Lie symmetry and its generation of conserved quantity of Appell equation in a dynamical system of the relative motion with Chetaev-type nonholonomic constraints*

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Lie symmetry and conserved quantity deduced from Lie symmetry of Appell equations in a dynamical system of relative motion with Chetaev-type nonholonomic constraints are studied. The differential equations of motion of the Appell equation for the system, the definition and criterion of Lie symmetry, the condition and the expression of generalized Hojman conserved quantity deduced from Lie symmetry for the system are obtained. The condition and the expression of Hojman conserved quantity deduced from special Lie symmetry for the system under invariable time are further obtained. An example is given to illustrate the application of the results.

Keywords: Chetaev-type nonholonomic constraints, dynamics of relative motion, Appell equation, Lie symmetry

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1. Introduction

In 1894, the German scholar Hertz first distinguished holonomic constraints and nonholonomic constraints, dividing mechanical systems into holonomic systems and nonholonomic systems. Hence, nonholonomic mechanics was born. With the development of modern science and technology, the study of dynamics in complex mechanical systems is becoming more and more important. The movements of these complex systems include the movement of the carrier and the movement relative to the carrier. In the early 20th century, the Lagrange equation in a holonomic system of the rotary motion with a constant speed was studied by Whittaker. In 1984, the generalized Nielsen equation in a nonholonomic system of the rotary motion with a constant speed was reported by Mei Feng-Xiang in the third General Mechanics Conference of China. In 1899, Appell obtained equations of motion of mechanical systems that are essentially different from other researchers. This kind of equation is not only applicable in holonomic systems, but also in nonholonomic systems, and it is not only applied in generalized coordinates, but also applied in quasi coordinates. The modern symmetry theories in a constrained mechanical system are mainly of three types: Noether symmetry,[1–6] Lie symmetry,[7–16] and Mei symmetry.[17–26] In recent decades, Chinese scholars have achieved important results in studying symmetry theories of Appell systems and their applications.[27–36] In Refs. [37] and [38], symmetry theories in a dynamical system of the relative motion with nonholonomic constraints were introduced by Mei Feng-Xiang. But so far, no study of Lie symmetry and conserved quantity deduced from Lie symmetry of Appell equations in a dynamical system of relative motion with Chetaev-type nonholonomic constraints has been reported. This paper presents a study of the definition and criterion for Lie symmetry and conserved quantity deduced from Lie symmetry of Appell equations in a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type. The condition and the expression of generalized Hojman conserved quantity deduced from Lie symmetry have been researched. With the special infinitesimal transformations in which time is invariable, the condition and the expression of Hojman conserved quantity deduced from special Lie symmetry have been obtained. Finally, an example is given to illustrate the application of the results.

2. Differential equations of the system

Suppose that the configuration of the mechanical system is determined by N generalized coordinates q_s $(s=1,2,\ldots,n)$, the g two-sided ideal nonholonomic constraint equations of Chetaev's type subjected by the system

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0,\quad (\beta=1,2,\ldots,g). \tag{1}$$

Also, assume the constraints are compatible and independent of each other. Chetaev's conditions of constraint equations (1) putting on the virtual displacement δq_s are

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s=0,\quad (\beta=1,2,\ldots,g). \tag{2}$$

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Suppose the speed of the origin of a moving coordinate system O that is fixed on the sports reference materials is $\boldsymbol{\nu}_0$ and the angular velocity of reference materials is $\boldsymbol{\omega}$ which are known functions of time t. The acceleration energy for the system and Appell equations in a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type are respectively

$$\begin{aligned} S_r &= S_r(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}), &(3)\\ \frac{\partial S_r}{\partial \ddot{q}_s} &= Q_s - \frac{\partial}{\partial q_s}(V^0 + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s \\ &\quad + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \quad (s = 1, 2, \ldots n), &(4) \end{aligned}$$

where Q_s are the generalized forces, $V^0 = M(\dot{\boldsymbol{\nu}}_0 + \boldsymbol{\omega} \times \boldsymbol{\nu}_0) \cdot \boldsymbol{r}'_c$ is the potential energy in a uniform field, $V^\omega = -\boldsymbol{\omega} \cdot \theta^0 \cdot \boldsymbol{\omega}/2$ is called the potential energy of centrifugal force, $Q_s^{\dot{\omega}} = -(\dot{\boldsymbol{\omega}} \times m_i \boldsymbol{r}'_i) \cdot (\partial \boldsymbol{r}'_i / \partial q_s)$ is called the generalized Coriolis inertial force,

$$\Gamma_s = 2\boldsymbol{\omega} \cdot \left(m_i \frac{\partial \boldsymbol{r}'_i}{\partial q_s} \times \frac{\partial \boldsymbol{r}'_i}{\partial q_k} \right) \dot{q}_k$$

is called the generalized gyroscopic force, λ_β is the constraint multiplier.

From Eqs. (1) and (4), we can obtain all λ_β as a function of t, $\boldsymbol{q}$, and $\dot{\boldsymbol{q}}$. Hence, equation (4) can be written as

$$\begin{aligned} &\frac{\partial S_r}{\partial \ddot{q}_s} = Q_s - \frac{\partial}{\partial q_s}(V^0 + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s + \Lambda_s, \\ &(s = 1, 2, \ldots n), \end{aligned} \quad (5)$$

where

$$\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \quad (s = 1, 2, \ldots, n) \quad (6)$$

are generalized constraint reaction forces.

Equation (5) is an equation of the relative motion for a holonomic system corresponding to Eqs. (1) and (4) of the relative motion in a dynamical system with nonholonomic constraints of Chetaev's type. The equation of the relative motion in a dynamical system with nonholonomic constraints of Chetaev's type can be found in solutions of the equations of the relative motion in a holonomic system. Provided the initial conditions of motion satisfy the constraint equations of relative motion in a dynamical system with nonholonomic constraints of Chetaev's type, then the solution of Eq. (5) of relative motion in a holonomic system gives the motions of Eqs. (1) and (4) of the relative motion in a dynamical system with nonholonomic constraints of Chetaev's type. From Eq. (5), all generalized accelerations can be solved, written as follows:

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, 2, \ldots, n). \quad (7)$$

3. Lie symmetry and its generation of conserved quantity

Introducing the infinitesimal transformations of groups of time and generalized coordinates as

$$t^* = t + \Delta t, \quad q_s^*(t^*) = q_s(t) + \Delta q_s, \quad (s = 1, 2, \ldots, n), \quad (8)$$

equation (8) can be extended to

$$\begin{aligned} &t^* = t + \varepsilon \xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad q_s^*(t^*) = q_s(t) + \varepsilon \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ &(s = 1, 2, \ldots, n), \end{aligned} \quad (9)$$

where ε is an infinitesimal parameter, ξ_0 and ξ_s are infinitesimal generators. The invariance of the differential equations under the infinitesimal transformations of groups (9) is called Lie symmetry.

The determining equation of Lie symmetry of Appell equations Eq. (5) for a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type can be written as:

$$\begin{aligned} &X^{(2)} \left\{ \frac{\partial S_r}{\partial \ddot{q}_s} \right\} \\ &= X^{(1)} \left[Q_s - \frac{\partial}{\partial q_s}(V^0 + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s + \Lambda_s \right], \end{aligned} \quad (10)$$

where

$$X^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \left(\frac{\mathrm{d}\xi_s}{\mathrm{d}t} - \dot{q}_s \dot{\xi}_0 \right) \frac{\partial}{\partial \dot{q}_s}, \quad (11)$$

$$X^{(2)} = X^{(1)} + \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}\xi_s}{\mathrm{d}t} - \dot{q}_s \dot{\xi}_0 \right) - \ddot{q}_s \dot{\xi}_0 \right] \frac{\partial}{\partial \ddot{q}_s}. \quad (12)$$

The determining equation of Lie symmetry for Eq. (7) can be written as

$$\ddot{\xi}_s - \dot{q}_s \ddot{\xi}_0 - 2\dot{\xi}_0 \alpha_s = \frac{\partial \alpha_s}{\partial t} \xi_0 + \frac{\partial \alpha_s}{\partial q_k} \xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k} (\dot{\xi}_k - \dot{q}_k \dot{\xi}_0), \quad (13)$$

where the two determining equations (10) and (13) of Lie symmetry are equivalent.

Definition 1 If infinitesimal generators ξ_0 and ξ_s satisfy the determining equations (10) or (13), then the symmetry is Lie symmetry of the relative motion for a holonomic system (5) or (7) corresponding to Eqs. (1) and (4) of the relative motion in a nonholonomic system with Chetaev's type.

The invariance of the constraint equation (1) in a nonholonomic system under the infinitesimal transformations of groups (9) is expressed as the following constraint equation

$$X^{(1)} \left\{ f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \right\} = 0. \quad (14)$$

Definition 2 If infinitesimal generators ξ_0 and ξ_s satisfy the determining equations (10) or (13) and restriction equation (14), then the symmetry is weak Lie symmetry of the relative motion for a holonomic system (5) or (7) corresponding to Eqs. (1) and (4) of the relative motion in a nonholonomic system with Chetaev-type constraints.

Considering the restriction on infinitesimal generators ξ_0 and ξ_s by Chetaev's condition, so we have the following additional restriction equation

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\left(\xi_s-\dot{q}_s\xi_0\right)=0. \tag{15}$$

Definition 3 If infinitesimal generators ξ_0 and ξ_s satisfy the determining equations (10) or (13), restriction equation (14), and additional restriction equation (15), then the symmetry is strong Lie symmetry of the relative motion for a holonomic system (5) or (7) corresponding to Eqs. (1) and (4) of the relative motion in a nonholonomic system with Chetaev's type.

For a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type, generalized Hojman conserved quantity can be deduced directly from the Lie symmetry.

Proposition 1 For a holonomic system (7) corresponding to a dynamical system of relative motion with nonholonomic constraints of Chetaev's type, if the generators ξ_0 and ξ_s under the special infinitesimal transformations satisfy Eq. (13), and there exists a function $\mu=\mu\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right)$, which holds true for

$$\frac{\partial \alpha_s}{\partial \dot{q}_s}+\frac{\mathrm{d}}{\mathrm{d}t}\ln\mu=0, \tag{16}$$

then generalized Hojman conserved quantity deduced directly from the Lie symmetry of Appell equations in a dynamical system of relative motion with nonholonomic constraints of Chetaev's type is

$$\begin{aligned} I_{\mathrm{H}} &= \frac{1}{\mu}\frac{\partial}{\partial t}\left(\mu\xi_0\right)+\frac{1}{\mu}\frac{\partial}{\partial q_s}\left(\mu\xi_s\right)+\frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left[\mu\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)\right] \\ &= \text{const.} \end{aligned} \tag{17}$$

Proof Find the derivative of I_{H} with respect to t, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{H}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right)+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) \\ &+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s}+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)\right] \\ &+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right), \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t}=\frac{\partial\dot{\xi}_0}{\partial t}-\frac{\partial\alpha_k}{\partial t}\frac{\partial\xi_0}{\partial\dot{q}_k}, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s}=\frac{\partial\dot{\xi}_s}{\partial q_s}-\frac{\partial\alpha_k}{\partial q_s}\frac{\partial\xi_s}{\partial\dot{q}_k}, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right) \\ &=\frac{\partial}{\partial\dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)-\frac{\partial}{\partial q_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right) \\ &-\frac{\partial\alpha_k}{\partial\dot{q}_s}\frac{\partial}{\partial\dot{q}_k}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right). \end{aligned} \tag{19}$$

Substituting Eqs. (19) into Eq. (18) and using Eq. (13), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{H}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) \\ &+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\right]+\ddot{\xi}_0+\frac{\partial\alpha_s}{\partial\dot{q}_s}\dot{\xi}_0 \\ &+\frac{\partial^2\alpha_s}{\partial\dot{q}_s\partial t}\xi_0+\frac{\partial^2\alpha_s}{\partial\dot{q}_s\partial q_k}\xi_k \\ &+\frac{\partial^2\alpha_s}{\partial\dot{q}_s\partial\dot{q}_k}(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0). \end{aligned} \tag{20}$$

Taking the partial derivatives of condition (16) for $t,q_k,\dot{q}_k$, substituting them into Eq. (20), and using condition (13), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{H}}=0. \tag{21}$$

Proposition 2 For a holonomic system (7) corresponding to a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type, if the generators ξ_0 and ξ_s under the special infinitesimal transformations satisfy Eq. (13) and restriction equation (14), and there exists a function $\mu=\mu\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right)$ satisfying Eq. (16), then the generalized Hojman conserved quantities (17) can be deduced directly from the weak Lie symmetry of a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type.

Proposition 3 For a holonomic system (7) corresponding to a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type, if the generators ξ_0 and ξ_s under the special infinitesimal transformations satisfy Eq. (13), restriction equation (14), and special additional restriction equation (15), and there exists a function $\mu=\mu\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right)$ satisfying Eq. (16), then the generalized Hojman conserved quantities (17) can be deduced from the strong Lie symmetry of a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type.

If $\xi_0=0$, then transformations (9) become the special infinitesimal transformations in which the time is invariable, which are expressed as

$$t^*=t,\quad q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}). \tag{22}$$

The determining equation of Lie symmetry for Eq. (7) under the special infinitesimal transformations (22) can be expressed as

$$\ddot{\xi}_s=\frac{\partial\alpha_s}{\partial q_k}\xi_k+\frac{\partial\alpha_s}{\partial\dot{q}_k}\dot{\xi}_k. \tag{23}$$

The invariance of the constraint equation (1) in a nonholonomic system under the special infinitesimal transformations (22) is expressed as the following restriction equation

$$X^{(1)}\left\{f_\beta\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right)\right\}=0. \tag{24}$$

Considering the restriction on infinitesimal generators ξ_s by Chetaev's condition, so we have the following additional restriction equation

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\xi_s = 0. \tag{25}$$

Deduction 1 If the generators ξ_s under the infinitesimal transformations (22) satisfy Eq. (23), and there exists a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ which holds true for

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\mu = 0, \tag{26}$$

then Hojman conserved quantity deduced directly from the Lie symmetry of Appell equations in a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type is

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu\frac{\mathrm{d}}{\mathrm{d}t}\xi_s\right) = \text{const.} \tag{27}$$

Deduction 2 For a holonomic system (7) corresponding to a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type, if the generators ξ_s under the special infinitesimal transformations satisfy Eq. (23) and restriction equation (24), and there exists a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying Eq. (26), then Hojman conserved quantities (27) are deduced directly from the weak Lie symmetry of dynamical system of the relative motion with nonholonomic constraints of Chetaev's type.

Deduction 3 For a holonomic system (7) corresponding to a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type, if the generators ξ_s under the special infinitesimal transformations satisfy Eq. (23), restriction equation (24), and special additional restriction equation (25), and there exists a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying Eq. (26), then Hojman conserved quantities (27) are deduced from the strong Lie symmetry of a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type.

4. An illustrative example

In the following, we give an example, which is only to illustrate the application of the above results.

The carrier with angular velocity ω around the vertical axis q_3 for a fixed axis rotation, a particle of quality as m is moving without friction on the carrier. It is easy to know the acceleration energy in a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type, the generalized gyroscopic force, the generalized Coriolis inertial force, the potential energy in a uniform field, and the potential energy of centrifugal force are respectively

$$S_r = \frac{1}{2}m\left(\ddot{q}_1^2 + \ddot{q}_2^2 + \ddot{q}_3^2\right), \tag{28}$$

$$\Gamma_1 = 2m\omega\dot{q}_2, \quad \Gamma_2 = -2m\omega\dot{q}_1, \quad \Gamma_3 = 0, \tag{29}$$

$$Q_s^{\dot{\omega}} = 0, \quad (s = 1, 2, 3), \tag{30}$$

$$V^0 = 0, \tag{31}$$

$$V^\omega = -\frac{1}{2}m\left(q_1^2 + q_2^2\right)\omega^2. \tag{32}$$

Suppose constrained equations of Chetaev's type and generalized forces of the system are respectively

$$f = \dot{q}_1^2 + \dot{q}_2^2 - \dot{q}_3^2 = 0, \tag{33}$$

$$\begin{aligned} &Q_1 = -m\omega^2 q_1 - 2m\omega\dot{q}_2, \quad Q_2 = -m\omega^2 q_2 + 2m\omega\dot{q}_1, \\ &Q_3 = -mg. \end{aligned} \tag{34}$$

Try to study Lie symmetry and conserved quantity deduced from Lie symmetry of the system.

From Eq. (4), we get

$$\begin{aligned} &m\ddot{q}_1 = 2\lambda\dot{q}_1, \quad m\ddot{q}_2 = 2\lambda\dot{q}_2, \\ &m\ddot{q}_3 = -mg - 2\lambda\dot{q}_3. \end{aligned} \tag{35}$$

Combining Eqs. (35) and (33), we obtain

$$\lambda = -\frac{mg}{4\dot{q}_3}. \tag{36}$$

Substituting Eq. (36) into Eq. (35), we obtain

$$\begin{aligned} &m\ddot{q}_1 = -\frac{mg\dot{q}_1}{2\dot{q}_3}, \quad m\ddot{q}_2 = -\frac{mg\dot{q}_2}{2\dot{q}_3}, \\ &m\ddot{q}_3 = -\frac{mg}{2}. \end{aligned} \tag{37}$$

The determining equation (23) of special Lie symmetry gives

$$\begin{aligned} &\ddot{\xi}_1 = -\dot{\xi}_1\frac{g}{2\dot{q}_3} + \dot{\xi}_3\frac{g\dot{q}_1}{2\dot{q}_3^2}, \\ &\ddot{\xi}_2 = -\dot{\xi}_2\frac{g}{2\dot{q}_3} + \dot{\xi}_3\frac{g\dot{q}_2}{2\dot{q}_3^2}, \\ &\ddot{\xi}_3 = 0. \end{aligned} \tag{38}$$

The restriction equation (24) gives

$$\dot{q}_1\dot{\xi}_1 + \dot{q}_2\dot{\xi}_2 - \dot{q}_3\dot{\xi}_3 = 0. \tag{39}$$

The additional restriction equation (25) gives

$$\dot{q}_1\xi_1 + \dot{q}_2\xi_2 - \dot{q}_3\xi_3 = 0. \tag{40}$$

Take the infinitesimal generator

$$\xi_1 = \xi_2 = \xi_3 = 1, \tag{41}$$

$$\xi_1 = \xi_2 = 0, \xi_3 = \left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2gq_3\right)^2. \tag{42}$$

Obviously, they all satisfy Eqs. (38) and (39), and they are all the infinitesimal generators of weak Lie symmetry, then the systems are special Lie symmetrical.

According to the expression (26), we have

$$-\frac{g}{\dot{q}_3} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\mu = 0. \tag{43}$$

We can solve

$$\mu = \frac{1}{\dot{q}_3^2}, \tag{44}$$

$$\mu = \frac{1}{\dot{q}_3^2}\left(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2+2gq_3\right). \tag{45}$$

Substituting Eqs. (41) and (45) into Eq. (26), we obtain

$$I_{\mathrm{H}_1} = 2g\left(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2+2gq_3\right)^{-1} = \text{const.} \tag{46}$$

Substituting Eqs. (39) and (41) into Eq. (26), we obtain

$$I_{\mathrm{H}_2} = 4g\left(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2+2gq_3\right) = \text{const.} \tag{47}$$

The infinitesimal generators (41) and (42) respectively satisfy the restriction equations (39), but they do not satisfy the additional restriction equations (40), so they are not the infinitesimal generators of strong Lie symmetry for a system. From Proposition 2 and Proposition 3, we know I_{H_1} and I_{H_2} are generalized Hojman conserved quantities of Lie symmetry and weak Lie symmetry for a system, but they are not generalized Hojman conserved quantities of strong Lie symmetry.

5. Conclusion

The determining equations (10) and (13), three propositions and three deductions corresponding to them for Lie symmetry of Appell equations in a dynamical system of the relative motion with Chetaev-type nonholonomic constraints are the main results of this paper. The results in this paper can be applied to Appell equations in a dynamical system of the relative motion with holonomic constraints, Appell equations in a mechanical system with nonholonomic constraints of Chetaev's type, Appell equations in a mechanical system with non-Chetaev nonholonomic constraints, Appell equations in a mechanical system with holonomic constraints, and so on.

References

[1] Ge W K 2002 *Acta Phys. Sin.* **51** 1156 (in Chinese)

[2] Zhang Y and Mei F X 2004 *Acta Phys. Sin.* **53** 2419 (in Chinese)

[3] Wu H B and Mei F X 2006 *Acta Phys. Sin.* **55** 3825 (in Chinese)

[4] Zheng S W, Tang Y F and Fu J L 2006 *Chin. Phys.* **15** 243

[5] Shang M and Mei F X 2007 *Chin. Phys.* **16** 3161

[6] Fu J L, Nie N M, Huang J F, Jiménez S, Tang Y F, Vázquez L and Zhao W J 2009 *Chin. Phys.* B **18** 2634

[7] Mei F X and Shang M 2000 *Acta Phys. Sin.* **49** 1901 (in Chinese)

[8] Luo S K 2003 *Acta Phys. Sin.* **52** 2941 (in Chinese)

[9] Chen X W, Liu C M and Li Y M 2006 *Chin. Phys.* **15** 470

[10] Luo S K 2007 *Chin Phys. Lett.* **24** 2463

[11] Zhang Y 2007 *Acta Phys. Sin.* **56** 3054 (in Chinese)

[12] Cai J L and Mei F X 2008 *Acta Phys. Sin.* **57** 5369 (in Chinese)

[13] Cai J L, Luo S K and Mei F X 2008 *Chin. Phys.* B **17** 3170

[14] Cai J L 2008 *Chin. Phys. Lett.* **25** 1523

[15] Fang J H 2010 *Chin. Phys.* B **19** 40301

[16] Xie Y L, Jia L Q and Yang X F 2011 *Acta Phys. Sin.* **60** 030201 (in Chinese)

[17] Mei F X 1998 *Analytical Mechanics Topics* (Beijing: Beijing Institute of Technology Press)

[18] Lou Z M 2004 *Acta Phys. Sin.* **53** 2046 (in Chinese)

[19] Zheng S W, Xie J F and Chen W C 2008 *Chin. Phys. Lett.* **25** 809

[20] Cai J L 2009 *Acta Phys. Sin.* **58** 22 (in Chinese)

[21] Wu H B and Mei F X 2009 *Chin. Phys.* B **18** 3145

[22] Fang J H 2009 *Acta Phys. Sin.* **58** 3617 (in Chinese)

[23] Zheng S W, Xie J F, Chen X W and Du X L 2010 *Acta Phys. Sin.* **59** 5209 (in Chinese)

[24] Jiang W A, Li Z J and Luo S K 2011 *Chin. Phys.* B **20** 030202

[25] Jiang W A and Luo S K 2011 *Acta Phys. Sin.* **60** 060201 (in Chinese)

[26] Cai J L 2011 *Nonlinear Dyn.* **69** 487

[27] Mei F X, Liu D and Luo Y 1991 *Advanced Analytical Mechanics* (Beijing: Beijing Institute of Technology Press)

[28] Mei F X 2001 *Chin. Phys.* **10** 177

[29] Li R J, Qiao Y F and Meng J 2002 *Acta Phys. Sin.* **51** 1 (in Chinese)

[30] Luo S K 2002 *Acta Phys. Sin.* **51** 712 (in Chinese)

[31] Luo S K 2008 *Advances in the Study of Dynamics of Constrained Systems* (Beijing: Science Press)

[32] Jia L Q, Xie J F and Zheng S W 2008 *Chin. Phys.* B **17** 17

[33] Li Y C, Xia L L, Wang X M and Liu X W 2010 *Acta Phys. Sin.* **59** 3639 (in Chinese)

[34] Jia L Q, Xie Y L, Zhang Y Y, Cui J C and Yang X F 2010 *Acta Phys. Sin.* **59** 7552 (in Chinese)

[35] Jia L Q, Xie Y L and Luo S K 2011 *Acta Phys. Sin.* **60** 040201 (in Chinese)

[36] Yang X F, Sun X T, Wang X X, Zhang M L and Jia L Q 2011 *Acta Phys. Sin.* **60** 111101 (in Chinese)

[37] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press)

[38] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press)