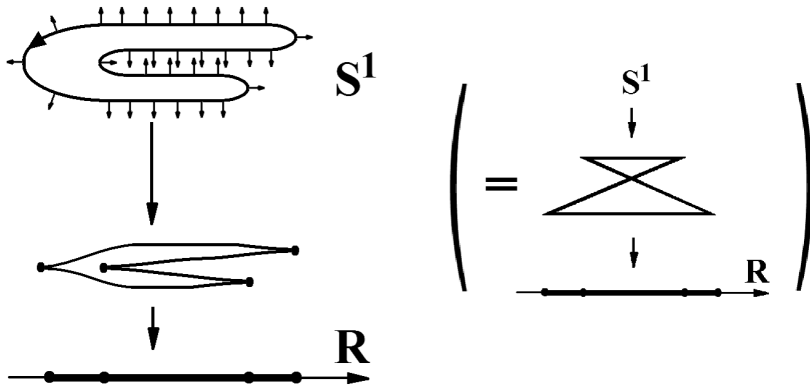


FIGURE 2. The \bowtie -fold map generating $\text{SFold}(1, 0)$

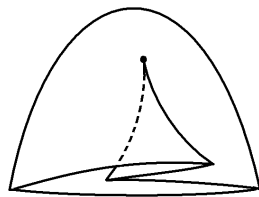


FIGURE 3. An extension with a cusp of the \bowtie -fold map

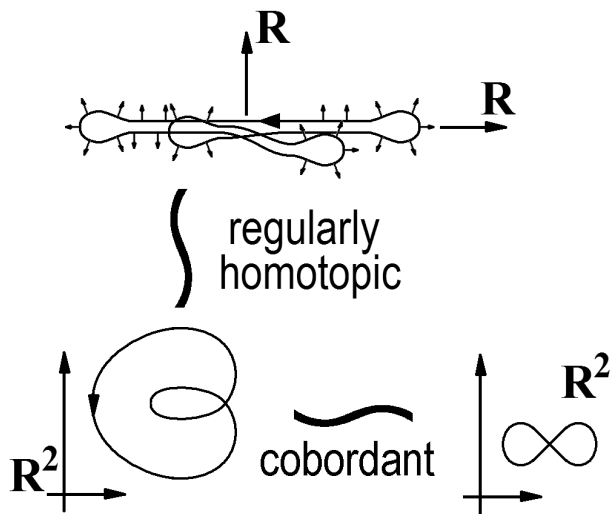


FIGURE 4. The “figure 8” immersion generating $\text{SI}(1, 1)$

Theorem 2.2. *The above homomorphism*

$$\mathfrak{m}: \text{SFold}(n, 0) \rightarrow \text{SI}(n, 1), [f] \mapsto [\bar{f}]$$

is an isomorphism for $n \geq 1$.

Proof. It suffices to show that the homomorphism \mathfrak{m} is surjective, since the groups on the both sides are known to be isomorphic to π_n^S .

Let $F: N^n \looparrowright \mathbf{R}^{n+1}$ be an immersion of an oriented n -manifold N^n in \mathbf{R}^{n+1} . Then, the regular homotopy class of F corresponds to the homotopy class of the induced stable framing of N^n . Therefore, due to Ando's h -principle [3, Corollary 2], we can deform F by regular homotopy into an immersion such that it followed by the projection $p: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$, $(y_1, \dots, y_{n+1}) \mapsto (y_1, \dots, y_n)$ becomes a fold map. Denote this resulting immersion by F' .

At each point x of the fold set $S(p \circ F')$ of the fold map $p \circ F'$, we can choose a normal vector $n(x)$ of $S(p \circ F') \subset N^n$ so that $dF_x(n(x))$ coincides with $(\partial/\partial y_{n+1})_{F'(x)}$. This defines a normal vector field n on each component of $S(p \circ F') \subset N^n$ since the normal bundle of $S(p \circ F') \subset N^n$ is orientable [15]. Thus we can choose the induced orientation σ_i of each component S_i of $S(p \circ F')$ such that (n, σ_i) agrees with the orientation of N^n .

Let ν be the normal vector field of $F(N^n) \subset \mathbf{R}^{n+1}$. Then, $(dp(\nu), d(p \circ F')(\sigma_i))$ agrees with the orientation of \mathbf{R}^n or with the opposite one, for each component S_i of $S(p \circ F')$. Denote by S_- the set of components of $S(p \circ F')$ on which $(dp(\nu), d(p \circ F')(\sigma_i))$ accords with the orientation of \mathbf{R}^n , and put $S_+ := S(p \circ F') \setminus S_-$. The left and right pictures of Figure 5 describe the situations of the normal (2-dimensional) disk at fold points of $F'(S_+) \subset \mathbf{R}^{n+1}$ and of $F'(S_-) \subset \mathbf{R}^{n+1}$, respectively.

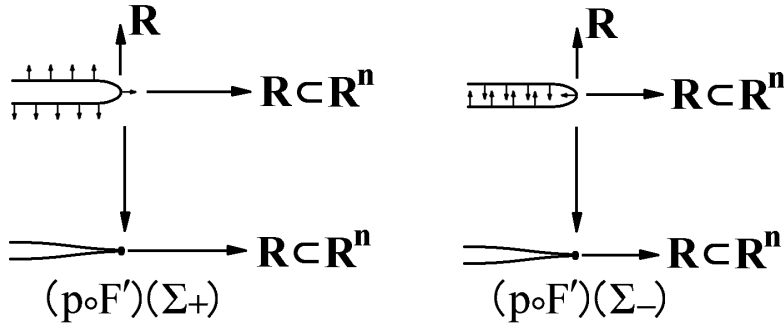


FIGURE 5. S_+ and S_-

Now, we modify F' near S_- by bordism. Inside the 2-dimensional normal disk of $S_- \subset \mathbf{R}^{n+1}$ at a point $x \in S_-$, the modification is described as Figure 6. This process changes F' by bordism and consequently we have an immersion of $(S_- \times S^1) \sharp N^n$ into \mathbf{R}^{n+1} , bordant to F' , which we denote by F'' . Clearly, the composition $p \circ F''$ is a fold map into \mathbf{R}^n and $\mathfrak{m}(p \circ F'')$ agrees with F'' bordant to F .

□

Remark 2.3. In the above proof, in order to obtain the inverse operation of \mathfrak{m} , we first deform F by regular homotopy into F' and then deform F'

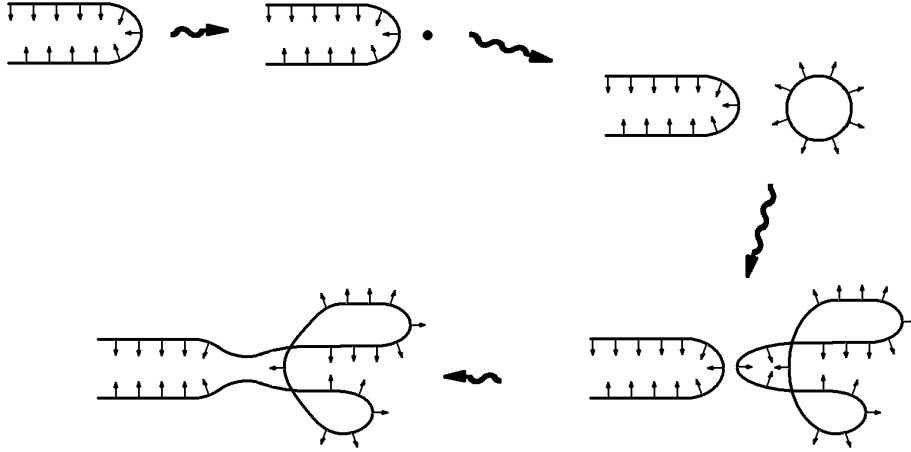


FIGURE 6. A bordism around $S_- \subset \mathbf{R}^{n+1}$, in the 2-dimensional normal disk

by bordism into F'' free from the S_- part. The regular homotopy alone is not enough here. This can be seen also from the following. If we regard the S^1 of the top of Figure 2 as an immersed (embedded) circle in \mathbf{R}^2 , then the immersion belongs to the trivial regular homotopy class, but its projection represents the generator of $\text{SFold}(1, 0)$ as a fold map. To obtain the correct inverse of the generator of $\text{SFold}(1, 0)$, we need to eliminate the S_- part by bordism as shown in Figure 6.

3. THE COMPOSITIONS OF FOLD MAPS

In the case of codimension one immersions of closed oriented manifolds the isomorphism $\text{SI}(n, 1) \rightarrow \pi_n^S$ is given through the Pontrjagin-Thom construction (see §2), which enables us to understand various algebraic operations geometrically. In Koschorke [10], for such codimension one immersions, the operations corresponding to the composition and to the Toda bracket in π_n^S are introduced. In this section, we interpret Koschorke's operations in terms of equi-dimensional fold maps via the isomorphism $\mathbf{m}: \text{SFold}(n, 0) \rightarrow \text{SI}(n, 1)$, explained in the previous section.

3.1. The composition. Let $f: N^n \looparrowright \mathbf{R}^{n+1}$ be an immersion of a closed oriented manifold N^n . Then we can extend f to an immersion from the total space of its normal line bundle, diffeomorphic to $N^n \times \mathbf{R}$. Denote this extension by $f': N^n \times \mathbf{R} \looparrowright \mathbf{R}^{n+1}$. For the symbol “ \looparrowright ” used below, see §2.1.

Let $\alpha \in \pi_a^S$ and $\beta \in \pi_b^S$. Let $i: A^a \rightarrow \mathbf{R}^a$ and $j: B^b \rightarrow \mathbf{R}^b$ be the corresponding fold maps, respectively. Suppose that $b \geq 1$. Then, in view of

Koschorke [10, §1] we see that the immersion corresponding to the composition $\alpha \circ \beta \in \pi_{a+b}^S$ is defined as

$$i * j: A^a \times B^b \xrightarrow{(\text{Id}, j)} A^a \times \mathbf{R}^b = (A^a \times \mathbf{R}) \times \mathbf{R}^{b-1} \xrightarrow{((\bar{i})', \text{Id})} \mathbf{R}^{a+1} \times \mathbf{R}^{b-1} = \mathbf{R}^{a+b}.$$

If $b = 0$, then $\beta \in \pi_0^S = \mathbf{Z}$ or j is represented by some integer s . Then we consider the immersion $i * j$ to be the union of s copies of i (each shifted in the last coordinate of \mathbf{R}^a for convenience).

Remark 3.1. We easily see from the above construction that the fold set $S(i * j)$ of the composition $i * j$ equals $A^a \times S(j) \subset A^a \times B^b$. Furthermore, we see that the immersion $(i * j)|_{S(i * j)}$ equals $\bar{i} \bar{*} (j|_{S(j)}): A^a \times S(j) \looparrowright \mathbf{R}^{a+b}$, where $\bar{*}$ stands for Koschorke's $*$ -product [10] of codimension one immersions that represents the composition of the corresponding stable homotopy classes under the isomorphism $\text{SI}(n, 1) \rightarrow \pi_n^S$. Thus, we can see that $\overline{i * j}$ equals $\bar{i} \bar{*} \bar{j}$, from which we can easily deduce the associativity of the operation $*$ for fold maps.

Example 3.2. (1) The fold map $T^2 \rightarrow \mathbf{R}^2$ in Figure 7, that is obtained by putting the \bowtie -fold map $S^1 \rightarrow \mathbf{R}^1$ (Figure 2) in each fibre of the normal line bundle of the “figure 8” immersion $S^1 \looparrowright \mathbf{R}^2$, represents the generator $\pi_2^S \approx \mathbf{Z}_2$ (cf. [12]), since $\eta \circ \eta$ generates $\pi_2^S \approx \mathbf{Z}_2$ (cf. [21, p.189]).

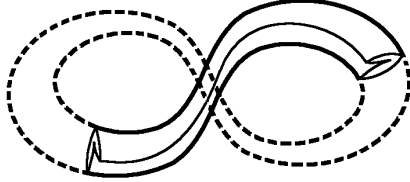


FIGURE 7. A fold map $T^2 \rightarrow \mathbf{R}^2$ generating $\pi_2^S \approx \mathbf{Z}_2$

- (2) The fold map $T^3 \rightarrow \mathbf{R}^3$ obtained by putting the \bowtie -fold map in each fibre of the normal line bundle of the “8 by 8” immersion $T^2 \looparrowright \mathbf{R}^3$ in Figure 8, represents $\eta \circ \eta \circ \eta$ which is known to be equal to 4ν for a generator ν of order 8 in $\pi_3^S \approx \mathbf{Z}_8 \oplus \mathbf{Z}_3$ [21, (5.5)].
- (3) We can repeat the similar construction in higher dimensions. However, the fold map $T^4 \rightarrow \mathbf{R}^4$ obtained by putting the \bowtie -fold map in each fibre of the normal line bundle of the “8 by 8 by 8” immersion $T^3 \looparrowright \mathbf{R}^4$ is null bordant, since $\eta \circ \eta \circ \eta \circ \eta \in \pi_4^S = 0$.

3.2. The Toda bracket. Let $\alpha \in \pi_a^S$, $\beta \in \pi_b^S$ and $\gamma \in \pi_c^S$. Let $i: A^a \rightarrow \mathbf{R}^a$, $j: B^b \rightarrow \mathbf{R}^b$ and $k: C^c \rightarrow \mathbf{R}^c$ be the corresponding fold maps, respectively.

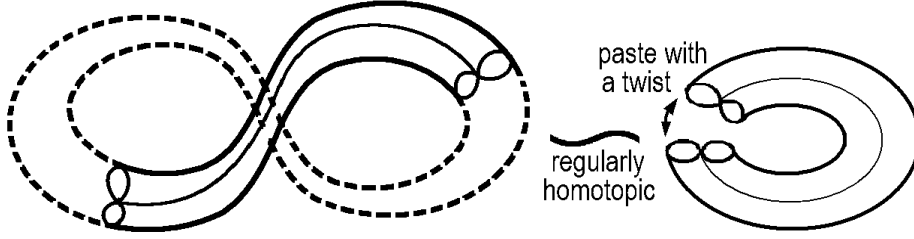


FIGURE 8. The “8 by 8” immersion $T^2 \looparrowright \mathbf{R}^3$, that generates $\text{SI}(2, 1)$

Suppose $\alpha \circ \beta = 0$ and $\beta \circ \gamma = 0$. Then, again in view of Koschorke [10, §1], the Toda bracket $\langle \alpha, \beta, \gamma \rangle$ is understood in terms of the fold maps i, j and k , as follows.

It follows from $\alpha \circ \beta = 0$ that $i * j: A^a \times B^b \rightarrow \mathbf{R}^{a+b}$ is null-bordant. Thus, we can take a null bordism, that is, a fold map from an $(a + b + 1)$ -dimensional manifold X^{a+b+1} with $\partial X^{a+b+1} = A^a \times B^b$

$$\ell_+: X^{a+b+1} \rightarrow \mathbf{R}^{a+b} \times [0, \infty)$$

such that ℓ_+ coincides with $(i * j) \times \text{Id}$ on a collar $\partial X^{a+b+1} \times [0, \epsilon) \subset X^{a+b+1}$. Similarly, by $\beta \circ \gamma = 0$ we can take a null bordism

$$\ell_-: Y^{b+c+1} \rightarrow \mathbf{R}^{b+c} \times (-\infty, 0]$$

of $j * k$. Thus, we have two null bordisms of $i * j * k$:

$$\ell_+ * k: X^{a+b+1} \times C^c \rightarrow \mathbf{R}^{a+b+c} \times [0, \infty)$$

and

$$i * \ell_-: A^a \times Y^{b+c+1} \rightarrow \mathbf{R}^{a+b+c} \times (-\infty, 0].$$

By pasting them along the common boundaries, we have a fold map from the closed manifold $(X^{a+b+1} \times C^c) \cup_{\partial} (A^a \times Y^{b+c+1})$ to $\mathbf{R}^{a+b+c+1}$. All fold maps constructed in this way form the Toda bracket $\langle \alpha, \beta, \gamma \rangle \subset \pi_{a+b+c+1}^S$.

Example 3.3. Choose a generator $\iota \in \pi_0^S \approx \mathbf{Z}$. Then, the corresponding fold map is the map $\{\text{one point}\} \rightarrow \mathbf{R}^0$. We can check that the above construction for $\langle 2\iota, \eta, 2\iota \rangle$ provides the immersion as in Figure 9, which is same as the fold map (representing $\eta \circ \eta$) in Figure 7. Thus, we can show the relation $\langle 2\iota, \eta, 2\iota \rangle = \eta \circ \eta \in \pi_2^S$ [21, Corollary 3.7] by this example.

4. A FOLD MAP WHICH GENERATES THE STABLE 6-STEM

Here, by using the composition in §3, we construct a fold map $S^3 \times S^3 \rightarrow \mathbf{R}^6$ which represents the generator of the stable 6-stem isomorphic to \mathbf{Z}_2 .

First we need the following observation.

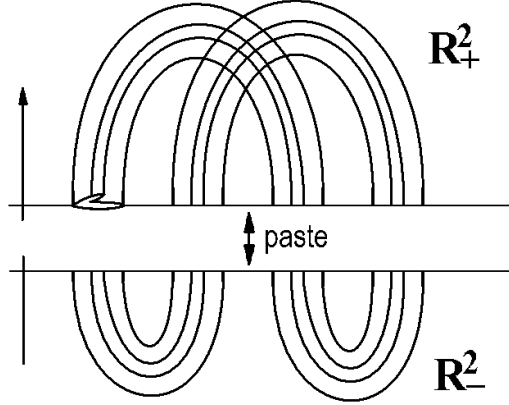


FIGURE 9. $\langle 2\iota, \eta, 2\iota \rangle: T^2 \rightarrow \mathbf{R}^2$

Proposition 4.1. *Let $F: S^3 \looparrowright \mathbf{R}^6$ be a self-transverse immersion with odd number of double points. Note that F has trivial normal bundle. Then, an immersion $f: S^3 \looparrowright \mathbf{R}^4$ obtained by compressing F into \mathbf{R}^4 represents a generator of order 8 in $\text{SI}(3, 1) \approx \pi_3^S \approx \mathbf{Z}_8 \oplus \mathbf{Z}_3$.*

Proof. Assume that f represents an even element in $\text{SI}(3, 1) \approx \mathbf{Z}_{24}$. Then, by [20, Proposition 4.5], f is bordant to the compression of an embedding $S^3 \hookrightarrow \mathbf{R}^6$, which implies that F is bordant (as an immersion) to an embedding. This, however, is impossible since the parity of the number of double points of a self-transverse immersion $S^3 \looparrowright \mathbf{R}^6$ is invariant up to bordism. \square

Let X be two copies of the 3-disk D^3 in 6-space intersecting with each other at exactly one point, whose “boundary” consists of the two copies of the 2-sphere. Then, by connecting with X two barycentric standard 3-spheres in \mathbf{R}^6 , (after suitably smoothing it) we obtain an immersion $F: S^3 \looparrowright \mathbf{R}^6$ with one double point (see Figure 10).

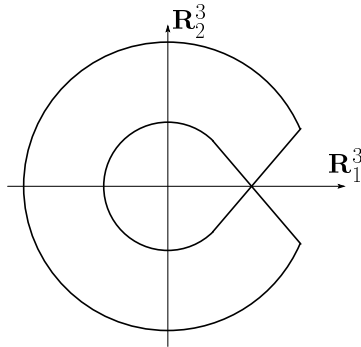


FIGURE 10. The fold map $F: S^3 \looparrowright \mathbf{R}^6$ with one double point

By Proposition 4.1, the composition of F and the projection $\mathbf{R}^6 \rightarrow \mathbf{R}^3$ in an appropriate direction (e.g. the projection $\mathbf{R}^6 = \mathbf{R}_1^3 \times \mathbf{R}_2^3 \rightarrow \mathbf{R}_1^3$ in

Figure 10) becomes a fold map $f: S^3 \rightarrow \mathbf{R}^3$ which represents a generator ν of order 8 in $\text{SFold}(3, 0) \approx \mathbf{Z}_8 \oplus \mathbf{Z}_3$ (this is very similar to Example 2.1, see §2).

Thus, by mapping a copy of S^3 via f into each normal disk of the (trivial) normal bundle of the immersion $F: S^3 \looparrowright \mathbf{R}^6$, we obtain a fold map $G: S^3 \times S^3 \rightarrow \mathbf{R}^6$ which represents $\nu \circ \nu \in \pi_6^S$ (see §3.1). Since $\nu \circ \nu$ generates $\pi_6^S \approx \mathbf{Z}_2$ by [21, p. 189], we see that G represents a generator of π_6^S .

Remark 4.2. Ando [6, §6] gives another explicit construction of a fold map $S^3 \times S^3 \rightarrow \mathbf{R}^6$ which represents the generator of π_6^S .

Remark 4.3. Let $F': S^3 \looparrowright \mathbf{R}^4$ be an immersion obtained by compressing the above F into \mathbf{R}^4 (see Proposition 4.1) and $j: \mathbf{R}^6 \rightarrow \mathbf{R}^7$ be the inclusion. If we immerse a copy of S^3 via F' into each normal 4-disk of the (trivial) normal bundle of the immersion $j \circ F': S^3 \looparrowright \mathbf{R}^7$, then we obtain an immersion $S^3 \times S^3 \looparrowright \mathbf{R}^7$ which represents the generator under $\text{SI}(6, 1) \approx \pi_6^S \approx \mathbf{Z}_2$.

Acknowledgements. The second-named author is partially supported by the Grant-in-Aid for Scientific Research (C), JSPS, Japan.

REFERENCES

- [1] Pyotr M. Akhmet'ev: *A geometric approach to stable homotopy groups of spheres: the Adams-Hopf invariants*, Fundam. Prikl. Mat. 13 (2007), no. 8, 3–15; translation in J. Math. Sci. (N. Y.) 159 (2009), 753–760.
- [2] Pyotr M. Akhmet'ev and Peter J. Eccles: *The relationship between framed bordism and skew-framed bordism*, Bull. London Math. Soc. 39 (2007), 473–481.
- [3] Y. Ando: *Folding maps and the surgery theory on manifolds*, J. Math. Soc. Japan 53 (2001), 357–382.
- [4] Y. Ando: *Fold-maps and the space of base point preserving maps of spheres*, J. Math. Kyoto Univ. 41 (2001), 693–737.
- [5] Y. Ando: *Existence theorems of fold-maps*, Japan J. Math. (N.S.) 30 (2004), 29–73.
- [6] Y. Ando: *Stable homotopy groups of spheres and higher singularities*, J. Math. Kyoto Univ. 46 (2006), 147–165.
- [7] Ja. M. Èliäsberg: *On singularities of folding type*, Math. USSR-Izv. 4 (1970), 1119–1134.
- [8] B. Kalmar: *Cobordism invariants of fold maps*, in: Real and complex singularities, 103–115, Contemp. Math., 459, Amer. Math. Soc., Providence, RI, 2008.

- [9] S. Kikuchi and O. Saeki: *Remarks on the topology of folds*, Proc. Amer. Math. Soc. 123 (1995), 905–908.
- [10] U. Koschorke: *Multiple points of immersions, and the Kahn-Priddy theorem*, Math. Z. 169 (1979), 223–236.
- [11] U. Koschorke: *Vector fields and other vector bundle morphisms — A singularity approach*, Lecture Notes in Math. 847, Springer-Verlag, 1981.
- [12] U. Pinkall: *Regular homotopy classes of immersed surfaces*, Topology 24 (1985), 421–434.
- [13] R. Rimányi and A. Szűcs: *Pontrjagin-Thom-type construction for maps with singularities*, Topology 37 (1998), 1177–1191.
- [14] R. Sadykov: *Fold maps, framed immersions and smooth structures*, Trans. Amer. Math. Soc. (in press; arXiv:0803.3780).
- [15] O. Saeki: *Notes on the topology of folds*, J. Math. Soc. Japan 44 (1992), 551–566.
- [16] O. Saeki: *Fold maps on 4-manifolds*, Comment. Math. Helv. 78 (2003), 627–647.
- [17] O. Saeki: *Topology of singular fibers of differentiable maps*, Lecture Notes in Math. 1854, Springer-Verlag, Berlin, 2004.
- [18] O. Szűcs: *Cobordism of maps with simplest singularities*, Lecture Notes in Math. 788, 223–244, Springer-Verlag, 1980.
- [19] O. Szűcs: *Cobordism of singular maps*, Geom. Topol. 12 (2008), 2379–2452.
- [20] M. Takase: *An Ekholm-Szűcs type formula for codimension one immersions of 3-manifolds up to bordism*, Bull. London Math. Soc. 32 (2007), 163–178.
- [21] H. Toda: *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49 Princeton University Press, Princeton, N.J. 1962.
- [22] R. Wells: *Cobordism groups of immersions*, Topology 5 (1966), 281–294.

DEPARTMENT OF GENERAL EDUCATION, NAGANO NATIONAL COLLEGE OF TECHNOLOGY, 716 TOKUMA, NAGANO, 381-8550 JAPAN
E-mail address: hirato@ge.nagano-nct.ac.jp

DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE, SHINSHU UNIVERSITY, MATSUMOTO, 390-8621 JAPAN
E-mail address: takase@math.shinshu-u.ac.jp