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A COST AND PIPELINE TRADE-OFF IN A TRANSPORTATION PROBLEM

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Abstract: The present paper deals with a trade off between cost and pipeline at a given time in a transportation problem. The time lag between commissioning a project and the time when the last consignment of goods reaches the project site is an important factor. This motivates the study of a bi-criteria transportation problem at a pivotal time T. An exhaustive set E of all independent cost-pipeline pairs (called efficient pairs) at time T is constructed in such a way that each pair corresponds to a basic feasible solution and in turn, gives an optimal transportation schedule. A convergent algorithm has been proposed to determine non-dominated cost pipeline pairs in a criteria space instead of scanning the decision space, where the number of such pairs is large as compared to those found in the criteria space.

Keywords: Transportation problem, Combinatorial optimization, Bottleneck transportation problem, Bi-criteria transportation problem, Efficient points

MSC: 90B06, 90C05, 90C08

1. INTRODUCTION

The cost minimization transportation problem is defined as:

$$\min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \tag{P$_1$}$$

subject to the following constraints:

$$\left.\begin{aligned} \sum_{j \in J} x_{ij} = a_i,\ a_i > 0,\ i \in I, \\ \sum_{i \in I} x_{ij} = b_j,\ b_j > 0,\ j \in J, \\ x_{ij} \geq 0,\ \forall\ (i,j) \in I \times J \end{aligned}\right\} \tag{1.1}$$

where I is the index set of supply points, J is the index set of destinations, x_{ij} is the amount of the product transported from i^{th} supply point to j^{th} destination, c_{ij} is the per unit of cost of transportation on $(i,j)^{th}$ route, a_i is the availability of the product at i^{th} supply point and b_j is the requirement of the same at j^{th} destination. Here the aim is to minimize the cost of transporting goods totalling $\sum_{i \in I} a_i = \sum_{j \in J} b_j$.
A time minimization transportation problem (TMTP), which is a special case of bottleneck linear programming problem, has been studied by Arora et al. [2, 3], Bhatia et al. [5], Garfinkel et al. [7], Hammer [9], Prakash [13] , Sharma et al. [17]and Szwarc [18]. In (TMTP), the transportation of goods from sources to destinations is done in parallel, and its prime aim is to supply to destinations with the required quantity within a shortest possible time. Mathematically, this problem can be formulated as follows:

$$\min_{X \in S} \{\max t_{ij}(x_{ij}) \mid x_{ij} > 0\}, \tag{1.2}$$

where

$$S = \{X = \{x_{ij}\} \mid X \text{ satisfies (1.1)}\} \tag{1.3}$$

and t_{ij} is the time of transportation from i^{th} supply point to j^{th} destination.
In the recent past, the transportation problem with more than one objective has been solved by many researchers like Bhatia et al. [4], Glickman and Berger [8], Khurana et al. [10], Malhotra and Puri [11], Purushotam et al. [16], Prakash [12, 14, 15]. Bhatia et al. [4] developed an enumerative technique to obtain successive time-cost commodity in pipeline trade-off relationship in a transportation

problem. Prakash [12] has solved the transportation problem with two objectives after having accorded first and second priorities to the minimization of total cost and duration of transportation, respectively. Purushotam et al. [16] dealt with this problem in reverse order of priorities. In the present paper, we have discuss cost-pipeline trade off at a pivotal time T.

Cost-pipeline tradeoff relationship in a transportation problem is relevant in a project planning, which requires transportation of raw materials/machinary etc. to the site of the project before it starts functioning. Besides, taking care of transportation costs, the goods reaching on the last day have also to be taken into consideration because the commissioning of the project is influenced in the sense that some time is consumed even after the last consignment of goods reaches the site, as some formalities are required to be completed before processing them onto the machine. This time lag between the arrival of the last consignment of goods and the time of initiation of the project indirectly means cost to the decision maker. As in some situations, early initiation of project is desired, which is possible when the quantity of goods reaching just before the initiation of the project is very small, but this in turn means high cost of transportation. Therefore the problem of interest is trade- off between total cost of transportation and pipeline at a time T (time of transportation) such that if early initiation of the project is desired, i.e. at some time $T^*(< T)$, means higher cost of transportation, such a time (T) is referred to as pivotal time. The proposed algorithm determines all the independent, non-dominated cost-pipeline pairs called efficient pairs, which correspond to basic feasible solutions (BFS) starting from the minimum cost solution at a pivotal time T chosen. The determination of extreme points of the non-dominated set in objective space in preference to the decision space is justified; as asserted by Aneja and Nair [1], the number of extreme points of the feasible set in objective space is in general lesser than the that in the decision space. The proposed algorithm finds all such pairs in criteria space. Process terminates when no more new efficient extreme points are available.

This paper is organized as follows : Definitions and notations are given in Section 2, theoretical results have been proved in Section 3. Section 4 discusses the procedure to solve the problem and the paper concludes with a numerical example discussed in Section-5.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

For any $X \in S$, let $T = \max\{t_{ij} \mid x_{ij} > 0\}$.
At any time T, define following cost minimization transportation problem, whose optimal solution yields minimum transportation cost at the given time T.

$$\min_{X \in S} \sum_{i \in I} \sum_{j \in J} c'_{ij} x_{ij} \tag{P_2}$$

where

$$c'_{ij} = \begin{cases} c_{ij} & \text{if } t_{ij} \leq T, \\ \infty & \text{if } t_{ij} > T \end{cases}$$

Let the optimal value of the problem (P_2) be denoted by Z.

Pivotal time. Time T is called pivotal time if for any other time of transportation T^*(say), $T^* > T \implies Z^* < Z$ and $T^* < T \implies Z^* > Z$, where Z and Z^* are minimum transportation costs given by (P_2) at times T and T^*, respectively.

Remark 2.1. *T is a pivotal time of transportation if in each optimal basic feasible solution (OBFS) of the problem (P_2) there exists a cell (i, j) with $t_{ij} = T,\ x_{ij} > 0$.*

Pipeline. The pipeline at pivotal time T corresponding to an optimal basic feasible solution $X = \{x_{ij}\}$ of (P_2) is given by $p = \sum_{\{(i,j)/t_{ij}=T\}} x_{ij}$.

Construct related problem $(RP - T)$ as:

$$\min_{X \in S} \sum_{i \in I} \sum_{j \in J} c^*_{ij} x_{ij} \tag{RP-T}$$

where

$$c^*_{ij} = \begin{cases} 0 & \text{if } t_{ij} < T, \\ 1, \text{ if } t_{ij} = T, & \\ \infty & \text{if } t_{ij} > T. \end{cases}$$

Remark 2.2. *The optimal value of the problem (RP-T) yields minimum pipeline at time T.*

So, mathematically the problem can be formulated as

$$\min_{X \in S}(Z,\ p), \tag{2.1}$$

where Z is the minimum cost of transportation problem and p is the minimum pipeline at time T, yielded by optimal solution of (P_2) and (RP-T), respectively.

Pair. A cost pipeline pair at given time of transportation T is denoted by $(T : Z,\ p)$.

Dominated pair. A pair $(T : Z,\ p)$ is called dominated pair at pivotal time T if there exists a pair $(T : Z^*,\ p^*)$ such $(Z,\ p) \geq (Z^*,\ p^*)$ i.e. $Z \geq Z^*$ and $p \geq p^*$ with strict inequality holding least at one place.

Non-dominated pair. A pair which is not dominated is called a non-dominated pair.

Efficient point. A solution yielding a non-dominated pair is called an efficient point.

3. NOTATIONS

$\mathbf{q^{th}}$ **Efficient pair** $(\mathbf{T}:\mathbf{Z_q},\ \mathbf{p_q}),\ (\mathbf{q}\geq\mathbf{2})$**.** The q^{th} efficient pair is that member of L_{q-1} for which $Z_q=\min\{Z \mid (T:Z,\ p)\in L_{q-1}\}$, where the set L_q for $q\geq 1$ is defined below.

For $q\geq 1$, following notations are introduced.
X^q is the set of all basic feasible solutions yielding the q^{th} efficient pair $=\{X^{qh}\mid h=1 \text{ to } s_q\}$.
B^{qh} is the set of basic cells of solution X^{qh}.
Δ_{ij} is the relative cost co-efficient for a basic feasible solution of problem (P_2) for a cell (i,j).
Δ'_{ij} is the relative cost co-efficient for a basic feasible solution of problem (RP-T) for the cell (i,j).
$\hat{X}^{qh}$ is the basic feasible solution derived from X^{qh} by a single pivot operation such that the corresponding time of transportation remains T, that is

$$\max_{\{(i,j)\ \mid\ \hat{x}_{ij}^{qh}>0\}} t_{ij}=T.$$

For each $h=1,2\ldots s_q$, define

$$N^{qh}=\left\{(i,j)\notin B^{qh}\ \middle|\ \begin{array}{l}\Delta_{ij}<0,\ \Delta'_{ij}>0,\ \hat{x}_{ij}^{qh}=x_{lm}^{qh},\ \hat{x}_{lm}^{qh}=0,\ (l,m)\in B^{qh}\\ \text{and}\quad \max\limits_{\{(r,w)\ \mid\ \hat{x}_{rw}^{qh}>0\}} t_{rw}=T\end{array}\right\}$$

Collecting those nonbasic cells, so that the entry of which into the current basis corresponds to a q^{th} efficient pair, increases the cost of transportation and reduces the pipeline.
$D_{qh}=\{(T:Z,p)|Z=Z_q-\Delta_{ij}x_{lm}^{qh},\ p=p_q-\Delta'_{ij}x_{lm}^{qh},\ (l,m)\in B^{qh},\ (i,j)\in N^{qh}\}$, is the collection of all the pairs $(T:Z,p)$, where Z is the increased cost and p is the reduced pipelineobtained by entering nonbasic cells from the set N^{qh} in a single pivot operation.

$$D_q=\bigcup_{h=1}^{s_q} D_{qh}$$

$L'_q=L_{q-1}-\{(T:Z_q,\ p_q)\}$ for $q\geq 2$, where $L'_1=\emptyset$ and the list L_q is given by:

$L_q=L'_q\cup D_q-\{(T:Z,\ p)\mid (T:Z,\ p)$ is a dominated pair in $L'_q\cup D_q\}$

E is the set of efficient pairs $(T:Z_i,\ p_i), i=1$ to M.

4. THEORETICAL DEVELOPMENT

This section discusses the main theoretical results, which lead to the convergence of procedure given in Section-4.

Theorem 4.3. *There exists a basic feasible solution yielding the first efficient pair* $(T : Z_1,\ p_1)$.

Proof : Let X_0 be an optimal basic feasible solution (OBFS) of problem (P_2) giving Z_1 as the minimum cost at pivotal time T. Let B_0 be the set of basic cells of the solution X_0. Construct set

$$N_0 = \{(i,j) \notin B_0 \mid \Delta_{ij} = 0,\ \Delta'_{ij} > 0\}$$

If $N_0 = \emptyset$, then X_0 gives the first efficient pair $(T : Z_1,\ p_1)$, else choose $(s,t) \in N_0$ such $\Delta'_{st} = \max\{\Delta'_{ij} \mid (i,j) \in N_0\}$ and enter cell (s,t) into basis B_0. This will reduce the pipeline at the same cost Z_1. Let the basic feasible solution thus obtained be X_1 with basis B_1. Let $N_1 = \{(i,j) \notin B_1 \mid \Delta_{ij} = 0,\ \Delta'_{ij} > 0\}$. Again, if $N_1 = \emptyset,\ X_1$ gives the pair $(T : Z_1, p_1)$ and when $N_1 \neq \emptyset$, entry of cell $(d,e) \in N_1$, into basis B_1, where $\Delta'_{de} = \max\{\Delta'_{ij} \mid (i,j) \in N_1\}$ further decreases the pipeline at cost Z_1. Continuing likewise, a sequence of solutions is constructed till a stage is reached at which $N_g = \emptyset$.
Denote X_g by X^{11}. Thus, X^{11} is a basic feasible solution with basis B^{11}, giving pair $(T : Z_1,\ p_1)$.

Remark 4.4. *If set* $H = \{(i,j) \notin B^{11} \mid \Delta_{ij} = 0,\ \Delta'_{ij} = 0\}$, *then* X^1 *is the set of all basic feasible solutions, each obtained as a result of entering a cell of* H *into basis* B^{11}.

Corollary 4.5. *Every efficient pair* $(T : Z_q,\ p_q),\ q \geq 2$, *is attainable at a basic feasible solution.*

Proof : The second efficient pair $(T : Z_2,\ p_2)$ is that member of L_1 for which $Z_2 = \min\{Z \mid (T : Z,\ p) \in L_1\}$. Thus $(T : Z_2,\ p_2)$ is attained at a basic feasible solution. By definition of $D_q,\ L_q\ (q \geq 2)$, every pair $(T : Z_q,\ p_q),\ q \geq 3$ also corresponds to a basic feasible solution.

Remark 4.6. *Corollary 4.5 justifies that* E *is a finite set.*

Theorem 4.7. *For any efficient pair* $(T : Z_q,\ p_q),\ p_q$ *is the minimum pipeline at cost* Z_q *and* Z_q *is the minimum cost at pipeline* p_q.

Proof : It is sufficient to show for each $h = 1$ to s_q, the sets

$$S_{1h} = \left\{ (i,j) \notin B^{qh} \;\middle|\; \begin{array}{l} \Delta_{ij} = 0,\ \Delta'_{ij} > 0,\ \hat{x}^{qh}_{ij} = x^{qh}_{lm},\ \hat{x}^{qh}_{lm} = 0,\ (l,m) \in B^{qh} \\ \text{and} \displaystyle\max_{\{(r,w)\ \mid\ \hat{x}^{qh}_{rw} > 0\}} t_{rw} = T \end{array} \right\}$$

and

$$S_{2h} = \left\{ (i,j) \notin B^{qh} \;\middle|\; \begin{array}{l} \Delta_{ij} > 0,\ \Delta'_{ij} = 0,\ \hat{x}^{qh}_{ij} = x^{qh}_{lm},\ \hat{x}^{qh}_{lm} = 0,\ (l,m) \in B^{qh} \\ \text{and} \displaystyle\max_{\{(r,w)\ \mid\ \hat{x}^{qh}_{rw} > 0\}} t_{rw} = T \end{array} \right\}$$

are both empty. Suppose, on the contrary, that $S_{1h} \neq \emptyset$ for some $h,\ 1 \leq h \leq s_q$. Then there exists a cell (i,j) in S_{1h} such that $\Delta_{ij} = 0,\ \Delta'_{ij} > 0,\ \hat{x}^{qh}_{ij} = x^{qh}_{lm},\ (l,m) \in B^{qh},\ \hat{x}^{qh}_{lm} = 0$. The entry of such a cell (i,j) into the basis of the solution X^{qh} will result in a pair $(T : Z_q,\ p)$ with $p < p_q$. This contradicts the non-dominance of $(T : Z_q,\ p_q)$. Hence $S_{1h} = \emptyset$. Similarly $S_{2h} = \emptyset\ \forall\ h = 1$ to s_q.

Corollary 4.8. *If $(T : Z,\ p)$ is a pair and $Z = Z',\ p \neq p'$ where $(T : Z',\ p')$ is an efficient pair, then $(T : Z,\ p)$ is dominated.*

Theorem 4.9. *A non-dominated pair $(T : Z,\ p)$ not in E satisfies the relation*

$$Z = \sum_{i=1}^{M} \lambda_i Z_i,\ p \leq \sum_{i=1}^{M} \lambda_i p_i,\ \ \sum_{i=1}^{M} \lambda_i = 1,\ \lambda_i \geq 0,\ i = 1 \text{ to } M.$$

Proof : Since $(T : Z,\ p)$ is a non-dominated pair not in E, therefore $(T : Z,\ p) \notin \bigcup_{i=1}^{M} L_i$ and $(T : Z,\ p)$ does not correspond to a basic feasible solution. Thus $(T : Z,\ p)$ is yielded by a feasible solution. Clearly $Z_1 < Z < Z_M$. There exist scalars λ_i not all zero such that $Z = \sum_{i=1}^{M} \lambda_i Z_i,\ \sum_{i=1}^{M} \lambda_i = 1,\ \lambda_i \geq 0,\ i = 1$ to M. Let $p^* = \sum_{i=1}^{M} \lambda_i p_i$. Since $(T : Z,\ p)$ is non-dominated, $p \leq p^* = \sum_{i=1}^{M} \lambda_i p_i$.

Theorem 4.10. *If $(T : Z,\ p)$ is a pair with $Z \neq Z_i,\ i = 1$ to M and $Z_1 < Z < Z_M$ then there exists $p' \leq p$ such that $(T : Z,\ p')$ is a non-dominated pair.*

Proof : Since $Z \neq Z_i,\ i = 1$ to M, therefore $(T : Z,\ p) \notin E$.
Let $Z_k < Z < Z_{k+1},\ k \in \{1,\ 2, \ldots M\}$. Then there exists λ such that $Z = \lambda Z_k + (1 - \lambda) Z_{k+1},\ 0 < \lambda < 1$. Consider $p^* = \lambda p_k + (1 - \lambda) p_{k+1}$. Clearly $p_{k+1} < p^* < p_k$. Also $p^* \leq p$, because if $p^* > p$ then $(T : Z,\ p^*)$ is dominated by $(T : Z,\ p)$ which is a contradiction as $(T : Z,\ p^*)$ being a convex combination of adjacent efficient pairs, must be non-dominated. Setting $p^* = p'$, the desired result is obtained.

Theorem 4.11. *E is the exhaustive set of efficient pairs.*

Proof : The proof is divided into two pairs.
Case I. No efficient pair other than the ones in E can be derived from a dominated pair.
Suppose on the contrary that $(T : Z',\ p')$ is an efficient pair derived from a dominated pair $(T : Z,\ p)$, such that $(T : Z',\ p') \neq (T : Z_i,\ p_i),\ i = 1$ to M. Now $(T : Z',\ p')$ is a non-dominated pair not in E and so from Corollary 4.3, it follows that it does not correspond to a basic feasible solution. This violates the very character of an efficient pair.
Case II. Any pair $(T : Z,\ p)$ derived from an efficient pair $(T : Z_q,\ p_q)$ with

$Z < Z_q,\ p > p_q$ is either identical with one of the efficient pairs $(T : Z_i,\ p_i),\ i =$ 1 to $q-1$ or is a dominated pair or is a convex combination of efficient pairs $(T : Z_i,\ p_i),\ i = 1$ to M. Now Z_1 being the global minimum cost at time T, $Z_1 \leq Z < Z_q$. If $p > p_1$, then $(T : Z,\ p)$ is dominated by $(T : Z_1,\ p_1)$ and the conclusion follows. Let now $p \leq p_1$. Thus $p_q < p \leq p_1$. Two exhaustive cases arise:
(i) $Z = Z_i$. for some $i \in \{1,\ 2, \ldots q-1\}$ In this case $p = p_1$ because if $p_1 > p$ then $(T : Z_1,\ p_1)$ is dominated by $(T : Z_1,\ p)$ which contradicts the efficient character of $(T : Z_1,\ p_1)$. Thus in this case $(T : Z,\ p)$ is identical with $(T : Z_1,\ p_1)$
(ii) $Z \neq Z_i$. Let $Z_r < Z < Z_{r+1}$ where $r \geq 1,\ q \geq r+1$. Let $Z = \lambda Z_r + (1-\lambda)Z_{r+1}$ where $0 < \lambda < 1$ and $p^* = \lambda p_r + (1-\lambda)p_{r+1}$. Thus $p_{r+1} < p^* < p_r$. If $p^* > p$, then $(T : Z,\ p^*)$ is dominated by $(T : Z,\ p)$ which is a contradiction, because $(T : Z,\ p^*)$ is a convex combination of two adjacent efficient pairs and therefore must be a non-dominated pair. Thus $p^* \leq p$, if $p^* < p$ then clearly $(T : Z,\ p)$ is a dominated pair and if $p^* = p$, then $(T : Z,\ p)$ is a convex combination of efficient pairs in E.

Theorem 4.12. *The last pair in E gives the global minimum pipeline at pivotal time T.*

Proof : Since $(T : Z_M,\ p_M)$ is the last pair, L_{M-1} is a singleton, viz. $L_{M-1} = \{(T : Z_M,\ p_M)\}$ and $D_M = \emptyset$. Thus $N^{Mh} = \emptyset$. Therefore there does not exist any cell $(i,j) \notin B^{Mh}$ with $\Delta_{ij} < 0,\ \Delta'_{ij} > 0$, the entry of which into the basis of solution X^{Mh} results in a solution with time of transportation equal to T. Thus the only possibility for cell $(i,j) \notin B^{Mh}$ with $\Delta' > 0$ are $\Delta_{ij} = 0,\ \Delta'_{ij} > 0$ or $\Delta_{ij} > 0,\ \Delta'_{ij} > 0$. In both cases, the fact that $(T : Z_M,\ p_M)$ is efficient, is contradicted. Hence $\Delta'_{ij} \not> 0$. Thus the set P given by

$$P = \left\{ (i,j) \notin B^{Mh},\ h = 1 \text{ to } s_M \;\middle|\; \begin{array}{l} \Delta'_{ij} > 0,\ \hat{x}^{Mh}_{ij} = x^{Mh}_{lm},\ \hat{x}^{Mh}_{lm} = 0,\ (l,m) \in B^{Mh} \\ \text{and} \displaystyle\max_{\{(r,w)\ \mid\ \hat{x}^{Mh}_{rw} > 0\}} t_{rw} = T \end{array} \right\}$$

is empty. Thus the pipeline cannot be reduced further at pivotal time T of transportation. Hence p_M is the global minimum pipeline at pivotal time T.

5. ALGORITHM FOR FINDING COST AND PIPELINE TRADEOFF PAIRS

Initially set $r = 0$ and $l = 0$. Let the partition of various time routes be given by $t_0 > t_1 > t_2 \cdots > t_k$, where $t_k = \min\{t_{ij} \mid i \in I, j \in J\}$

Step 0. Solve the problem (P_2) at time $T = t_l$ and go to the next step.

Step 1. If every optimal basic feasible solution of problem (P_2) yield time T, then declare T as pivotal time and go to the next step, otherwise set $l = l + 1$ and go to Step 0.

Step 2 (Finding Ist efficient pair; $(T : Z_1,\ p_1)$). Read optimal basic feasible solution of problem (P_2) corresponding to time T with minimum cost as Z_1 and pipeline p_1 and corresponding basis as B_r.

Step (2.a) Construct $N_r = \{(i,j) \notin B_r \mid \Delta_{ij} = 0, \Delta'_{ij} > 0\}$, if $N_r = \emptyset$, then go to Step (2.c), otherwise go to Step (2.b).

Step (2.b) Choose $(s,t) \in N_r$ such that $\Delta'_{st} = \max\{\Delta'_{ij} \mid (i,j) \in N_r\}$ and enter cell (s,t) into the basis B_r, set $r = r + 1$ and obtain new basic feasible solution as X_r with basis B_r and go to Step (2.a).

Step (2.c) Record $(T : Z_1,\ p_1)$ as the first efficient pair and the corresponding basic feasible solution as X^{11} with basis B^{11}. Construct the set $H = \{(i,j) \notin B^{11} \mid \Delta_{ij} = 0,\ \Delta'_{ij} = 0\}$ (as mentioned in Remark 4.2), compute $X^{1h},\ h = 2, 3 \ldots s_1$ and set $X^1 = \{X^{1h},\ h = 1, 2 \ldots s_1\}$, go to the next step.

Step 3. Initially record $E = \{(T : Z_1,\ p_1)\}$, set q=1 and go to the next step.

Step 4. Construct the set $N^{qh},\ h = 1, 2 \ldots s_q$. If $N^{qh} = \emptyset$ for all $h = 1, 2 \ldots s_q$, go to terminal step, otherwise go to Step 5.

Step 5. Construct $D_q,\ L'_q$ and L_q. Then note the $(q+1)^{\text{th}}$ efficient pair given as $(T : z_{q+1},\ p_{q+1})$, where $z_{q+1} = \min\{Z \mid (T : Z,\ p) \in L_q\}$ and set $E = E \cup \{(T : Z_{q+1},\ p_{q+1})\}$, set $q = q + 1$ and go to Step 4.

Step 6 (Terminal step). Set E is the exhaustive set of efficient pairs corresponding to the pivotal time T (as proved in Theorem 4.9).

Theorem 5.13. *The algorithm terminates in finite number of steps.*

Proof. By virtue of Corollory 4.5, it follows that each efficient pair at pivotal time T corresponds to an extreme point of the set of feasible solutions of (P_2). As there is a finite number of extreme points of the feasible set, and the choice of sets L'_q and L_q is such that none of the extreme points is repeatedly examined, the proposed algorithm terminates in finite number of steps.

6. NUMERICAL ILLUSTRATION

Consider a 4×6 transportation problem given by Table 1. The upper left corner in each cell gives the time of transportation on the corresponding route and the lower right corner gives the per unit cost on that route.

Table 1

				$a_i \downarrow$
12 / 19	13 / 32	34 / 8	40 / 12	17
7 / 70	18 / 30	36 / 40	7 / 60	27
11 / 40	20 / 8	45 / 20	21 / 22	28
$b_j \rightarrow$ 10	18	25	19	

The partition of various time routes is given by $t^0(=45) > t^1(=40) > t^2(=36) > t^3(=34) > t^4(=21) > t^5(=20) > t^6(=18) > t^7(=13) > t^8(=12) > t^9(=11) > t^{10}(=7)$. **Step 0.** Solve the problem (P_2) at time $T = t^0(=45)$ and go to Step 1.

Step 1. An optimal feasible solution is depicted in the follwoing table.

Table 2

				$a_i \downarrow$
12 / **10** / 19	13 / 32	34 / **7** / 8	40 / 12	17
7 / 70	18 / **9** / 30	36 / **18** / 40	7 / 60	27
11 / 40	20 / **9** / 8	45 / 20	21 / **19** / 22	28
$b_j \rightarrow$ 10	18	25	19	

Since the (OFS) of this problem does not yield time $T(=45)$, therefore $T = 45$ is not pivotal time. Set $l = 1$ and go to Step 0
Step 0. Solve the problem (P_2) at time $T = t^1(=40)$ and go to Step 1.
Step 1. This problem has two alternate optimal feasible solutions and are depicted

in Tables 3 and 4 below.

Table 3

					$a_i \downarrow$
	12 **10** 19	13 32	34 **7** 8	40 12	17
	7 70	18 **9** 30	36 **18** 40	7 60	27
	11 40	20 **9** 8	**M** 20	21 **19** 22	28
$b_j \rightarrow$	10	18	25	19	

Table 4

					$a_i \downarrow$
	12 **10** 19	13 32	34 8	40 **7** 12	17
	7 70	18 **2** 30	36 **25** 40	7 60	27
	11 40	20 **16** 8	**M** 20	21 **12** 22	28
$b_j \rightarrow$	10	18	25	19	

From Tables 3 and 4, it is observed that (OFS) of problem (P_2) at time $T = 40$ yield the time of transportation as 36 and 40, respectively. Therefore, $T = 40$ is not a pivotal time. Set $l = 2$ and go to Step 0.

Step 0. Solve the problem (P_2) at time $T = t^2 (= 36)$ and go to Step 1.

Step 1. An optimal feasible solution of this problem is depicted in Table 5.

Table 5

					$a_i \downarrow$
	12 **10** 19	13 32	34 **7** 8	**M** 12	17
	7 70	18 **9** 30	36 **18** 40	7 60	27
	11 40	20 **9** 8	**M** 20	21 **19** 22	28
$b_j \rightarrow$	10	18	25	19	

Since there is no alternate optimal solution to the one depicted in Table 5, $T = 36$ is a pivotal time. Go to Step 2.

Step 2. Note the pair $(T : Z,\ p) = (36 : 1726,\ 18)$ and the corresponding basis as B_0 and go to Step (2.a).

Step (2.a). Corresponding to the basis B_0, depicted in Table 5, the value of Δ'_{ij} and Δ_{ij} for $(i, j) \notin B_0$ is calculated bz using the formula $\Delta'_{ij} = u'_i + v'_j - c'_{ij}$ and $\Delta_{ij} = u_i + v_j - c_{ij}$ respectively, and is shown in the table below. Initially $u_1 = u'_1 = 0$.

Table 6

(i,j)	(1,2)	(1,4)	(2,1)	(2,4)	(3,1)	(3,3)
Δ'_{ij}	-1	-M-1	1	0	1	1-m
Δ_{ij}	-34	0	-19	-16	-11	-2

Since the set $N_0 = \{(i, j) \notin B_0 \mid \Delta_{ij} = 0,\ \Delta'_{ij} > 0\} = \emptyset$, the current recorded pair $(36 : 1726,\ 18)$ is efficient, go to Step (2.c)

Step (2.c). Record the first efficient pair as $(T : Z,\ p) = (36 : 1726,\ 18)$ with the corresponding solution as $X^1 = X^{11} = \{x^{11}_{11} = 10, x^{11}_{13} = 7, x^{11}_{22} = 9, x^{11}_{23} = 18, x^{11}_{32} = 9, x^{11}_{34} = 19\}$ and go to Step 3.

Step 3. Record $E = \{(36 : 1726,\ 18)\}$, set $q = 1$ and go to Step 4.

Step 4. Construct the set $N^{11} = \{(2, 1), (3, 1)\}$ from Table 6 and go to Step 5.

Step 5. Construct the set $D_1(= D_{11}) = \{(36 : 1825, 9), (36 : 1916, 8)\}$, by entering nonbasic cells from the set N^{11} successively in single pivot operation. since there are no dominated pairs in D_1, we have $L_1 = D_1$ and

$$(T : Z_2,\ p_2) = \min\{Z \mid (T : Z,\ p) \in L_1\},$$

we get $(T : Z_2,\ p_2) = (36 : 1825,\ 9)$. Update the set $E = \{(36 : 1726,\ 18), (36 : 1825,\ 9)\}$, set $q = 2$ and go to Step 4.

Step 4. The table depicting second efficient pair $(T : Z_2,\ p_2) = (36 : 1825,\ 9)$ is shown below

Table 7

				$a_i \downarrow$
	12 **1** 19	13 32	34 **16** 8	**M** 12	17
	7 70	18 **18** 30	36 **9** 40	7 60	27
	11 **9** 40	20 8	**M** 20	21 **19** 22	28
$b_j \rightarrow$	10	18	25	19	

Construct the set $N^{21} = \{(2,1),(2,4)\}$ on the same lines as constructed above the set N_0. Since $N^{21} \neq \emptyset$, go to Step 5.
Step 5. Constructing the set $D_2 = \{(36:1844,\ 8),\ (36:1852,\ 8)\}$, we have

$$L_2' = L_1 \setminus \{(36:1825,\ 9)\} = (36:1916,\ 8)$$

and $L_2 = \{(36:1844,\ 8)\}$. Therefore, the third efficient pair is given as $(36:1844,\ 8)$, update the set $E = \{(36:1726,\ 18),(36:1825,\ 9),(36:1844,\ 8)\}$. Set $q = 3$ and go to Step 4.
Step 4. Table depicting the third efficient pair is given below

Table 8

					$a_i \downarrow$
	12 19	13 32	34 **17** 8	**M** 12	17
	7 **1** 70	18 **18** 30	36 **8** 40	7 60	27
	11 **9** 40	20 8	**M** 20	21 **19** 22	28
$b_j \rightarrow$	10	18	25	19	

Since the set $N^{3h} = \emptyset$, go to Step 6.
Step 6. The exhaustive set of efficient pairs corresponding to the pivotal time $T = 36$ is given by $E = \{(36:1726,\ 18),(36:1825,\ 9),(36:1844,\ 8)\}$

7. CONCLUDING REMARKS

1. In this paper, we have presented an algorithm, which enumerates all the independent, non-dominated cost-pipeline pairs called efficient pairs, which correspond to basic feasible solutions (BFS) starting from the minimum cost solution at pivotal time T chosen. When the time taken for transportation is not pivotal, there always exists an alternate solution of problem (P_2), which yields zero unit of pipeline at time T, thereby reducing the total time of transportation from T to time $T' < T$ and yielding a unique cost pipeline pair with zero pipeline; hence, there is no need to provide nondominated solution.

2. It may be noted that in order to find all efficient pairs, cost minimization transportation problems (P_2) and RP-T are being solved repeatedly and by corollary 4.3, each such pair corresponds to an extreme point of S. Therefore, only finite number of such problems are to be solved. The best polynomial running time for cost minimization transportation problem is $o(m\log n(m + n\log n))$, where $|I| = m,\ |J| = n$ (Orlin [6]). Hence, the proposed algorithm is a polynomial time algorithm.

3. This problem can further be explored in the case when decision variable are taken as bounded.

4. The problem of finding cost pipeline efficient pairs with positive pipelines at a given particular time which may not be a pivotal time, can be explored further.

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