CANONICAL FORMS OF BOREL FUNCTIONS ON THE MILLIKEN SPACE

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ABSTRACT: The goal of this paper is to canonize Borel measurable mappings $\Delta: \Omega^\omega \to \mathbb{R}$, where Ω^ω is the Milliken space, i.e., the space of all increasing infinite sequences of pairwise disjoint nonempty finite sets of ω. Our main result refers to the metric topology on the Milliken space. The result is a common generalization of a theorem of Taylor (cf. Theorem 0.4) and a theorem of Prömel and Voigt (cf. Theorem 0.7).

0. Introduction

Ramsey's Theorem [Ra30] is an important extension of the *pigeon-hole principle*: If $\omega = P_0 \cup \ldots \cup P_{k-1}$ is a partition of ω into finitely many pieces, then for some $i < k$, P_i is infinite.

THEOREM 0.1. (Ramsey R) *Let $l \in \omega$. If $[\omega]^l = P_0 \cup \ldots \cup P_{k-1}$ is a partition of $[\omega]^l$ into finitely many pieces, there is an infinite set $A \in [\omega]^\omega$ such that $[A]^l \subseteq P_i$ for some $i < k$.*

Ramsey's Theorem can be viewed as a canonization of finite-range functions on $[\omega]^l$. Later P. Erdös and R. Rado [ErRa50] canonized arbitrary such functions.

THEOREM 0.2. (Erdös-Rado ER) *If $k \in \omega$ and $f: [\omega]^k \to \omega$, then there exists an infinite set $X \subseteq \omega$ and a set $\Delta(f, X) \subseteq 0, \ldots, k - 1$ such that if $\{x_0, \ldots, x_{k-1}\}$ and $\{y_0, \ldots, y_{k-1}\}$ are in $[X]^k$ with $x_0 < \ldots < x_{k-1}$ and $y_0 < \ldots < y_{k-1}$, then*

$$f(\{x_0, \ldots, x_{k-1}\}) = f(\{y_0, \ldots, y_{k-1}\}) \text{ iff } x_i = y_i \text{ for all } i \in \Delta(f, X).$$

About twenty years later N. Hindman [Hi74] analysed the space of all finite subsets of ω. He found the following famous result. For $\kappa \leq \omega$ and $a \in \Omega^\omega$ let $(a)^\kappa$ denote the collection of all increasing sequences of κ pairwise disjoint nonempty finite subsets of ω, which are obtained by unions of some $a(i)$, $i \in \omega$.

THEOREM 0.3. (Hindman) *Let $k \in \omega$. If $f\colon [\omega]^{<\omega} \to k$, then there exists $a \in \Omega^{\omega}$ such that f is constant on $(a)^1$.*

This theorem was the basis of the work of K. R. Milliken and A. D. Taylor mentioned below. Taylor proved a canonical partition relation for finite subsets of ω that generalizes Hindman's Theorem in much the same way that the Erdös-Rado Theorem generalizes Ramsey's Theorem. In his proof Taylor used a n-dimensional version of Theorem 0.3, which was obtained independently also by Milliken (see Lemma 2.5, H^n).

The following result of Taylor [Ta76] was stimulating for a part of this work.

THEOREM 0.4. (Taylor T) *If $f\colon [\omega]^{<\omega} \to \omega$, then there exists $a \in \Omega^{\omega}$ such that exactly one of (a) - (e) holds:*

(a) If $m, n \in (a)^1$, then $f(m) = f(n)$.

(b) If $m, n \in (a)^1$, then $f(m) = f(n)$ iff $\min(m) = \min(n)$.

(c) If $m, n \in (a)^1$, then $f(m) = f(n)$ iff $\max(m) = \max(n)$.

(d) If $m, n \in (a)^1$, then $f(m) = f(n)$ iff $\min(m) = \min(n)$ and $\max(m) = \max(n)$.

(e) If $m, n \in (a)^1$, then $f(m) = f(n)$ iff $m = n$.

F. Galvin and K. Prikry have shown in [GaPr73] that a similar result to Theorem 0.1 is valid for finite partitions of $[\omega]^{\omega}$ - with the restriction that all pieces of the partitions must be Borel.

THEOREM 0.5. (Galvin-Prikry GP) *Let $k > 0$ and $[\omega]^{\omega} = P_0 \cup \ldots \cup P_{k-1}$, where each P_i is Borel. Then there is an infinite set $A \in [\omega]^{\omega}$ and $i < k$ with $[A]^{\omega} \subseteq P_i$.*

The power set of ω can be identified with the *Cantor space* 2^{ω}. It can be endowed with the product topology of the discrete topology on ω. It is a well-known fact that this topological space is completely metrizable. Thus, we can interpret the spaces $[\omega]^1$ and $[\omega]^{\omega}$ in the theorems above as topological spaces with the relative topology of $[\omega]^{\leqslant\omega}$. For distinction we call this topology the *metric topology* of $[\omega]^{\leqslant\omega}$.

A subset $P \subsetneq [\omega]^{\omega}$ is called *Ramsey* iff there is an infinite set $A \in [\omega]^{\omega}$ such that either $[A]^{\omega} \subsetneq P$ or else $[A]^{\omega} \cap P = \varnothing$. By Theorem 0.5 every Borel set is Ramsey.

J. Silver [Si70] extended the result of Galvin-Prikry to analytic sets. Subsequent to Silver's investigation A. Mathias [Ma68] obtained a new proof of the same result.

For stronger results E. Ellentuck has introduced a finer topology on $[\omega]^{\omega}$ which is called *Ellentuck topology*. For any $a \in [\omega]^{<\omega}$ and $A \in [\omega]^{\omega}$ with $a < A$ let $[a, A]^{\omega} = \{S \in [\omega]^{\omega}: a \subsetneq S \subsetneq a \cup A\}$. The Ellentuck topology then has as basic open sets all the sets of the form $[a, A]^{\omega}$ for $a < A$. Note that there are continuum many pairwise disjoint ones of them. Clearly the Ellentuck topology is finer than the metric topology.

Call a set $P \subsetneq [\omega]^{\omega}$ *completely Ramsey* iff for every $a < A$ there is $B \in [A]^{\omega}$ with $[a, B]^{\omega} \subsetneq P$ or $[a, B]^{\omega} \cap P = \varnothing$. Ellentuck [El74] has shown the following main result, which is slightly stronger than the theorem of Galvin-Prikry.

THEOREM 0.6. (Ellentuck) *Let $P \subsetneq [\omega]^{\omega}$. Then P is completely Ramsey, if P has the Baire property in the Ellentuck topology.*

Moreover Galvin [El74] made the observation, that every completely Ramsey set has the Baire property. Therewith also the converse of Theorem 0.6 holds. An analogous result with respect to a finer topology – the *Σ-topology* – was proven by Milliken (see Theorem 4.4 in [Mi75]). Especially, we take notice of a corollary of Milliken's result (M): Let $k > 0$ and $\Omega^{\omega} = P_0 \cup \ldots \cup P_{k-1}$ be a partition of Ω^{ω} into finitely many pieces, where each P_i is Borel. Then there exists $a \in \Omega^{\omega}$ and $i < k$ with $(a)^{\omega} \subsetneq P_i$.

P. Pudlák and V. Rödl [PuRö82] canonized Borel-measurable mappings on $[\omega]^{\omega}$ with a countable range. The following result of H. J. Prömel and B. Voigt [PrVo85] gives the canonization of such functions with arbitrary range.

THEOREM 0.7. (Prömel-Voigt PV) *Let $\Delta: [\omega]^{\omega} \to \mathbb{R}$ be a Borel-measurable mapping. Then there exists $A \in [\omega]^{\omega}$ and there exists γ:*

$[A]^{<\omega} \to \{s, m\}$ such that the mapping Γ: $[A]^{\omega} \to [A]^{\leqslant\omega}$ with $\Gamma(X) = \{k \in X: \gamma(X \cap k) = s\}$ has the following properties:

(a) $\Gamma(X) \subseteq X$ for all $X \in [A]^{\omega}$,

(b) for no $X, Y \in [A]^{\omega}$ there exists $k \in \Gamma(Y)$ such that $\Gamma(X) = \Gamma(Y) \cap k$, i.e., no $\Gamma(X)$ is a proper initial segment of some $\Gamma(Y)$,

(c) for all $X, Y \in [A]^{\omega}$ it follows that $\Delta(X) = \Delta(Y)$ iff $\Gamma(X) = \Gamma(Y)$.

The following Figure 0.1 shows the relation between the theorems mentioned above. Here $A \to B$ means that A generalizes B.

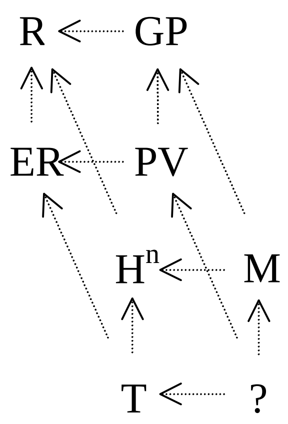

FIGURE 0.1.

All of these implications are pretty obvious and well-known. It was natural to search for a theorem, which stands at the place of the interrogation sign. The purpose of this paper is to provide such a theorem.

First, we give some definitions to be able to formulate our Main Theorem. Let $\Omega^{<\omega}$ denote the space of all increasing finite sequences of pairwise disjoint nonempty finite subsets of ω.

DEFINITION. Let γ: $\Omega^{<\omega} \to \{sm, min\text{-}sep, max\text{-}sep, min\text{-}max, sss, vss\}$. For $m \in [\omega]^{<\omega}$ let $sm(m) = \emptyset$, $min\text{-}sep(m) = \{\min(m)\}$, $max\text{-}sep(m) = \{\max(m)\}$, $min\text{-}max(m) = \{\min(m), \max(m)\}$ and $sss(m) = vss(m) = m$.

Let $x \in \Omega^{\omega}$. Define $\Gamma_\gamma(x)$ as follows: Let $k(0) = 0$ and $\langle k(i): 0 < i < N \leqslant \omega\rangle$ increasingly enumerate those k such that $\gamma(x \restriction (k-1)) = vss$. Moreover let $k(N) = \omega$, if $N < \omega$. Now let $\Gamma_\gamma(x) = \langle \bigcup_{k(i) \leqslant j < k(i+1)} \gamma(x \restriction j)(x(j)): i < N\rangle$.

Now we give our main result:

MAIN THEOREM. (MT) *For every Borel measurable mapping* $\Delta\colon \Omega^{\omega} \to \mathbb{R}$ *there exist* $\gamma\colon \Omega^{<\omega} \to \{sm, min\text{-}sep, max\text{-}sep, min\text{-}max, sss, vss\}$ *and* $a \in \Omega^{\omega}$ *such that for all* $x, y \in (a)^{\omega}$

$$\Delta(x) = \Delta(y) \text{ iff } \Gamma_{\gamma}(x) = \Gamma_{\gamma}(y).$$

REMARK. Moreover for the guaranteed $a \in \Omega^{\omega}$ it even holds that for no $x, y \in (a)^{\omega}$ the set $\Gamma_{\gamma}(x)$ is a proper initial segment of $\Gamma_{\gamma}(y)$ (see Lemma 2.36).

We give the proof of the Main Theorem in chapter 2. In chapter 1 we show that every analytic subset of the Milliken space is completely H-Ramsey – a property that will be used in chapter 2. For the implications $MT \to T$ and $MT \to PV$ see section A in the appendix. The implication $MT \to M$ is obvious.

1. THE MILLIKEN SPACE

Hindman's Theorem can be stated in equivalent form speaking about integers and their sums rather than finite sets and their unions. This was first mentioned by Graham and Rothschild [GrRo71]. The sum of two integers written in binary notation looks like the characteristic function of the union of two sets, provided the integers in binary are sufficiently spread out so that no carrying occurs upon addition. But the proof of Hindman's Theorem shows that the integers can be chosen with such a property. Milliken [Mi75] stated and proved his result in the sum notation.

The following results up to 1.15 (except for Hindman's Theorem) are essentially Milliken's results in the finite set notation, except for difference between his notation of Σ-Ramseyness and our notation of H-Ramseyness. Also see [To98] for an axiomatic treatment of these arguments. First of all let us expand our notation.

We begin by establishing some notation. The set of nonnegative integers is denoted by ω, and we identify each element of ω with the set of its predecessors as usual, for instance $k = \{0, ..., k - 1\}$. For the set of all subsets of X which have the same cardinality $k < \omega$ we write $[X]^k$. The collection of all finite resp. countably infinite subsets of X is denoted by $[X]^{<\omega}$ resp. $[X]^{\omega}$. Moreover let $[X]^{\leqslant\omega} = [X]^{<\omega} \cup [X]^{\omega}$. If $A \in [\omega]^{<\omega}$ and $B \in [\omega]^{\leqslant\omega}$, then we write $A < B$ iff $max(A) < min(B)$. Finally, if s is a mapping, we will write $dom(s)$ to denote the domain of s and $ran(s)$ to denote the range of s.

DEFINITION. Let Ω^{ω} denote the collection of all mappings $x\colon \omega \to [\omega]^{<\omega}$ with $x(i) < x(j)$ for all $i < j$. If $k \in \omega$, let Ω^k denote all mappings $s\colon k \to [\omega]^{<\omega}$ with the same property. Let $\Omega^{<\omega} = \bigcup_{i\in\omega} \Omega^i$ and $\Omega^{\leqslant\omega} = \Omega^{<\omega} \cup \Omega^{\omega}$. Moreover we define ω_{max} to be the mapping $\omega \to [\omega]^{<\omega}$ with $i \mapsto \{i\}$ for every $i \in \omega$.

DEFINITION. If $a \in \Omega^{\omega}$, let $(a)^{\omega}$ denote the the set of all mappings $x \in \Omega^{\omega}$ such that for every $i \in \omega$ there exists an $A \in [\omega]^{<\omega}$ with $x(i) = \bigcup_{j\in A} a(j)$. Analogously, if $k \in \omega$, define $(a)^k$ to be the set of all mappings $s \in \Omega^k$ with the property that for every $i \in dom(s)$ there exists an $A \in [\omega]^{<\omega}$ with $s(i) = \bigcup_{j\in A} a(j)$. Additionally, let $(a)^{<\omega} = \bigcup_{i\in\omega} (a)^i$. Finally, if $s \in \Omega^{<\omega}$ and $a \in \Omega^{\omega}$, we use $(s, a)^{\omega}$ to denote the set of mappings x

$\in \Omega^{\omega}$ such that $x(i) = s(i)$ for every $i \in dom(s)$ and for some $b \in (a)^{\omega}$, $x(i + dom(s)) = b(i)$ for every $i \in \omega$.

Assume that $s, t \in \Omega^{<\omega}$ and $a, b \in \Omega^{\omega}$. We abbreviate $s \in (t)^{<\omega}$ resp. $s \in (b)^{<\omega}$ resp. $a \in (b)^{\omega}$ as $s \ll t$ resp. $s \ll b$ resp. $a \ll b$.

Now let $s \in \Omega^{<\omega}$ and $t \in \Omega^{\leqslant\omega}$. If s and t are nonempty, we write $s < t$ iff $s(dom(s)\text{-}1) < t(0)$. For the following definition suppose $s < t$, if s and t are nonempty. Then we use $s \hat{} t$ to denote the mapping $\langle s(i)\text{: } i < dom(s), t(i)\text{: } i < dom(t)\rangle$. Moreover for every $k \in \omega$ let $t \restriction k$ denote the mapping $\langle t(i)\text{: } i < k\rangle$ and $t \upharpoonright k$ denote the mapping $\langle t(i)\text{: } i \geqslant k\rangle$.

Note that $(a)^{\omega}$ and $[a]^{\omega}$, $t \restriction k$ and $t \upharpoonright k$ as well as $s \ll t$ and $s < t$ have different meanings. Regard Ω^{ω} as a topological space endowed with the neighborhood system consisting of sets of the form $(s, a)^{\omega}$ where $s \in \Omega^{<\omega}$ and $a \in \Omega^{\omega}$. We will call Ω^{ω} the *Milliken space* and its topology the *H-Ellentuck topology*. The following results will refer to this topology till we revoke it.

Finally, we want to establish some abbreviations for simpler notation. If p is a mapping with $ran(p) \leqslant 1$, we will write p instead of $p(0)$ or $\varnothing$.

For the remainder of this paper let the lower case letters m, n be elements of Ω^1, p, q be elements of $\Omega^{\leqslant 1}$, r, s, t be elements of $\Omega^{<\omega}$, a, b, c, x, y, z be elements of Ω^{ω} and i, j, k, l be elements of ω. Furthermore, let indexed letters be elements of the same space as the corresponding non-indexed letters. Moreover we stipulate that, whenever we write a concatenation like $s \hat{} m$, we have $s < m$ for nonempty s.

DEFINITION. Let $R \subseteq \Omega^{\omega}$. We say *$a$ accepts s* iff $(s, a)^{\omega} \subseteq R$ and *a rejects s* iff there is no $b \ll a$ which accepts s. Moreover we say *a decides s* iff a accepts s or a rejects s.

For the next few proofs let R be an arbitrary but fixed set.

LEMMA 1.1. *There exists a which decides every $s \ll a$.*

PROOF. Inductively, we construct $a_j \in \Omega^{\omega}$ for every $j < \omega$. By definition there is an a_0 such that a_0 decides $\varnothing$. Assume that $a_0, \ldots, a_j$ have been constructed with the property that for every $i \leqslant j$ a_i decides every $s \ll \langle a_0(0), \ldots, a_{j-1}(0)\rangle$. After 2^j steps we can find an $a_{j+1} \ll a_j \upharpoonright 1$

which decides every $s \ll \langle a_0(0), ..., a_j(0)\rangle$. Then $a = \langle a_j(0): j \in \omega\rangle$ has the desired property. ⊣

Now we repeat the result of Hindman as Theorem 1.2.

THEOREM 1.2. (Hindman [Hi74]) *If $f: [\omega]^{<\omega} \to k$, then there exists a such that f is constant on $(a)^1$.*

For a simpler proof also see [Ba74].

LEMMA 1.3. *If a decides every $s \ll a$ and a rejects $\emptyset$, then there is b $\ll a$ which rejects every $s \ll b$.*

PROOF. Inductively, we construct $b_j \in \Omega^\omega$ for every $j < \omega$. We begin by claiming that there is $b_0 \ll a$ such that b_0 rejects every $s \ll b_0$ with $dom(s) = 1$. To see this we define a mapping $d: (a)^1 \mapsto \{acc, rej\}$ by $d(s) = acc$ iff a accepts s. By using $f(\{x_0, ..., x_k\}) = d(\langle a(x_0) \cup ... \cup a(x_k)\rangle)$ Theorem 1.2 guarantees the existence of $b_0 \ll a$ such that d is constant on $(b_0)^1$. Now assume that b_0 accepts every $s \ll b_0$ with $dom(s) = 1$. Thus, $(s, b_0)^\omega \subseteq R$ for every $s \in (b_0)^1$. But then $(b_0)^\omega \subseteq R$, as $(b_0)^\omega = \bigcup\{(s, b_0)^\omega: s \in (b_0)^1\}$, contradicting that a rejects $\emptyset$.

For the inductive step, using the same arguments repeatedly we can construct $b_j \ll b_{j-1} \upharpoonright 1$ such that b_j rejects all $s \ll b_j$ with $dom(s) = j + 1$. The assertion follows by putting $b = \langle b_j(0): j \in \omega\rangle$. ⊣

DEFINITION. We call $R \subseteq \Omega^\omega$ *H-Ramsey* iff there is a such that $(\emptyset, a)^\omega \subseteq R$ or $(\emptyset, a)^\omega \subseteq \Omega^\omega \setminus R$.

LEMMA 1.4. *Every open set $R \subseteq \Omega^\omega$ is H-Ramsey.*

PROOF. Take an a which by Lemma 1.1 decides every $s \ll a$. If a accepts $\emptyset$, then $(a)^\omega \subseteq R$. Otherwise a rejects $\emptyset$, and by Lemma 1.3 we may assume that a rejects every $s \ll a$. Suppose that $b \in (a)^\omega \cap R$. Then there is a neighborhood $(r, c)^\omega$ such that $b \in (r, c)^\omega \subseteq R$. This implies that b accepts r and thus a cannot reject r, contradiction. ⊣

The following two lemmas we obtained by straightforward generalizations of the proofs of Lemma 1.1 and 1.3.

LEMMA 1.5. *For every s and a there exists* $b \ll a$ *such that b decides* $s \hat{}\, t$ *for every* $t \ll b$.

LEMMA 1.6. *If a decides* $s \hat{}\, t$ *for every* $t \ll a$ *and a rejects s, then there is* $b \ll a$ *which rejects* $s \hat{}\, t$ *for every* $t \ll b$.

DEFINITION. We call $C \subseteq \Omega^\omega$ *completely H-Ramsey* iff for every s and a there is $b \ll a$ such that $(s, b)^\omega \subseteq C$ or $(s, b)^\omega \subseteq \Omega^\omega \setminus C$.

The following lemma follows from Lemmas 1.5 and 1.6 as Lemma 1.4 did from 1.1 and 1.3.

LEMMA 1.7. *Every open set* $C \subseteq \Omega^\omega$ *is completely H-Ramsey.*

COROLLARY 1.8. *The complement of a completely H-Ramsey set is completely H-Ramsey.*

PROOF. Obvious from the definition. ⊣

DEFINITION. We call $C \subseteq \Omega^\omega$ *H-Ramsey null* iff for every s and a there is $b \ll a$ such that $(s, b)^\omega \subseteq \Omega^\omega \setminus C$.

LEMMA 1.9. *If* $C \subseteq \Omega^\omega$ *is nowhere dense, then it is H-Ramsey null.*

PROOF. By Lemma 1.7 and Corollary 1.8 the closure $\overline{C}$ of C is completely H-Ramsey. Then there is $b \ll a$ such that $(s, b)^\omega \subseteq \overline{C}$ or $(s, b)^\omega \subseteq \Omega^\omega \setminus \overline{C} \subseteq \Omega^\omega \setminus C$. Since $\overline{C}$ is nowhere dense, the former case cannot occur. ⊣

LEMMA 1.10. *If* $C \subseteq \Omega^\omega$ *is meager, then it is H-Ramsey null.*

PROOF. Let C_n be a sequence of nowhere dense sets whose union is C. We may assume that $C_n \subseteq C_{n+1}$ holds for all n.

Inductively, we construct $b_j \in \Omega^\omega$ for every $j < \omega$. For any s and a, by Lemma 1.9 we can get $b_0 \ll a$ such that $(s, b_0)^\omega \subseteq \Omega^\omega \setminus C_0$. Assume that $b_0, ..., b_j$ have been constructed such that $(s \hat{}\, t, b_j)^\omega \subseteq \Omega^\omega \setminus C_j$ for every $t \ll \langle b_0(0), ..., b_{j-1}(0)\rangle$. Then also by Lemma 1.9 we can find $b_{j+1} \ll b_j \restriction 1$ such that $(s \hat{}\, t, b_{j+1})^\omega \subseteq \Omega^\omega \setminus C_{j+1}$ for every $t \ll \langle b_0(0), ..., b_j(0)\rangle$.

Hence $b = \langle b_j(0): j \in \omega \rangle$ satisfies the assertion of the lemma. ⊣

COROLLARY 1.11. *Every subset of Ω^ω is nowhere dense iff it is meager iff it is H-Ramsey null.*

PROOF. Obvious from the definition of nowhere dense sets and Lemma 1.10. ⊣

LEMMA 1.12. *Every set $C \subseteq \Omega^\omega$ with the Baire property is completely H-Ramsey.*

PROOF. Any set C with the Baire property can be expressed as $C = C_0 \vartriangle C_1$ where C_0 is open, C_1 is meager and $\vartriangle$ is the symmetric difference. Since C_1 is meager, by Lemma 1.10 for any s and a we can choose $b \ll a$ so that $(s, b)^\omega \subseteq \Omega^\omega \setminus C_1$ and then by Lemma 1.7 there is $c \ll b$ such that $(s, c)^\omega \subseteq C_0$ or $(s, c)^\omega \subseteq \Omega^\omega \setminus C_0$. In the former case $(s, c)^\omega \subseteq C$ and in the latter $(s, c)^\omega \subseteq \Omega^\omega \setminus C$. ⊣

LEMMA 1.13. *Let $C \subseteq \Omega^\omega$. Then C is completely H-Ramsey iff C has the Baire property.*

PROOF. Let C be completely H-Ramsey. Then we claim that $N = C \setminus Int(C)$ is nowhere dense (so C has the Baire property). Indeed, if this fails, there are s and a such that $(s, a)^\omega \subseteq \overline{N}$. Let $b \ll a$ be such that $(s, b)^\omega \subseteq C$ or $(s, b)^\omega \subseteq \Omega^\omega \setminus C$. Since $(s, b)^\omega \cap N \neq \varnothing$, $(s, b)^\omega \subseteq \Omega^\omega \setminus C$ is impossible. So $(s, b)^\omega \subseteq C$, thus $(s, b)^\omega \subseteq Int(C)$ and $(s, b)^\omega \cap N = \varnothing$, giving a contradiction.

Hence, the assertion of the lemma follows by Lemma 1.12. ⊣

A *Souslin system* is a class of closed sets that are indexed by finite sequences of nonnegative integers. A *Souslin set* is one which can be expressed in the form $\bigcup_{f\in\omega^\omega} \bigcap_{k\in\omega} S_{f\restriction k}$ where $\{S_e\}_e$ is a Souslin system, $f \restriction k$ is the restriction of f to the predecessors of k, and ω^ω is the set of all functions mapping ω into ω.

LEMMA 1.14. *Every Souslin set $C \subseteq \Omega^\omega$ is completely H-Ramsey.*

PROOF. The Baire property is preserved under the Souslin operation - for a proof see [Ku66]. Since closed sets have the Baire property, Lemma 1.12 gives our result. ⊣

The following result and also the results of the remainder of this paper will refer to the *metric topology* on Ω^ω. Note that by definition the Milliken space is a subspace of $([\omega]^{<\omega})^\omega$. The latter one can be regarded as a topological space with the product topology of the discrete topology of $[\omega]^{<\omega}$. Hence the metric topology on Ω^ω is the relative topology on $([\omega]^{<\omega})^\omega$. Notice that it is completely metrizable and coarser than the H-Ellentuck topology.

THEOREM 1.15. *Every analytic set* $C \subseteq \Omega^\omega$ *is completely H-Ramsey.*

PROOF. Every analytic set is a Souslin set [Ku66]. Since closed sets in the metric topology on Ω^ω are also closed in the H-Ellentuck topology, Lemma 1.14 applies directly. ⊣

2. PROOF OF THE MAIN THEOREM

The proof of the Main Theorem requires some further results. Our first lemma is analogous to Lemma 1 in [PrVo85].

LEMMA 2.1. *Let $\Delta\colon \Omega^{\omega} \to \mathbb{R}$ be Borel-measurable. Then there exists a such that the restriction $\Delta \restriction (a)^{\omega}$ is a continuous mapping.*

PROOF. Let $(I_j)_{j\in\omega}$ be an enumeration of all open intervals in $\mathbb{R}$ which have rational endpoints. The I_j form a basis for the topology of the reals. Inductively, we construct $a_j \in \Omega^{\omega}$ for every $j < \omega$. Put $a_0 = \omega_{max}$ and assume by induction that $a_0, ..., a_j$ have been constructed such that for all $i < j$ and all $s \ll \langle a_0(0), ..., a_i(0)\rangle$ either $(s, a_{i+1})^{\omega} \subseteq \Delta^{-1}(I_i)$ or $(s, a_{i+1})^{\omega} \subseteq \Omega^{\omega} \setminus \Delta^{-1}(I_i)$. Since I_j is open, it follows that $\Delta^{-1}(I_j) \subseteq \Omega^{\omega}$ must be Borel and hence by Theorem 1.15 completely H-Ramsey. Hence we can get an $a_{j+1} \ll a_j \restriction 1$ such that for all $s \ll \langle a_0(0), ..., a_j(0)\rangle$ either $(s, a_{j+1})^{\omega} \subseteq \Delta^{-1}(I_j)$ or $(s, a_{j+1})^{\omega} \subseteq \Omega^{\omega} \setminus \Delta^{-1}(I_j)$.

We claim that $a = \langle a_j(0)\colon j \in \omega\rangle$ has the desired properties. To see this we shall prove that every inverse image $(\Delta \restriction (a)^{\omega})^{-1}\,(I_j)$ is the union of all open sets $(s, a)^{\omega}$ with the property that $s \ll \langle a_i(0)\colon i \leqslant j\rangle$ and $(s, a)^{\omega} \subseteq \Delta^{-1}(I_j)$. It is obvious that the union of these open sets is part of the inverse image. So let x be an arbitrary but fixed element of the inverse image. Therefore $x \ll a$ and $\Delta(x) \in I_j$. Let k be maximal such that $s = x \restriction k \ll \langle a_i(0)\colon i \leqslant j\rangle$. Then it follows $x \in (s, a)^{\omega} \subseteq (\Delta \restriction (a)^{\omega})^{-1}\,(I_j)$ and, thus, the assertion of the lemma. ⊣

REMARK. Suppose $\Delta\colon \Omega^{\omega} \to \mathbb{R}$ is Baire measurable with respect to the H-Ellentuck topology. The same argument, using Lemma 1.12 instead of Theorem 1.15, shows that $\Delta \restriction (a)^{\omega}$ is continuous with respect to the metric topology on Ω^{ω} for some a.

For the remainder of this section let $\Delta\colon \Omega^{\omega} \to \mathbb{R}$ be an arbitrary but fixed mapping.

DEFINITION. Let s, t and x be such that $s < t, x$ and $s = \langle s(0), ..., s(k)\rangle$. We abbreviate the mappings $s \restriction k\,\hat{}\,\langle s(k) \cup t(0)\rangle\,\hat{}\,t \restriction 1$ resp. $s \restriction k\,\hat{}\,\langle s(k) \cup x(0)\rangle\,\hat{}\,x \restriction 1$ as $s{\uparrow}\hat{}\,t$ resp. $s{\uparrow}\hat{}\,x$. Additionally, we define $s{\uparrow}\hat{}\,\varnothing$ to be $s{\uparrow}$. Moreover we use $s\square$ as a variable for s or $s{\uparrow}$.

Analogously to [PrVo85] we introduce now the terms *separating* and *mixing*.

DEFINITION. We say that *$s\square$ and $t\square$ are separated by a* iff $\Delta(s\square{}^\frown x) \neq \Delta(t\square{}^\frown y)$ for all $x, y \ll a$ with $s < x, y$ and $t < x, y$. Moreover *$s\square$ and $t\square$ are mixed by a* iff for no $b \ll a$ the sets $s\square$ and $t\square$ are separated by b. Finally, *$s\square$ and $t\square$ are decided by a* iff $s\square$ and $t\square$ are separated or mixed by a.

We stipulate that, whenever we write a concatenation like $s\square{}^\frown m\square$ resp. $s\square{}^\frown m\square{}^\frown n\square$, we have $s < m$ resp. $s < m < n$ for nonempty s, m, n.

COROLLARY 2.2. *For every s, t and a, there exists $b \ll a$ which decides $s\square$ and $t\square$. If $s\square$ and $t\square$ are decided by b, then they are also decided by each $c \ll b$, and c decides in the same way as b does.*

PROOF. Obvious from the definition. ⊣

LEMMA 2.3. (Transitivity of mixing) *Assume that $r\square$ and $s\square$, as well as $s\square$ and $t\square$ are mixed by a. Then also $r\square$ and $t\square$ are mixed by a.*

PROOF. Assume to the contrary that there exists $b \ll a$ which separates $r\square$ and $t\square$. We may assume without loss of generality that $r, s, t < b$. Consider the set $A = \{x \ll b\colon \exists y \ll b\ \Delta(r\square{}^\frown y) = \Delta(s\square{}^\frown x)\}$. Then A is analytic, so by Theorem 1.15 A is completely H-Ramsey. By definition of completely H-Ramsey there exists $c \ll b$ with $(c)^\omega \subseteq A$ or $(c)^\omega \cap A = \varnothing$. Both cases lead to a contradiction:

Assume first that $(c)^\omega \subseteq A$. Then for all $x \ll c$ there exists $y \ll b$ such that $\Delta(r\square{}^\frown y) = \Delta(s\square{}^\frown x)$. Since $r\square$ and $t\square$ are separated by b, it follows that $\Delta(r\square{}^\frown y) \neq \Delta(t\square{}^\frown z)$ for every $y, z \ll b$. Hence we get $\Delta(s\square{}^\frown x) \neq \Delta(t\square{}^\frown z)$ for all $x, z \ll c$, contradicting that $s\square$ and $t\square$ are mixed by a.

Otherwise if $(c)^\omega \cap A = \varnothing$, then $r\square$ and $s\square$ are separated by c. ⊣

LEMMA 2.4. *For every a there exists $b \ll a$ such that for every $s, t \ll b$ the sets $s\square$ and $t\square$ are decided by b.*

PROOF. Inductively, we construct $b_j \in \Omega^\omega$ for every $j < \omega$. By Corollary 2.2 there exists $b_0 \ll a$ such that $\varnothing$ and $\varnothing$ are decided by b_0. Assume that $b_0, \ldots, b_j$ have been constructed such that for every $i \leqslant j$

and for all $s, t \ll \langle b_k(0): k < i \rangle$ the sets $s\square$ and $t\square$ are decided by b_i. Some applications of Corollary 2.2 yield $b_{j+1} \ll b_j \upharpoonright 1$ such that the inductive assumption is also satisfied for $b_0, ..., b_{j+1}$. Then $b = \langle b_j(0): j \in \omega \rangle$ has the desired properties. ⊣

LEMMA 2.5. (Taylor [Ta76]) *If $k, l > 0$ and $f: (a)^l \to k$, then there exists $b \ll a$ such that f is constant on $(b)^l$.*

For a proof see Lemma 2.2 in [Ta76].

The following Lemma is modeled in the image of Theorem 2.1 in [Ta76].

LEMMA 2.6. *For every s and a, there exists $b \ll a$ such that exactly one of the following properties holds:*

(a) If $m, n \ll b$, then $s \hat{} m$ and $s \hat{} n$ are mixed by b.

(b) If $m, n \ll b$, then $s \hat{} m$ and $s \hat{} n$ are mixed by b iff $min(m) = min(n)$.

(c) If $m, n \ll b$, then $s \hat{} m$ and $s \hat{} n$ are mixed by b iff $max(m) = max(n)$.

(d) If $m, n \ll b$, then $s \hat{} m$ and $s \hat{} n$ are mixed by b iff $min(m) = min(n)$ and $max(m) = max(n)$.

(e) If $m, n \ll b$, then $s \hat{} m$ and $s \hat{} n$ are mixed by b iff $m = n$.

PROOF. Lemma 2.4 guarantees the existence of $b_0 \ll a$ such that $s \hat{} m$ and $s \hat{} n$ are decided by b_0 for every $m, n \ll b_0$. Let F be the set of all functions f such that $dom(f) = 3$ and $ran(f) \subseteq 2$. Define $g: (b_0)^3 \to F$ as follows:

$g(h)(0) = 0$ iff $s \hat{} \langle h(0) \cup h(1) \cup h(2) \rangle$ and $s \hat{} \langle h(0) \rangle$ are mixed by b_0.

$g(h)(1) = 0$ iff $s \hat{} \langle h(0) \cup h(1) \cup h(2) \rangle$ and $s \hat{} \langle h(2) \rangle$ are mixed by b_0.

$g(h)(2) = 0$ iff $s \hat{} \langle h(0) \cup h(1) \cup h(2) \rangle$ and $s \hat{} \langle h(0) \cup h(2) \rangle$ are mixed by b_0.

By Lemma 2.5 there exists $b_1 \ll b_0$ and a function $f = \langle f(0), f(1), f(2) \rangle \in F$ such that $g((b_1)^3) = \{f\}$. We claim first that f cannot be $\langle 0, 0, 1 \rangle$ or $\langle 1, 0, 1 \rangle$ or $\langle 0, 1, 1 \rangle$. The first two are ruled out by the observation that if $f(1) = 0$, then we must have $f(2) = 0$. Indeed, if $f(2) \neq 0$, then $s^\frown \langle (b_1(0) \cup b_1(1)) \cup b_1(2) \cup b_1(3)\rangle$ and $s^\frown \langle (b_1(0) \cup b_1(1)) \cup b_1(3)\rangle$ are separated by b_1. But since $f(1) = 0$, both of these are mixed with $s^\frown \langle b_1(3)\rangle$. By transitivity of mixing we get a contradiction. Similarly, the third one is ruled out since if $f(0) = 0$, then we must have $f(2) = 0$. This leaves five possibilities for f.

We will show that these five possibilities correspond to the five clauses (a) - (e) of this lemma. By construction we are guaranteed that exactly one case holds in the assertion.

Case (a). $f = \langle 0, 0, 0\rangle$. Let $b = \langle b_1(i): i > 1\rangle$ and $m, n \ll b$. Since $f(1) = 0$, $s^\frown m$ and $s^\frown \langle b_1(0) \cup b_1(1) \cup m\rangle$ as well as $s^\frown n$ and $s^\frown \langle b_1(0) \cup b_1(1) \cup n\rangle$ are mixed by b. Moreover because $f(0) = 0$, we have that $s^\frown \langle b_1(0) \cup b_1(1) \cup m\rangle$ and $s^\frown \langle b_1(0)\rangle$ as well as $s^\frown \langle b_1(0) \cup b_1(1) \cup n\rangle$ and $s^\frown \langle b_1(0)\rangle$ are mixed by b. By transitivity of mixing it follows that $s^\frown m$ and $s^\frown n$ are mixed by b whenever $m, n \ll b$, so b satisfies clause (a) of the lemma.

Case (b). $f = \langle 0, 1, 0\rangle$. Let $b = \langle b_1(3i) \cup b_1(3i + 1) \cup b_1(3i + 2): i < \omega\rangle$. Suppose first that $m, n \ll b$ with $min(m) = min(n)$. Then $m = \langle b(k) \cup p\rangle$ and $n= \langle b(k) \cup q\rangle$ for some k and some $p, q \ll b \upharpoonright k$. Since $f(0) = 0$, $s^\frown m$ and $s^\frown \langle b_1(3k)\rangle$ as well as $s^\frown n$ and $s^\frown \langle b_1(3k)\rangle$ are mixed by b. By transitivity of mixing we obtain that $s^\frown m$ and $s^\frown n$ are mixed by b.

Conversely, if $m, n \ll b$ and $min(m) < min(n)$, then $m = \langle b(k) \cup p\rangle$ for some k and $p \ll b \upharpoonright k$, and $b(k) < n$. Thus, $s^\frown m$ and $s^\frown \langle b(k) \cup n\rangle$ are mixed by b, since both are mixed with $s^\frown \langle b_1(3k)\rangle$ by virtue of the fact that $f(0) = 0$. But since $f(1) = 1$, we have that $s^\frown \langle b(k) \cup n\rangle$ and $s^\frown n$ are separated by b, and so - by transitivity of mixing - we must have that $s^\frown m$ and $s^\frown n$ are separated by b.

Thus, $s^\frown m$ and $s^\frown n$ are mixed by b iff $min(m) = min(n)$, so b satisfies clause (b) of the lemma.

Case (c). $f = \langle 1, 0, 0\rangle$. Let $b = \langle b_1(3i) \cup b_1(3i + 1) \cup b_1(3i + 2): i < \omega\rangle$ like in case (b). If $m, n \ll b$ and $max(m) = max(n)$, then $m = \langle p \cup b(k)\rangle$ and $n = \langle q \cup b(k)\rangle$ for some k and $p, q \ll b \upharpoonright k$. Since $f(1) = 0$ we have that $s^\frown m$ and $s^\frown \langle b_1(3k + 2)\rangle$ as well as $s^\frown n$ and $s^\frown \langle b_1(3k + 2)\rangle$ are mixed by b. Hence $s^\frown m$ and $s^\frown n$ are mixed by b.

Conversely, if $m, n \ll b$ and $max(m) < max(n)$, then $n = \langle q \cup b(k)\rangle$ for some k and $q \ll b \upharpoonright k$, and $m < b(k)$. Thus, $s^\frown n$ and $s^\frown \langle m \cup b(k)\rangle$ are mixed by b, because both are mixed with $s^\frown \langle b_1(3k + 2)\rangle$ since $f(1)$

$= 0$. But $s \hat{}\ \langle m \cup b(k)\rangle$ and $s \hat{}\ m$ are separated by b since $f(0) = 1$. So we must have that $s \hat{}\ m$ and $s \hat{}\ n$ are separated by b.

Hence $s \hat{}\ m$ and $s \hat{}\ n$ are mixed by b iff $max(m) = max(n)$, and hence b satisfies clause (c) of the lemma.

CLAIM 2.6.1. *Let s and a be such that* $s \hat{}\ m$ *and* $s \hat{}\ n$ *are decided by a for every* $m, n \ll a$. *If* $s \hat{}\ \langle h(0) \cup h(1) \cup h(2)\rangle$ *and* $s \hat{}\ \langle h(0)\rangle$ *are separated by a for all* $h \in (a)^3$, *then there exists* $b \ll a$ *such that* $s \hat{}\ m$ *and* $s \hat{}\ n$ *are separated by b for every* $m, n \ll b$ *with* $max(m) < max(n)$.

PROOF. Let a_0, a_1 be elements of Ω^ω. We construct b inductively. Put $b(0) = a(0)$ and suppose that $b(0), ..., b(k\text{-}1)$ have been constructed such that $s \hat{}\ m$ and $s \hat{}\ n$ are separated by a for all $m, n \ll b \upharpoonright k$ with $max(m) < max(n)$. Let $a_0 \ll a$ with $b(k) < a_0(0)$. Choose $\langle b(k)\rangle \ll a_0$ such that $s \hat{}\ m$ and $s \hat{}\ \langle p \cup b(k)\rangle$ are separated by a for every $m, p \ll b \upharpoonright k$. This is possible, since otherwise for all $\langle b(k)\rangle \ll a_0$ there would exist $m, p \ll b \upharpoonright k$ such that $s \hat{}\ m$ and $s \hat{}\ \langle p \cup b(k)\rangle$ are mixed by a. Hindman's Theorem would yield $a_1 \ll a_0$ and fixed $m, p \ll b \upharpoonright k$ such that $s \hat{}\ m$ and $s \hat{}\ \langle p \cup b(k)\rangle$ are mixed by a for every $\langle b(k)\rangle \ll a_1$. By transitivity of mixing (Lemma 2.3) we get that $s \hat{}\ \langle p \cup m\rangle$ and $s \hat{}\ \langle p \cup n\rangle$ are mixed by a for all $m, n \ll a_1$. Choosing $h = \langle p \cup a_1(0), a_1(1), a_1(2)\rangle$ we get a contradiction to the assumption of the lemma. This completes the construction of b. ⊣

Case (d). $f = \langle 1, 1, 0\rangle$. To handle case (d) we choose $b_2 \ll b_1$ as guaranteed to exist by Claim 2.6.1. Let $b = \langle b_1(3i) \cup b_1(3i + 1) \cup b_1(3i + 2)\text{: } i < \omega\rangle$. We claim that if $m, n \ll b$, then $s \hat{}\ m$ and $s \hat{}\ n$ are mixed by b iff $min(m) = min(n)$ and $max(m) = max(n)$.

Suppose first that $min(m) = min(n)$ and $max(m) = max(n)$. Then for some $i < j$ we have that $s \hat{}\ m$ and $s \hat{}\ \langle b_2(3i) \cup b_2(3j + 2)\rangle$ as well as $s \hat{}\ n$ and $s \hat{}\ \langle b_2(3i) \cup b_2(3j + 2)\rangle$ are mixed by b, since $f(2) = 0$. By transitivity of mixing we get that $s \hat{}\ m$ and $s \hat{}\ n$ are mixed by b.

For the converse, suppose that either $min(m) \neq min(n)$ or $max(m) \neq max(n)$. If $max(m) \neq max(n)$, then clearly $s \hat{}\ m$ and $s \hat{}\ n$ are separated by b, by construction according to Claim 2.6.1. Hence we can assume that $max(m) = max(n)$ and $min(m) < min(n)$. Let $m = \langle b(k) \cup p \cup b(l)\rangle$ for some $k < l$ and some $p \ll \langle b(i)\text{: } k < i < l\rangle$ and $b(k) < n$. But then $s \hat{}\ m$ and $s \hat{}\ \langle b_2(3k) \cup b_2(3l + 2)\rangle$ as well as $s \hat{}\ \langle b_2(3k) \cup b_2(3l + 2)\rangle$ and $s \hat{}\ \langle b(k) \cup n\rangle$ are mixed by b, since $f(2) = 0$. However, since $f(1) = 1$ we have that $s \hat{}\ \langle b(k) \cup n\rangle$ and $s \hat{}\ n$ are separated by b, and by the transitivity of mixing it follows that $s ^\frown m$ and $s ^\frown n$ are separated by b. Thus, we have shown that b satisfies clause (d) of the lemma.

DEFINITION. For some given s and a we will say that t and b are *compatible* iff $t < b$ and $s ^\frown \langle p \cup m\rangle$ and $s ^\frown \langle q \cup m\rangle$ are separated by a for every $m \ll b$ and for all $p, q \ll t$ with $max(p) \neq max(q)$. Note that p and q can be empty as agreed in the introduction. We will say that t and b are *very compatible* iff they are compatible and, moreover, there exists $n \ll b$ and there exists $c \ll b$ such that $t ^\frown n$ and c are compatible.

CLAIM 2.6.2. *Let s and a be such that $s ^\frown m$ and $s ^\frown n$ are decided by a for every $m, n \ll a$. Suppose that $s ^\frown \langle h(0) \cup h(1) \cup h(2)\rangle$ and $s ^\frown \langle h(0) \cup h(2)\rangle$ are separated by a for all $h \in (a)^3$. Then if t and b are compatible where $b, t \ll a$, then t and b are in fact very compatible.*

PROOF. Suppose that t and b are compatible but not very compatible. Then for every $m \ll b$ and for all $c \ll b$ with $m < c$ there exists $n \ll c$ and there exists $p, q \ll t ^\frown m$ such that $max(p) \neq max(q)$ and $s ^\frown \langle p \cup n\rangle$ and $s ^\frown \langle q \cup n\rangle$ are mixed by a. Notice that we cannot have both $p, q \ll t$ since t and b are compatible. Thus, we better use instead of any such q a mapping of the form $\langle q \cup m\rangle$ with the restriction $q \ll t$. Now two applications of Lemma 2.5 yield $c \ll b$ and fixed $p, q \ll t$ such that $s ^\frown \langle p \cup n\rangle$ and $s ^\frown \langle q \cup m \cup n\rangle$ are mixed by a for every $m ^\frown n \ll c$. We get mixing for all $m ^\frown n$ because of our assumption above. Choosing $h \in (c)^3$ we obtain that $s ^\frown \langle p \cup h(2)\rangle$ and $s ^\frown \langle q \cup h(0) \cup h(2)\rangle$ as well as $s ^\frown \langle p \cup h(2)\rangle$ and $s ^\frown \langle q \cup h(0) \cup h(1) \cup h(2)\rangle$ are mixed by a since $\langle h(0), h(2)\rangle, \langle h(0) \cup h(1), h(2)\rangle \ll c$. Thus, by transitivity of mixing $s ^\frown \langle q \cup h(0) \cup h(1) \cup h(2)\rangle$ and $s ^\frown \langle q \cup h(0) \cup h(2)\rangle$ are mixed by a, contradicting the condition imposed in the lemma. This completes the proof of the claim. ⊣

Case (e). $f = \langle 1, 1, 1\rangle$. To handle case (e) we construct $b_2 \ll b_1$ inductively. To this end we build a sequence $\{(b_2(i), c_i): i < \omega\}$ such that $b_2 \restriction (i + 1)$ and c_i are compatible for every $i < \omega$ with $c_i \in \Omega^\omega$. Let $b_2(0) = b_1(0) \cup b_1(1)$ and $c_0 = b_1 \restriction 2$. Notice that $b_2 \restriction 1$ and c_0 are compatible since $f(1) = 1$ and $f(2) = 1$. Suppose now that $b_2 \restriction (k + 1)$ and c_k have been constructed and are compatible. Since $f(2) = 1$, Claim 2.6.2 applies and hence we have that $b_2 \restriction (k + 1)$ and c_k are very compatible. Thus, there exists $\langle b_2(k + 1)\rangle \ll c_k$ and there exists $c_{k+1} \ll$

c_k such that $b_2 \restriction (k + 2)$ and c_{k+1} are compatible. This completes the construction.

Now we claim that if $m, n \ll b_2$ with $m \neq n$ and $max(m) = max(n)$, then we have that $s \,\hat{}\, m$ and $s \,\hat{}\, n$ are separated by b_2. To see this, let $b_2(k)$ be the last piece of b_2 occuring in $(m \cup n) \setminus (m \cap n)$. Then we can assume without loss of generality that $m = \langle p \cup b_2(k) \cup m_0 \rangle$ and $n = \langle q \cup m_0 \rangle$ for some $p, q \ll b_2 \restriction k$ and some $m_0 \ll b_2$ with $b_2(k) < m_0$. Since $b_2 \restriction (k + 1)$ and c_k are compatible, $m_0 \ll c_k$ and $max(p \cup b_2(k)) \neq max(q)$ we have that $s \,\hat{}\, \langle p \cup b_2(k) \cup m_0 \rangle$ and $s \,\hat{}\, \langle q \cup m_0 \rangle$ are separated by b_2. Thus, $s \,\hat{}\, m$ and $s \,\hat{}\, n$ are separated by b_2. Since $f(0) = 1$ and $b_2 \ll b_1$, Claim 2.6.1 applies and we can choose $b \ll b_2$ such that $s \,\hat{}\, m$ and $s \,\hat{}\, n$ are separated by b whenever $m, n \ll b$ and $max(m) < max(n)$.

Finally, notice that if $s \,\hat{}\, m$ and $s \,\hat{}\, n$ are separated by b, we must have $m \neq n$ by definition of separated. So we can conclude that $s \,\hat{}\, m$ and $s \,\hat{}\, n$ are mixed by b iff $m \neq n$.

This completes the proof of case (e) and with it, the proof of Lemma 2.6. ⊣

The following definition is based on the five cases of Lemma 2.6.

DEFINITION. We say that *s is strongly mixed by a* iff $s \,\hat{}\, m$ and $s \,\hat{}\, n$ are mixed by a for every $m, n \ll a$. Moreover *s is min-separated by a* iff for every $m, n \ll a$ the sets $s \,\hat{}\, m$ and $s \,\hat{}\, n$ are mixed by a iff $min(m) = min(n)$. Furthermore, *s is max-separated by a* iff for every $m, n \ll a$ the sets $s \,\hat{}\, m$ and $s \,\hat{}\, n$ are mixed by a iff $max(m) = max(n)$. Moreover we say that *s is min-max-separated by a* iff for every $m, n \ll a$ the sets $s \,\hat{}\, m$ and $s \,\hat{}\, n$ are mixed by a iff $min(m) = min(n)$ and $max(m) = max(n)$. Finally, *s is strongly separated by a* iff for every $m, n \ll a$ the sets $s \,\hat{}\, m$ and $s \,\hat{}\, n$ are mixed by a iff $m = n$.

Furthermore, we say *s is separated in some sense by a* iff s is min-separated, max-separated, min-max-separated or strongly separated by a. Moreover *s is completely decided by a* iff s is strongly mixed by a or s is separated in some sense by a.

COROLLARY 2.7. *For every s and a the following properties hold.*

(a) Let s be strongly mixed by a. Then $s \,\hat{}\, m \restriction$ is strongly mixed by a for every $m \ll a$.

(b) Let s be min-separated by a. Then $s \,\hat{}\, m \restriction$ is strongly mixed by a

for every $m \ll a$.

(c) *Let* s *be max-separated by* a. *Then* $s^\frown m\uparrow$ *is max-separated by* a *for every* $m \ll a$.

(d) *Let* s *be min-max-separated by* a. *Then* $s^\frown m\uparrow$ *is max-separated by* a *for every* $m \ll a$.

(e) *Let* s *be strongly separated by* a. *Then* $s^\frown m\uparrow$ *is strongly separated by* a *for every* $m \ll a$.

PROOF. Obvious from the definition. ⊣

LEMMA 2.8. *For every* s *and* a *the following properties hold.*

(a) *Let* $s\downarrow$ *be strongly mixed by* a. *Then* $s\downarrow$ *and* $s\downarrow^\frown m\downarrow$ *as well as* $s\downarrow^\frown m\downarrow$ *and* $s\downarrow^\frown n\downarrow$ *are mixed by* a *for every* $m, n \ll a$.

(b) *Let* s *be min-separated by* a. *Then* $s^\frown m\downarrow$ *and* $s^\frown n\downarrow$ *are mixed by* a *for every* $m, n \ll a$ *with* $\min(m) = \min(n)$.

(c) *Let* $s\downarrow$ *be max-separated by* a. *Then* $s\downarrow$ *and* $s\downarrow^\frown m\uparrow$ *as well as* $s\downarrow^\frown m\uparrow$ *and* $s\downarrow^\frown n\uparrow$ *are mixed by* a *for every* $m, n \ll a$.

(d) *Let* s *be min-max-separated by* a. *Then* $s^\frown m\uparrow$ *and* $s^\frown n\uparrow$ *are mixed by* a *for every* $m, n \ll a$ *with* $\min(m) = \min(n)$.

PROOF. *Case (a).* Let $s\downarrow$ be strongly mixed by a. First, we prove that $s\downarrow$ and $s\downarrow^\frown m\downarrow$ are mixed by a for every $m \ll a$. Assume to the contrary that $s\downarrow$ and $s\downarrow^\frown m$ resp. $s\downarrow$ and $s\downarrow^\frown m\uparrow$ are not mixed by a for some $m \ll a$. Hence there exists $b \ll a$ such that $s\downarrow$ and $s\downarrow^\frown m$ resp. $s\downarrow$ and $s\downarrow^\frown m\uparrow$ are separated by b. Since $s\downarrow$ is strongly mixed by a, by Corollary 2.2 we get that $s\downarrow^\frown m$ and $s\downarrow^\frown n$ are also mixed by b for every $m, n \ll a$.

Now choose k minimal such that $m < b(k)$. By definition of separation we must have that $s\downarrow^\frown \langle b(k)\rangle$ and $s\downarrow^\frown m$ resp. $s\downarrow^\frown \langle b(k)\rangle$ and $s\downarrow^\frown m\uparrow^\frown \langle b(k)\rangle$ are separated by b. However, since $\langle b(k)\rangle \ll a$, both facts contradict that $s\downarrow$ is strongly mixed by a.

By transitivity of mixing the second assertion, that $s\downarrow^\frown m\downarrow$ and $s\downarrow^\frown n\downarrow$ are mixed by a for every $m, n \ll a$, follows from the first one.

Case (b). Let s be min-separated by a. Assume to the contrary that $s^\frown m$ and $s^\frown n\uparrow$ resp. $s^\frown m\uparrow$ and $s^\frown n\uparrow$ are not mixed by a for some $m, n \ll$

a with $min(m) = min(n)$. Hence there exists $b \ll a$ such that $s ^\frown m$ and $s ^\frown n\uparrow$ resp. $s ^\frown m\uparrow$ and $s ^\frown n\uparrow$ are separated by b. Since s is min-separated by a, by Corollary 2.2 we get that $s ^\frown m$ and $s ^\frown n$ are also mixed by b for every $m, n \ll a$ with $min(m) = min(n)$.

Now choose k minimal such that $m, n < b(k)$. By definition of separation we must have that $s ^\frown m$ and $s ^\frown n\uparrow ^\frown \langle b(k)\rangle$ resp. $s ^\frown m\uparrow ^\frown \langle b(k)\rangle$ and $s ^\frown n\uparrow ^\frown \langle b(k)\rangle$ are separated by b. However, since $\langle b(k)\rangle \ll a$, both facts contradict that s is min-separated by a.

Case (c). Let $s\square$ be max-separated by a. First, we prove that $s\square$ and $s\square ^\frown m\uparrow$ are mixed by a for every $m \ll a$. Assume to the contrary that $s\square$ and $s\square ^\frown m\uparrow$ are not mixed by a for some $m \ll a$. Hence there exists $b \ll a$ such that $s\square$ and $s\square ^\frown m\uparrow$ are separated by b. Since $s\square$ is max-separated by a, by Corollary 2.2 we get that $s\square ^\frown m$ and $s\square ^\frown n$ are also mixed by b for every $m, n \ll a$ with $max(m) = max(n)$.

Now choose k minimal such that $m < b(k)$. By definition of separation we must have that $s\square ^\frown \langle b(k)\rangle$ and $s\square ^\frown m\uparrow ^\frown \langle b(k)\rangle$ are separated by b. However, since $\langle b(k)\rangle \ll a$, this contradicts that $s\square$ is max-separated by a.

By transitivity of mixing the second assertion, that $s\square ^\frown m\uparrow$ and $s\square ^\frown n\uparrow$ are mixed by a for every $m, n \ll a$, follows from the first one.

Case (d). Let s be min-max-separated by a. Assume to the contrary that $s ^\frown m\uparrow$ and $s ^\frown n\uparrow$ are not mixed by a for some $m, n \ll a$ with $min(m) = min(n)$. Hence there exists $b \ll a$ such that $s ^\frown m\uparrow$ and $s ^\frown n\uparrow$ are separated by b. Since s is min-max-separated by a, by Corollary 2.2 we get that $s ^\frown m$ and $s ^\frown n$ are also mixed by b for every $m, n \ll a$ with $min(m) = min(n)$ and $max(m) = max(n)$.

Now choose k minimal such that $m, n < b(k)$. By definition of separation we must have that $s ^\frown m\uparrow ^\frown \langle b(k)\rangle$ and $s ^\frown n\uparrow ^\frown \langle b(k)\rangle$ are separated by b. However, since $\langle b(k)\rangle \ll a$, this contradicts that s is min-max-separated by a. ⊣

LEMMA 2.9. *For every a there exists $b \ll a$ which completely decides every $s \ll b$.*

PROOF. Inductively, we construct $b_j \in \Omega^\omega$ for every $j < \omega$. By Lemma 2.6 there exists $b_0 \ll a$ such that b_0 completely decides $\varnothing$. Assume by induction that $b_0, ..., b_j$ have been constructed such that for every $i \leqslant j$

and all $s \ll \langle b_k(0): k < i \rangle$ the set b_i completely decides s. Some applications of Lemma 2.6 yield $b_{j+1} \ll b_j \restriction 1$ such that the inductive assumption is also satisfied for $b_0, \ldots, b_{j+1}$. Then $b = \langle b_j(0): j \in \omega \rangle$ has the desired properties. ⊣

DEFINITION. a is *canonical for Δ* iff it has the following properties:

(a) The mapping $\Delta \restriction (a)^\omega$ is continuous.

(b) If $s, t \ll a$, then $s\square$ and $t\square$ are decided by a.

(c) Every $s \ll a$ is completely decided by a.

(d) Let $s, t \ll a$. Then $s\square$ and $s\square\,\hat{}\,m\square$ are either mixed by a for all $m \ll a$ or separated by a for all $m \ll a$. Equally $s\square\,\hat{}\,m\square$ and $s\square\,\hat{}\,m\square$, $s\square\,\hat{}\,m\square$ and $t\square\,\hat{}\,m\square$, $s\square\,\hat{}\,m\square$ and $s\square\,\hat{}\,m\square\,\hat{}\,n\square$ as well as $s\square\,\hat{}\,m\square$ and $t\square\,\hat{}\,m\square\,\hat{}\,n\square$ are in each case either mixed by a for all $m, n \ll a$ or separated by a for all $m, n \ll a$.

(e) If $s \ll a$, then either for every $x \ll a$ and all $k \in \omega$ the set $s\square\,\hat{}\,(x \restriction k)$ is strongly mixed by a or for every $x \ll a$ there exists $k \in \omega$ such that $s\square\,\hat{}\,(x \restriction k)$ is separated in some sense by a.

(f) There exists b with $a = \langle b(3i) \cup b(3i+1) \cup b(3i+2): i < \omega \rangle$ such that the properties (a) to (e) are even true for b instead of a.

LEMMA 2.10. *There exists a which is canonical for Δ.*

PROOF. First, observe by Corollary 2.2 that if $s\square$ and $t\square$ are decided by a, then they are also decided by each $b \ll a$, and b decides in the same way as a does. Hence by Lemma 2.1, 2.4 and 2.9 we are guaranteed that there exists b_0, which satisfies the properties (a) to (c) of canonical.

Now we turn to property (d). Inductively, we construct $c_j \in \Omega^\omega$ for every $j < \omega$. By Lemma 2.5 we can find $c_0 \ll b_0$ such that the sets $\varnothing$ and $\varnothing\,\hat{}\,m$ are either mixed by c_0 for every $m \ll c_0$ or separated by c_0 for every $m \ll c_0$. Assume that $c_0, \ldots, c_j$ have been constructed such that for all $i \leqslant j$ and for all $s \ll \langle c_l(0): l < i \rangle$ the sets s and $s\,\hat{}\,m$ are either mixed by c_i for every $m \ll c_i$ or separated by c_i for every $m \ll c_i$. Again, invoking Lemma 2.5 there exists $c_{j+1} \ll c_j \restriction 1$ such that the inductive assumption is also satisfied for $c_0, \ldots, c_{j+1}$. Then $b_1 = \langle c_j(0): j \in \omega \rangle$ has the desired property. Applying some similar inductions, we get b_1 fulfilling (a) to (d) of canonical.

Now we turn to property (e). Inductively, we construct $c_j \in \Omega^\omega$ for every $j < \omega$. Consider the set $C = \{x \ll b_1\colon \forall k \in \omega\ x\ 1k$ *is strongly mixed by* $b_1\}$. Since C is closed, by Theorem 1.15 there exists $c_0 \ll b_1$ such that $(c_0)^\omega \subseteq C$ or $(c_0)^\omega \subseteq \Omega^\omega \setminus C$. Assume that $c_0, \ldots, c_j$ have been constructed such that for all $i \leqslant j$ and for all $s \ll \langle c_l(0)\colon l < i\rangle$ either for every $x \ll c_i$ and all $k \in \omega$ the set $s\square\,\hat{}\,(x\ 1k)$ is strongly mixed by c_i or for every $x \ll c_i$ there exists $k \in \omega$ such that $s\square\,\hat{}\,(x\ 1k)$ is separated in some sense by c_i. For every $s \ll \langle c_l(0)\colon l < j\rangle$ consider the sets $C_{s\square,\square} = \{x \ll c_j\colon \forall k \in \omega\ s\square\,\hat{}\,\langle c_j(0)\rangle\square\,\hat{}\,(x\ 1k)$ *is strongly mixed by* $c_j\}$. Again, all $C_{s\square,\square}$ are closed. Hence some applications of Theorem 1.15 yield $c_{j+1} \ll c_j \restriction 1$ such that the inductive assumption is also satisfied for $c_0, \ldots, c_{j+1}$. Then $b = \langle c_j(0)\colon j \in \omega\rangle$ satisfies the properties (a) to (e) of canonical.

Finally, let $a = \langle b(3i) \cup b(3i+1) \cup b(3i+2)\colon i < \omega\rangle$. Hence a has the properties (a) to (f). This completes the proof. ⊣

For the remainder of this paper let a be canonical for Δ.

LEMMA 2.11. *Let* $s \ll a$.

(a) *Let* s *be min-separated by* a. *If* $x, y \ll a$, *then* $\Delta(s\,\hat{}\,x) = \Delta(s\,\hat{}\,y)$ *implies* $\min(x(0)) = \min(y(0))$.

(b) *Let* $s\square$ *be max-separated by* a. *If* $x, y \ll a$, *then* $\Delta(s\square\,\hat{}\,x) = \Delta(s\square\,\hat{}\,y)$ *implies* $\max(x(0)) = \max(y(0))$.

(c) *Let* s *be min-max-separated by* a. *If* $x, y \ll a$, *then* $\Delta(s\,\hat{}\,x) = \Delta(s\,\hat{}\,y)$ *implies* $\min(x(0)) = \min(y(0))$ *and* $\max(x(0)) = \max(y(0))$.

(d) *Let* $s\square$ *be strongly separated by* a. *If* $x, y \ll a$, *then* $\Delta(s\square\,\hat{}\,x) = \Delta(s\square\,\hat{}\,y)$ *implies that there exists* k *such that* $x(0) = y(0) \cap k$ *or* $y(0) = x(0) \cap k$, *i.e., either* $x(0)$ *is an initial segment of* $y(0)$ *or conversely.*

PROOF. Let $x, y \ll a$ be such that $\Delta(s\square\,\hat{}\,x) = \Delta(s\square\,\hat{}\,y)$. Notice that we can assume without loss of generality that $\max(x(0)) < \max(y(0))$; since $\max(x(0)) = \max(y(0))$ together with the hypothesis of each of the four cases implies that $s\square\,\hat{}\,\langle x(0)\rangle$ and $s\square\,\hat{}\,\langle y(0)\rangle$ are mixed by a, and the assertion follows by Lemma 2.6.

First of all, we show that if $s\square$ is separated in some sense by a and

$max(x(0)) < max(y(0))$, we must have that $min(x(0)) = min(y(0))$. Therefor assume to the contrary that $min(x(0)) \neq min(y(0))$. We distinguish three cases.

First, let $max(x(0)) < min(y(0))$. Since $\Delta(s\square{}^\frown x) = \Delta(s\square{}^\frown y)$, we have that $s\square{}^\frown \langle x(0)\rangle$ and $s\square$ are mixed by a. Hence by (d) of canonical we must have that $s\square{}^\frown m$ and $s\square$ are mixed by a for all $m \ll a$. By transitivity of mixing it follows that $s\square{}^\frown m$ and $s\square{}^\frown n$ are mixed by a for every $m, n \ll a$. But this contradicts that $s\square$ is separated in some sense by a.

Next, suppose that $min(x(0)) < min(y(0))$ and $max(x(0)) > min(y(0))$. Let v be the part of $x(0)$ below $min(y(0))$. Since $\Delta(s\square{}^\frown x) = \Delta(s\square{}^\frown y)$, we have that $s\square{}^\frown \langle v\rangle\uparrow$ and $s\square$ are mixed by a. Thus, by (d) of canonical we must have that $s\square{}^\frown m\uparrow$ and $s\square$ are mixed by a for all $m \ll a$. Now let w denote the part of $y(0)$ less than or equal to $max(x(0))$. Hence we have that $s\square{}^\frown \langle x(0)\rangle$ and $s\square{}^\frown \langle w\rangle\uparrow$ are mixed by a. By transitivity of mixing $s\square{}^\frown \langle x(0)\rangle$ and $s\square$ are mixed by a. Therefore, by (d) of canonical we must have that $s\square{}^\frown m$ and $s\square$ are mixed by a for all $m \ll a$. Again, by transitivity of mixing it follows that $s\square{}^\frown m$ and $s\square{}^\frown n$ are mixed by a for every $m, n \ll a$. But this contradicts that $s\square$ is separated in some sense by a.

Finally, assume that $min(x(0)) > min(y(0))$. Let v be the part of $y(0)$ below $min(x(0))$. Since $\Delta(s\square{}^\frown x) = \Delta(s\square{}^\frown y)$, we have that $s\square$ and $s\square{}^\frown \langle v\rangle\uparrow$ are mixed by a. Thus, by (d) of canonical we must have that $s\square$ and $s\square{}^\frown m\uparrow$ are mixed by a for all $m \ll a$. Now let w denote the part of $y(0)$ less than or equal to $max(x(0))$. Hence we have that $s\square{}^\frown \langle x(0)\rangle$ and $s\square{}^\frown \langle w\rangle\uparrow$ are mixed by a. By transitivity of mixing $s\square{}^\frown \langle x(0)\rangle$ and $s\square$ are mixed by a. Therefore, by (d) of canonical we must have that $s\square{}^\frown m$ and $s\square$ are mixed by a for all $m \ll a$. Again, by transitivity of mixing it follows that $s\square{}^\frown m$ and $s\square{}^\frown n$ are mixed by a for every $m, n \ll a$. But this contradicts that $s\square$ is separated in some sense by a.

Therewith we must have $min(x(0)) = min(y(0))$. This already proves case (a) of this lemma.

Now we prove case (b) and (c) in one step. Therefor let $s\square$ be max-separated or min-max-separated by a. Recall that we can assume without loss of generality $min(x(0)) = min(y(0))$ and $max(x(0)) < max(y(0))$.

Let v be the part of $y(0)$ less than or equal to $max(x(0))$. Hence we have that $s\square{}^\frown \langle x(0)\rangle$ and $s\square{}^\frown \langle v\rangle\uparrow$ are mixed by a. Additionally, the cases (c) and (d) of Lemma 2.8 yield that $s\square\,\hat{}\,m\uparrow$ and $s\square\,\hat{}\,n\uparrow$ are mixed by a for all $m, n \ll a$ with $min(m) = min(n)$. Therefore $s\square\,\hat{}\,\langle v\rangle\uparrow$ and $s\square\,\hat{}\,\langle x(0)\rangle\uparrow$ are mixed by a, because $min(v) = min(x(0))$. By transitivity of mixing we get that $s\square\,\hat{}\,\langle x(0)\rangle$ and $s\square\,\hat{}\,\langle x(0)\rangle\uparrow$ are mixed by a. Moreover by (d) of canonical we must have that $s\square\,\hat{}\,m$ and $s\square\,\hat{}\,m\uparrow$ are mixed by a for every $m \ll a$. Altogether, we have that $s\square\,\hat{}\,\langle x(0)\rangle$ and $s\square\,\hat{}\,\langle x(0)\rangle\uparrow$, $s\square\,\hat{}\,\langle x(0)\rangle\uparrow$ and $s\square\,\hat{}\,\langle y(0)\rangle\uparrow$ as well as $s\square\,\hat{}\,\langle y(0)\rangle\uparrow$ and $s\square\,\hat{}\,\langle y(0)\rangle$ are mixed by a. Again, by transitivity of mixing we obtain that $s\square\,\hat{}\,\langle x(0)\rangle$ and $s\square\,\hat{}\,\langle y(0)\rangle$ are mixed by a. But this contradicts the fact that $s\square$ is max-separated or min-max-separated by a.

Hence we must have that $max(x(0)) = max(y(0))$, and the assertion follows by Lemma 2.6.

Finally, we prove case (d) of this lemma. Therefor let $s\square$ be strongly separated by a. Recall that we can assume without loss of generality $min(x(0)) = min(y(0))$ and $max(x(0)) < max(y(0))$. Suppose to the contrary that $x(0)$ is not an initial segment of $y(0)$.

Let b with $a \ll b$ be as in (f) of canonical. Moreover let v denote the longest common initial segment of $x(0)$ and $y(0)$. Choose k with $min(x(0) \vartriangle y(0)) \in b(k)$. Since $x(0)$ is not an initial segment of $y(0)$, we have that $s\square\,\hat{}\,\langle v\rangle\uparrow$ and $s\square\,\hat{}\,\langle v\rangle\uparrow\,\hat{}\,\langle b(k)\rangle\uparrow$ are mixed by b. Hence by (d) of canonical we must have that $s\square\,\hat{}\,m\uparrow$ and $s\square\,\hat{}\,m\uparrow\,\hat{}\,n\uparrow$ are mixed by b for all $m, n \ll b$. Furthermore, let w denote the part of $y(0)$ less than or equal to $max(x(0))$. Therewith we get that $s\square\,\hat{}\,\langle x(0)\rangle$ and $s\square\,\hat{}\,\langle w\rangle\uparrow$ are mixed by b, too. Since v is an initial segment of w, we get with the result above that $s\square\,\hat{}\,\langle w\rangle\uparrow$ and $s\square\,\hat{}\,\langle v\rangle\uparrow$ are mixed by b. Hence by transitivity of mixing $s\square\,\hat{}\,\langle x(0)\rangle$ and $s\square\,\hat{}\,\langle v\rangle\uparrow$ are mixed by b. Moreover since v is an initial segment of $x(0)$, property (d) of canonical yields that $s\square\,\hat{}\,m\uparrow$ and $s\square\,\hat{}\,m\uparrow\,\hat{}\,n$ are mixed by b for all $m, n \ll b$. Thus, equally $s\square\,\hat{}\,\langle y(0)\rangle$ and $s\square\,\hat{}\,\langle v\rangle\uparrow$ are mixed by b, and by transitivity we obtain that $s\square\,\hat{}\,\langle x(0)\rangle$ and $s\square\,\hat{}\,\langle y(0)\rangle$ are mixed by b. Since $a \ll b$, we must have that $s\square\,\hat{}\,\langle x(0)\rangle$ and $s\square\,\hat{}\,\langle y(0)\rangle$ are also mixed by a. But this contradicts that $s\square$ is strongly separated by a.

Hence we must have that $x(0)$ is an initial segment of $y(0)$. This completes the proof. ⊣

LEMMA 2.12. *Let $s, t \ll a$. Suppose $s\square$ and $t\square$ are mixed by a and $s\square$ is separated in some sense by a. If $x, y \ll a$ such that $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, then $max(x(0)) > min(y(0))$.*

PROOF. Let $x, y \ll a$ be such that $\Delta(s\square{}^\frown x) = \Delta(t\square{}^\frown y)$. Assume to the contrary that $max(x(0)) < min(y(0))$. Note that $max(x(0)) = min(y(0))$ is impossible by (f) of canonical.

Choose $0 < k \leqslant dom(t)$ maximal with $max(t(k\text{-}1)) \leqslant max(x(0))$ if possible, otherwise choose $k = 0$. Moreover if $k < dom(t)$, let v denote the part of $t(k)$ less than or equal to $max(x(0))$. Thus, if $k = dom(t)$ or $v = \varnothing$, we have that $s\square{}^\frown \langle x(0)\rangle$ and $t\upharpoonright k\square$ are mixed by a. Otherwise we have that $s\square{}^\frown \langle x(0)\rangle$ and $t\upharpoonright k{}^\frown \langle v\rangle\uparrow$ are mixed by a.

Moreover since $s\square$ and $t\square$ are mixed by a, there exist $x_0, y_0 \ll a$ with $s < x_0, y_0$ and $t < x_0, y_0$ such that $\Delta(s\square{}^\frown x_0) = \Delta(t\square{}^\frown y_0)$.

Now assume that we are in the first case, where $s\square{}^\frown \langle x(0)\rangle$ and $t\upharpoonright k\square$ are mixed by a. If $k < dom(t)$, we can choose $y_1 \ll a$ by $y_1 = \langle t(i)\colon k \leqslant i < dom(t)\rangle{}^\frown y_0$ such that $\Delta(s\square{}^\frown x_0) = \Delta(t\upharpoonright k\square{}^\frown y_1)$. By choice of k we have $s < x_0, y_1$ and $t\upharpoonright k < x_0, y_1$. Hence by (b) of canonical we must have that $s\square$ and $t\upharpoonright k\square$ are mixed by a.

Next, suppose that we are in the case, where $s\square{}^\frown \langle x(0)\rangle$ and $t\upharpoonright k{}^\frown \langle v\rangle\uparrow$ are mixed by a. Let w be the part of $t(k)$ above $max(v)$. If $k < dom(t) - 1$, choose $y_1 \ll a$ by $y_1 = \langle w\rangle{}^\frown \langle t(i)\colon k < i < dom(t)\rangle{}^\frown y_0$, otherwise choose $y_1 = \langle w\rangle{}^\frown y_0$. Therewith we have that $\Delta(s\square{}^\frown x_0) = \Delta(t\upharpoonright k{}^\frown \langle v\rangle\uparrow{}^\frown y_1)$ with $s < x_0, y_1$ and $t\upharpoonright k{}^\frown \langle v\rangle < x_0, y_1$. Thus, by (b) of canonical we get that $s\square$ and $t\upharpoonright k{}^\frown \langle v\rangle\uparrow$ are mixed by a.

Since $s\square$ and $t\square$ are mixed by a, by transitivity of mixing we can conclude that $s\square{}^\frown \langle x(0)\rangle$ and $s\square$ are mixed by a, contradicting all cases of Lemma 2.11. ⊣

LEMMA 2.13. *Let $s, t \ll a$. If $x, y \ll a$ with $min(x(0)) = min(y(0))$ such that $\Delta(s\square{}^\frown x) = \Delta(t\square{}^\frown y)$, then $s\square{}^\frown m\uparrow$ and $t\square{}^\frown m\uparrow$ are mixed by a for every $m \ll a$.*

PROOF. Let b with $a \ll b$ be as in (f) of canonical. Choose k with $min(x(0)) \in b(k)$. Since $\Delta(s\square{}^\frown x) = \Delta(t\square{}^\frown y)$, we have that $s\square{}^\frown \langle b(k)\rangle\uparrow$ and $t\square{}^\frown \langle b(k)\rangle\uparrow$ are mixed by b. By (d) of canonical we must have that $s\square{}^\frown m\uparrow$ and $t\square{}^\frown m\uparrow$ are mixed by b for every $m \ll b$. Since $a \ll b$, by Corollary 2.2 we also have that $s\square{}^\frown m\uparrow$ and $t\square{}^\frown m\uparrow$ are mixed by a for every $m \ll a$. ⊣

LEMMA 2.14. *Let $s, t \ll a$. Suppose s and t are mixed by a and both s and t are min-separated by a. If $x, y \ll a$ such that $\Delta(s{}^\frown x) = \Delta(t{}^\frown y)$,* *then* $min(x(0)) = min(y(0))$.

PROOF. Let $x, y \ll a$ be such that $\Delta(s ^\frown x) = \Delta(t ^\frown y)$. Assume to the contrary that $min(x(0)) \neq min(y(0))$. By symmetry we can suppose without loss of generality that $min(x(0)) < min(y(0))$. Moreover by Lemma 2.12 it suffices to prove that the assumption that $min(x(0)) < min(y(0))$ and $max(x(0)) > min(y(0))$ leads to a contradiction.

Let v be the part of $x(0)$ below $min(y(0))$. Since $\Delta(s ^\frown x) = \Delta(t ^\frown y)$, we must have that $s ^\frown \langle v\rangle\upharpoonleft$ and t are mixed by a. Moreover since s and t are mixed by a, by transitivity of mixing we get that $s ^\frown \langle v\rangle\upharpoonleft$ and s are mixed by a, contradicting case (a) of Lemma 2.11. ⊣

LEMMA 2.15. *Let* $s, t \ll a$. *Suppose s and t are mixed by a and both s and t are min-separated by a. Then* $s ^\frown m$ *and* $t ^\frown n$ *are mixed by a for all* $m, n \ll a$ *with* $min(m) = min(n)$.

PROOF. Since s and t are mixed by a, there exist $x, y \ll a$ such that $\Delta(s ^\frown x) = \Delta(t ^\frown y)$. By Lemma 2.14 we have that $min(x(0)) = min(y(0))$.

Moreover by Lemma 2.13 we get that $s ^\frown m\upharpoonleft$ and $t ^\frown m\upharpoonleft$ are mixed by a for every $m \ll a$. Additionally, case (b) of Lemma 2.8 yields that $s ^\frown m$ and $s ^\frown m\upharpoonleft$ as well as $t ^\frown m\upharpoonleft$ and $t ^\frown n$ are mixed by a for all $m, n \ll a$ with $min(m) = min(n)$. Thus, by transitivity of mixing we get that $s ^\frown m$ and $t ^\frown n$ are mixed by a for every $m, n \ll a$ with $min(m) = min(n)$. ⊣

LEMMA 2.16. *Let* $s, t \ll a$. *Suppose* $s\Box$ *and* $t\Box$ *are mixed by a and both* $s\Box$ *and* $t\Box$ *are max-separated by a. If* $x, y \ll a$ *such that* $\Delta(s\Box ^\frown x) = \Delta(t\Box ^\frown y)$, *then* $max(x(0)) = max(y(0))$.

PROOF. Let $x, y \ll a$ be such that $\Delta(s\Box ^\frown x) = \Delta(t\Box ^\frown y)$. Assume to the contrary that $max(x(0)) \neq max(y(0))$. By symmetry we can suppose without loss of generality that $max(x(0)) < max(y(0))$. Moreover by Lemma 2.12 it suffices to prove that the assumption that $max(x(0)) < max(y(0))$ and $max(x(0)) > min(y(0))$ leads to a contradiction.

So let w be the part of $y(0)$ less than or equal to $max(x(0))$. Since $\Delta(s\Box ^\frown x) = \Delta(t\Box ^\frown y)$, we have that $s\Box ^\frown \langle x(0)\rangle$ and $t\Box ^\frown \langle w\rangle\upharpoonleft$ are mixed by a. Additionally, case (c) of Lemma 2.8 yields that $t\Box$ and $t\Box ^\frown \langle w\rangle\upharpoonleft$ are mixed by a. Moreover since $s\Box$ and $t\Box$ are mixed by a, by transitivity of mixing it follows that $s\Box$ and $s\Box ^\frown \langle x(0)\rangle$ are mixed by a, contradicting case (b) of Lemma 2.11. ⊣

LEMMA 2.17. *Let* $s, t \ll a$. *Suppose* $s\square$ *and* $t\square$ *are mixed by* a *and both* $s\square$ *and* $t\square$ *are max-separated by* a. *Then* $s\square{}^\frown m$ *and* $t\square{}^\frown n$ *are mixed by* a *for all* $m, n \ll a$ *with* $max(m) = max(n)$.

PROOF. Since $s\square$ and $t\square$ are mixed by a, there exist $x, y \ll a$ such that $\Delta(s\square{}^\frown x) = \Delta(t\square{}^\frown y)$. By Lemma 2.16 we have that $max(x(0)) = max(y(0))$. Hence by definition of mixing we must have that $s\square{}^\frown \langle x(0)\rangle$ and $t\square{}^\frown \langle y(0)\rangle$ are mixed by a. Moreover we have that $t\square{}^\frown \langle x(0)\rangle$ and $t\square{}^\frown \langle y(0)\rangle$ are mixed by a, because $t\square$ is max-separated by a. By transitivity of mixing we get that $s\square{}^\frown \langle x(0)\rangle$ and $t\square{}^\frown \langle x(0)\rangle$ are mixed by a. Thus, (d) of canonical yields that $s\square{}^\frown m$ and $t\square{}^\frown m$ are mixed by a for all $m \ll a$.

Again, since $t\square$ is max-separated by a, we have that $t\square{}^\frown m$ and $t\square{}^\frown n$ are mixed by a for every $m, n \ll a$ with $max(m) = max(n)$. Finally, by transitivity of mixing we obtain that $s\square{}^\frown m$ and $t\square{}^\frown n$ are mixed by a for all $m, n \ll a$ with $max(m) = max(n)$. ⊣

LEMMA 2.18. *Let* $s, t \ll a$. *Suppose* s *and* t *are mixed by* a *and both* s *and* t *are min-max-separated by* a. *If* $x, y \ll a$ *such that* $\Delta(s^\frown x) = \Delta(t^\frown y)$, *then* $min(x(0)) = min(y(0))$ *and* $max(x(0)) = max(y(0))$.

PROOF. Let $x, y \ll a$ be such that $\Delta(s^\frown x) = \Delta(t^\frown y)$. First, assume to the contrary that $min(x(0)) \neq min(y(0))$. By symmetry we can suppose without loss of generality that $min(x(0)) < min(y(0))$. Moreover by Lemma 2.12 it suffices to prove that the assumption that $min(x(0)) < min(y(0))$ and $max(x(0)) > min(y(0))$ leads to a contradiction.

Let v be the part of $x(0)$ below $min(y(0))$. Since $\Delta(s^\frown x) = \Delta(t^\frown y)$ we must have that $s^\frown \langle v\rangle\upharpoonright$ and t are mixed by a. Since s and t are mixed by a, by transitivity of mixing we get that $s^\frown \langle v\rangle\upharpoonright$ and s are mixed by a, contradicting case (c) of Lemma 2.11.

Hence we must have $min(x(0)) = min(y(0))$. Now assume to the contrary that $max(x(0)) \neq max(y(0))$. Equally by symmetry we can suppose without loss of generality that $max(x(0)) < max(y(0))$.

Let w be the part of $y(0)$ less than or equal to $max(x(0))$. Therewith we have that $s^\frown \langle x(0)\rangle$ and $t^\frown \langle w\rangle\upharpoonright$ are mixed by a. Since $min(x(0)) = min(y(0))$, Lemma 2.13 yields that $s^\frown \langle w\rangle\upharpoonright$ and $t^\frown \langle w\rangle\upharpoonright$ are mixed by a.

By transitivity of mixing we get that $s\,\hat{}\,\langle x(0)\rangle$ and $s\,\hat{}\,\langle w\rangle\upharpoonright$ are mixed by a, contradicting case (c) of Lemma 2.11. This completes the proof of the lemma. ⊣

LEMMA 2.19. *Let $s, t \ll a$. Suppose s and t are mixed by a and both s and t are min-max-separated by a. Then $s\,\hat{}\,m\upharpoonright$ and $t\,\hat{}\,m\upharpoonright$ are mixed by a for every $m \ll a$. Moreover $s\,\hat{}\,m$ and $t\,\hat{}\,n$ are mixed by a for all $m, n \ll a$ with $min(m) = min(n)$ and $max(m) = max(n)$.*

PROOF. Since s and t are mixed by a, there exist $x, y \ll a$ such that $\Delta(s\,\hat{}\,x) = \Delta(t\,\hat{}\,y)$. By Lemma 2.18 we get that $min(x(0)) = min(y(0))$ and $max(x(0)) = max(y(0))$. Therefore, Lemma 2.13 yields that $s\,\hat{}\,m\upharpoonright$ and $t\,\hat{}\,m\upharpoonright$ are mixed by a for every $m \ll a$, which is our first assertion.

Additionally, by definition of mixing we have that $s\,\hat{}\,\langle x(0)\rangle$ and $t\,\hat{}\,\langle y(0)\rangle$ are mixed by a. Moreover we have that $t\,\hat{}\,\langle x(0)\rangle$ and $t\,\hat{}\,\langle y(0)\rangle$ are mixed by a, because t is min-max-separated by a. By transitivity of mixing we get that $s\,\hat{}\,\langle x(0)\rangle$ and $t\,\hat{}\,\langle x(0)\rangle$ are mixed by a. Thus, (d) of canonical yields that $s\,\hat{}\,m$ and $t\,\hat{}\,m$ are mixed by a for all $m \ll a$.

Again, since t is min-max-separated by a, we have that $t\,\hat{}\,m$ and $t\,\hat{}\,n$ are mixed by a for every $m, n \ll a$ with $min(m) = min(n)$ and $max(m) = max(n)$. Finally, by transitivity of mixing we obtain that $s\,\hat{}\,m$ and $t\,\hat{}\,n$ are mixed by a for all $m, n \ll a$ with $min(m) = min(n)$ and $max(m) = max(n)$. ⊣

LEMMA 2.20. *Let $s, t \ll a$. Suppose $s□$ and $t□$ are mixed by a and both $s□$ and $t□$ are strongly separated by a. If $x, y \ll a$ such that $\Delta(s□\,\hat{}\,x) = \Delta(t□\,\hat{}\,y)$, then there exists k such that $x(0) = y(0) \cap k$ or $y(0) = x(0) \cap k$, i. e., either $x(0)$ is an initial segment of $y(0)$ or conversely.*

PROOF. Let $x, y \ll a$ be such that $\Delta(s□\,\hat{}\,x) = \Delta(t□\,\hat{}\,y)$. First, assume to the contrary that $min(x(0)) \neq min(y(0))$. By symmetry we can suppose without loss of generality that $min(x(0)) < min(y(0))$. Moreover by Lemma 2.12 it suffices to prove that the assumption that $min(x(0)) < min(y(0))$ and $max(x(0)) > min(y(0))$ leads to a contradiction.

Let v be the part of $x(0)$ below $min(y(0))$. Since $\Delta(s□\,\hat{}\,x) = \Delta(t□\,\hat{}\,y)$, we must have that $s□\,\hat{}\,\langle v\rangle\upharpoonright$ and $t□$ are mixed by a. Moreover since $s□$ and $t□$ are mixed by a, by transitivity of mixing we get that $s□\,\hat{}\,\langle v\rangle\upharpoonright$ and $s□$ are mixed by a, contradicting case (d) of Lemma 2.11.

Hence we must have that $min(x(0)) = min(y(0))$. Now assume to the contrary that neither $x(0)$ is an initial segment of $y(0)$ nor conversely. By symmetry we can suppose without loss of generality that $x(0)$ is not an initial segment of $y(0)$.

Let v denote the longest common initial segment of $x(0)$ and $y(0)$. Moreover choose k with $min(x(0) \vartriangle y(0)) \in a(k)$. Since $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, we have that either $s\square\,\hat{}\,\langle v\rangle\uparrow\,\hat{}\,\langle a(k)\rangle\square$ and $t\square\,\hat{}\,\langle v\rangle\uparrow$ or $s\square\,\hat{}\,\langle v\rangle\uparrow$ and $t\square\,\hat{}\,\langle v\rangle\uparrow\,\hat{}\,\langle a(k)\rangle\uparrow$ are mixed by a. Additionally, by Lemma 2.13 we have that $s\square\,\hat{}\,\langle v\rangle\uparrow$ and $t\square\,\hat{}\,\langle v\rangle\uparrow$ are mixed by a, because $min(x(0)) = min(y(0))$. Thus, by transitivity of mixing we get in the first case that $s\square\,\hat{}\,\langle v\rangle\uparrow\,\hat{}\,\langle a(k)\rangle\square$ and $s\square\,\hat{}\,\langle v\rangle\uparrow$, in the second case that $t\square\,\hat{}\,\langle v\rangle\uparrow$ and $t\square\,\hat{}\,\langle v\rangle\uparrow\,\hat{}\,\langle a(k)\rangle\uparrow$ are mixed by a. Both cases contradict case (d) of Lemma 2.11. ⊣

Now we want to analyse the case that $s\square$ is strongly separated by a. Since a is canonical for Δ, we are able to distinguish exactly two possibilities.

DEFINITION. Let $s \ll a$. Suppose that $s\square$ is strongly separated by a. We say that *$s\square$ is still strongly separated by a* iff $s\square\,\hat{}\,m$ and $s\square\,\hat{}\,m\uparrow$ are mixed by a for every $m \ll a$. Moreover *$s\square$ is very strongly separated by a* iff $s\square\,\hat{}\,m$ and $s\square\,\hat{}\,m\uparrow$ are separated by a for every $m \ll a$.

COROLLARY 2.21. *Let $s \ll a$.*

(a) Let $s\square$ be still strongly separated by a. Then $s\square\,\hat{}\,m\uparrow$ is still strongly separated by a for every $m \ll a$.

(b) Let $s\square$ be very strongly separated by a. Then $s\square\,\hat{}\,m\uparrow$ is very strongly separated by a for every $m \ll a$.

PROOF. Obvious from the definition. ⊣

LEMMA 2.22. *Let $s, t \ll a$. Suppose $s\square$ and $t\square$ are mixed by a and both $s\square$ and $t\square$ are very strongly separated by a. If $x, y \ll a$ such that $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, then $x(0) = y(0)$.*

PROOF. Let $x, y \ll a$ be such that $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$. By Lemma 2.20 we have that $x(0)$ is an initial segment of $y(0)$ or conversely. Moreover by symmetry we can suppose without loss of generality that

$x(0)$ is an initial segment of $y(0)$.

Assume to the contrary that $max(x(0)) < max(y(0))$. Since $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, we have that $s\square\,\hat{}\,\langle x(0)\rangle$ and $t\square\,\hat{}\,\langle x(0)\rangle\uparrow$ are mixed by a. By (d) of canonical we get that $s\square\,\hat{}\,m$ and $t\square\,\hat{}\,m\uparrow$ are mixed by a for all $m \ll a$. Since $min(x(0)) = min(y(0))$, by Lemma 2.13 we also have that $s\square\,\hat{}\,m\uparrow$ and $t\square\,\hat{}\,m\uparrow$ are mixed by a for all $m \ll a$. Finally, by transitivity of mixing we can conclude that $s\square\,\hat{}\,m$ and $s\square\,\hat{}\,m\uparrow$ are mixed by a for every $m \ll a$. But this contradicts our assumption that $s\square$ is very strongly separated by a. ⊣

LEMMA 2.23. *Let $s, t \ll a$. Suppose $s\square$ and $t\square$ are mixed by a and both $s\square$ and $t\square$ are very strongly separated by a. Then $s\square\,\hat{}\,m$ and $t\square\,\hat{}\,m$ are mixed by a for all $m \ll a$.*

PROOF. Otherwise by (d) of canonical we would have that $s\square\,\hat{}\,m$ and $t\square\,\hat{}\,m$ are separated by a for every $m \ll a$. By Lemma 2.22 this would contradict that $s\square$ and $t\square$ are mixed by a, so the assertion follows. ⊣

LEMMA 2.24. *Let $s, t \ll a$. Suppose $s\square$ and $t\square$ are mixed by a and $s\square$ is strongly mixed by a. Moreover assume that $t\square$ is either min-separated, min-max-separated or strongly separated by a. If $x, y \ll a$ such that $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, then $max(x(0)) < min(y(0))$.*

PROOF. Let $x, y \ll a$ be such that $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$. Assume to the contrary that $max(x(0)) > min(y(0))$. By Lemma 2.12 we must have that $min(x(0)) < max(y(0))$. We distinguish three cases.

For the first case suppose that $min(x(0)) > min(y(0))$ and $min(x(0)) < max(y(0))$. Let v denote the part of $y(0)$ below $min(x(0))$. Since $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, we have that $s\square$ and $t\square\,\hat{}\,\langle v\rangle\uparrow$ are mixed by a. Moreover since $s\square$ and $t\square$ are mixed by a, by transitivity of mixing we get that $t\square$ and $t\square\,\hat{}\,\langle v\rangle\uparrow$ are mixed by a. But this contradicts case (a), (c) and (d) of Lemma 2.11.

Next, assume that $min(x(0)) \leqslant min(y(0))$ and $max(x(0)) \geqslant max(y(0))$. Let v be the part of $x(0)$ less than or equal to $max(y(0))$. Since $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, we have that $s\square\,\hat{}\,\langle v\rangle\square$ and $t\square\,\hat{}\,\langle y(0)\rangle$ are mixed by a. Additionally, by (a) of Lemma 2.8 we have that $s\square$ and $s\square\,\hat{}\,\langle v\rangle\square$ are mixed by a, because $s\square$ is strongly mixed by a. Moreover since $s\square$ and $t\square$ are mixed by a, by transitivity of mixing we obtain that $t\square$ and $t\square\,\hat{}$

$\langle y(0)\rangle$ are mixed by a. Equally, this contradicts case (a), (c) and (d) of Lemma 2.11.

Finally, suppose that $min(x(0)) \leqslant min(y(0))$ and $max(x(0)) < max(y(0))$. Let v denote the part of $y(0)$ less than or equal to $max(x(0))$. Since $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, we have that $s\square\,\hat{}\,\langle x(0)\rangle$ and $t\square\,\hat{}\,\langle v\rangle\!\uparrow$ are mixed by a. Additionally, by (a) of Lemma 2.8 we have that $s\square$ and $s\square\,\hat{}\,\langle x(0)\rangle$ are mixed by a, because $s\square$ is strongly mixed by a. Moreover since $s\square$ and $t\square$ are mixed by a, by transitivity of mixing we get that $t\square$ and $t\square\,\hat{}\,\langle v\rangle\!\uparrow$ are mixed by a, a contradiction as above. ⊣

LEMMA 2.25. *Let $s, t \ll a$. Suppose $s\square$ and $t\square$ are mixed by a, $s\square$ is strongly mixed by a and $t\square$ is max-separated by a. If $x, y \ll a$ such that $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, then $max(x(0)) < max(y(0))$.*

PROOF. Let $x, y \ll a$ be such that $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$. Assume to the contrary that $max(x(0)) \geqslant max(y(0))$. By Lemma 2.12 we must have $min(x(0)) < max(y(0))$. We distinguish two cases.

First, suppose that $max(x(0)) > max(y(0))$ and $min(x(0)) < max(y(0))$. Let v denote the part of $x(0)$ less than or equal to $max(y(0))$. Since $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, we have that $s\square\,\hat{}\,\langle v\rangle\!\uparrow$ and $t\square\,\hat{}\,\langle y(0)\rangle$ are mixed by a. Moreover by (a) of Lemma 2.8 we have that $s\square\,\hat{}\,\langle v\rangle\!\uparrow$ and $s\square$ are mixed by a, because $s\square$ is strongly mixed by a. Finally, since $s\square$ and $t\square$ are mixed by a, by transitivity of mixing we can conclude that $t\square\,\hat{}\,\langle y(0)\rangle$ and $t\square$ are mixed by a. But this contradicts case (b) of Lemma 2.11.

Next, assume that $max(x(0)) = max(y(0))$. By definition of mixing we have that $s\square\,\hat{}\,\langle x(0)\rangle$ and $t\square\,\hat{}\,\langle y(0)\rangle$ are mixed by a. Since $s\square$ is strongly mixed by a, by (a) of Lemma 2.8 we get that also $s\square\,\hat{}\,\langle x(0)\rangle$ and $s\square$ are mixed by a. Moreover we have that $s\square$ and $t\square$ are mixed by a. Therefore, by transitivity of mixing we get that $t\square\,\hat{}\,\langle y(0)\rangle$ and $t\square$ are mixed by a, which equally contradicts case (b) of Lemma 2.11. ⊣

LEMMA 2.26. *Let $s, t \ll a$. Suppose s and t are mixed by a, s is min-separated by a and t is min-max-separated by a. If $x, y \ll a$ such that $\Delta(s\,\hat{}\,x) = \Delta(t\,\hat{}\,y)$, then $min(x(0)) = min(y(0))$ and $max(x(0)) < max(y(0))$.*

PROOF. Let $x, y \ll a$ be such that $\Delta(s \hat{\ } x) = \Delta(t \hat{\ } y)$. First of all, we prove that we must have $min(x(0)) = min(y(0))$. For that purpose assume to the contrary that $min(x(0)) \neq min(y(0))$. Two applications of Lemma 2.12 yield that $max(x(0)) > min(y(0))$ and $min(x(0)) < max(y(0))$. We distinguish two more cases.

First, suppose that $min(x(0)) < min(y(0))$ and $max(x(0)) > min(y(0))$. Let v denote the part of $x(0)$ below $min(y(0))$. Since $\Delta(s \hat{\ } x) = \Delta(t \hat{\ } y)$, we have that $s \hat{\ } \langle v \rangle \uparrow$ and t are mixed by a. Moreover s and t are mixed by a. Hence by transitivity of mixing we get that $s \hat{\ } \langle v \rangle \uparrow$ and s are mixed by a. But this contradicts case (a) of Lemma 2.11.

Next, assume that $min(x(0)) > min(y(0))$ and $min(x(0)) < max(y(0))$. Let v be the part of $y(0)$ below $min(x(0))$. Since $\Delta(s \hat{\ } x) = \Delta(t \hat{\ } y)$, we have that s and $t \hat{\ } \langle v \rangle \uparrow$ are mixed by a. Moreover s and t are mixed by a. Hence by transitivity of mixing we get that t and $t \hat{\ } \langle v \rangle \uparrow$ are mixed by a. This contradicts case (c) of Lemma 2.11.

Hence we have $min(x(0)) = min(y(0))$. Now we show that we also have that $max(x(0)) < max(y(0))$. Therefor assume to the contrary that $max(x(0)) \geqslant max(y(0))$.

Let v denote the part of $x(0)$ less than or equal to $max(y(0))$. Since $\Delta(s \hat{\ } x) = \Delta(t \hat{\ } y)$, we have that $s \hat{\ } \langle v \rangle \square$ and $t \hat{\ } \langle y(0) \rangle$ are mixed by a. Additionally, by (b) of Lemma 2.8 we get that $s \hat{\ } \langle y(0) \rangle \uparrow$ and $s \hat{\ } \langle v \rangle \square$ are mixed by a, because $min(v) = min(y(0))$. Moreover since $min(x(0)) = min(y(0))$, Lemma 2.13 yields that $s \hat{\ } \langle y(0) \rangle \uparrow$ and $t \hat{\ } \langle y(0) \rangle \uparrow$ are also mixed by a. Altogether, by transitivity of mixing we get that $t \hat{\ } \langle y(0) \rangle$ and $t \hat{\ } \langle y(0) \rangle \uparrow$ are mixed by a. But this contradicts case (c) of Lemma 2.11. ⊣

LEMMA 2.27. *Let $s, t \ll a$. Suppose s and t are mixed by a, s is min-separated by a and t is min-max-separated by a. Then $s \hat{\ } m\uparrow$ and $t \hat{\ } m\uparrow$ are mixed by a for every $m \ll a$.*

PROOF. Since s and t are mixed by a, there exist $x, y \ll a$ such that $\Delta(s \hat{\ } x) = \Delta(t \hat{\ } y)$. By Lemma 2.26 we must have that $min(x(0)) = min(y(0))$. Hence Lemma 2.13 yields that $s \hat{\ } m\uparrow$ and $t \hat{\ } m\uparrow$ are mixed by a for every $m \ll a$. ⊣

LEMMA 2.28. *Let $s, t \ll a$.*

(a) *Suppose* $s\square$ *and* $t\square$ *are mixed by a and both* $s\square$ *and* $t\square$ *are still strongly separated by a. Then* $s\square\,\hat{}\,m\square$ *and* $t\square\,\hat{}\,m\square$ *are mixed by a for every* $m \ll a$.

(b) *Suppose* $s\square$ *and* $t\square$ *are mixed by a,* $s\square$ *is still strongly separated by a and* $t\square$ *is very strongly separated by a. Then* $s\square\,\hat{}\,m\square$ *and* $t\square\,\hat{}\,m\upharpoonleft$ *are mixed by a for every* $m \ll a$. *Moreover* $s\square\,\hat{}\,m\square$ *and* $t\square\,\hat{}\,m$ *are separated by a for every* $m \ll a$.

(c) *Suppose* $s\square$ *and* $t\square$ *are mixed by a and both* $s\square$ *and* $t\square$ *are very strongly separated by a. Then* $s\square\,\hat{}\,m\upharpoonleft$ *and* $t\square\,\hat{}\,m\upharpoonleft$ *are mixed by a for every* $m \ll a$. *Moreover* $s\square\,\hat{}\,m$ *and* $t\square\,\hat{}\,m\upharpoonleft$ *are separated by a for every* $m \ll a$.

PROOF. Since $s\square$ and $t\square$ are mixed by a, there exist $x, y \ll a$ such that $\Delta(s\square\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$. In each of the three cases both $s\square$ and $t\square$ are strongly separated by a. Hence by Lemma 2.20 we must have that $x(0)$ is an initial segment of $y(0)$ or conversely. Since $min(x(0)) = min(y(0))$, by Lemma 2.13 we have that $s\square\,\hat{}\,m\upharpoonleft$ and $t\square\,\hat{}\,m\upharpoonleft$ are mixed by a for every $m \ll a$.

The rest of the result follows directly by the definition of being still and very strongly separated, using the transitivity of mixing. ⊣

LEMMA 2.29. *Let* $s, t \ll a$. *Suppose s and* $t\square$ *are mixed by a and s is min-separated by a. Then* $t\square$ *is neither max-separated nor strongly separated by a.*

PROOF. Since s and $t\square$ are mixed by a, there exist $x, y \ll a$ such that $\Delta(s\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$. Assume to the contrary that $t\square$ is either max-separated or strongly separated by a. Two applications of Lemma 2.12 yield that $min(x(0)) < max(y(0))$ and $max(x(0)) > min(y(0))$. We distinguish five cases.

For the first case suppose that $min(x(0)) < min(y(0))$ and $max(x(0)) > min(y(0))$. Let v be the part of $x(0)$ below $min(y(0))$. Since $\Delta(s\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, we have that $s\,\hat{}\,\langle v\rangle\upharpoonleft$ and $t\square$ are mixed by a. Moreover s and $t\square$ are also mixed by a. Hence by transitivity of mixing we obtain that $s\,\hat{}\,\langle v\rangle\upharpoonleft$ and s are mixed by a. But this contradicts case (a) of Lemma 2.11.

Next, assume that $t\square$ is max-separated by a and $min(x(0)) > min(y(0))$ as well as $min(x(0)) < max(y(0))$. Choose k with $min(x(0)) \in a(k)$. Let

w denote the part of $y(0)$ less than or equal to $max(a(k))$. If $max(a(k)) < max(y(0))$, we get that $s ^\frown \langle a(k)\rangle\square$ and $t\square ^\frown \langle w\rangle\uparrow$ are mixed by a, because $\Delta(s ^\frown x) = \Delta(t\square ^\frown y)$. Otherwise, we must have that $max(a(k)) = max(y(0))$, since $min(x(0)) < max(y(0))$. Then we have that $s ^\frown \langle a(k)\rangle\square$ and $t\square ^\frown \langle y(0)\rangle$ are mixed by a. In the former case, by (c) of Lemma 2.8 we get that $t\square ^\frown \langle w\rangle\uparrow$ and $t\square$ are mixed by a. Moreover since s and $t\square$ are mixed by a, by transitivity of mixing we obtain that $s ^\frown \langle a(k)\rangle\square$ and s are mixed by a. This contradicts case (a) of Lemma 2.11. If we are in the latter case, we additionally have that $t\square ^\frown \langle y(0)\rangle$ and $t\square ^\frown \langle a(k)\rangle$ are mixed by a, because $t\square$ is max-separated by a and $max(a(k)) = max(y(0))$. By transitivity of mixing we can conclude that $s ^\frown \langle a(k)\rangle\square$ and $t\square ^\frown \langle a(k)\rangle$ are mixed by a. Moreover by (d) of canonical we must have that $s ^\frown m\square$ and $t\square ^\frown m$ are mixed by a for all $m \ll a$. Finally, by (b) of Lemma 2.8 we have that $s ^\frown m\square$ and $s ^\frown n\square$ are mixed by a for every $m, n \ll a$ with $min(m) = min(n)$. Again, by transitivity of mixing we obtain that $t\square ^\frown m$ and $t\square ^\frown n$ are mixed by a for all $m, n \ll a$ with $min(m) = min(n)$. But this contradicts that $t\square$ is max-separated by a.

For the third case suppose that $t\square$ is strongly separated by a and $min(x(0)) > min(y(0))$ as well as $min(x(0)) < max(y(0))$. Let v be the part of $y(0)$ below $min(x(0))$. Since $\Delta(s ^\frown x) = \Delta(t\square ^\frown y)$, we have that s and $t\square ^\frown \langle v\rangle\uparrow$ are mixed by a. Moreover s and $t\square$ are mixed by a. Hence by transitivity of mixing we obtain that $t\square$ and $t\square ^\frown \langle v\rangle\uparrow$ are mixed by a. But this contradicts case (d) of Lemma 2.11.

Now assume that $t\square$ is max-separated by a and $min(x(0)) = min(y(0))$. By Lemma 2.13 we get that $s ^\frown m\uparrow$ and $t\square ^\frown m\uparrow$ are mixed by a for every $m \ll a$. Additionally, by (c) of Lemma 2.8 we have that $t\square ^\frown m\uparrow$ and $t\square$ are mixed by a for all $m \ll a$, because $t\square$ is max-separated by a. Moreover since s and $t\square$ are mixed by a, by transitivity of mixing we obtain that $s ^\frown m\uparrow$ and s are mixed by a for every $m \ll a$. This contradicts case (a) of Lemma 2.11.

Finally, suppose that $t\square$ is strongly separated by a and $min(x(0)) = min(y(0))$. Equally, by Lemma 2.13 we get $s ^\frown m\uparrow$ and $t\square ^\frown m\uparrow$ are mixed by a for every $m \ll a$. Additionally, by (b) of Lemma 2.8 we have that $s ^\frown m\uparrow$ and $s ^\frown n\uparrow$ are mixed by a for all $m, n \ll a$ with $min(m) = min(n)$. Thus, by transitivity of mixing we get that $t\square ^\frown m\uparrow$ and $t\square ^\frown n\uparrow$ are mixed by a for every $m, n \ll a$ with $min(m) = min(n)$. But this contradicts case (d) of Lemma 2.11. ⊣

LEMMA 2.30. *Let $s, t \ll a$. Suppose $s\square$ and $t\square$ are mixed by a and $s\square$ is max-separated by a. Then $t\square$ is neither min-max-separated nor strongly separated by a.*

PROOF. Since $s\square$ and $t\square$ are mixed by a, there exist $x, y \ll a$ such that $\Delta(s\square\hat{}\ x) = \Delta(t\square\hat{}\ y)$. Assume to the contrary that $t\square$ is either min-max-separated or strongly separated by a. Two applications of Lemma 2.12 yield that $max(x(0)) > min(y(0))$ and $min(x(0)) < max(y(0))$. Now we distinguish three cases.

For the first case suppose that $min(x(0)) < min(y(0))$ and $max(x(0)) > min(y(0))$. Let b with $a \ll b$ be as in (f) of canonical. Moreover let v denote the part of $x(0)$ below $min(y(0))$. Choose k with $min(y(0)) \in b(k)$. Additionally, let w denote the part of $x(0)$ less than or equal to $max(b(k))$. Since $\Delta(s\square\hat{}\ x) = \Delta(t\square\hat{}\ y)$, we have that $s\square\hat{}\ \langle v\rangle\uparrow$ and $t\square$ as well as $s\square\hat{}\ \langle w\rangle\uparrow$ and $t\square\hat{}\ \langle b(k)\rangle\uparrow$ are mixed by b. By (c) and (f) of canonical we have that $s\square$ is also max-separated by b. Therefore, by (c) of Lemma 2.8 we get that $s\square\hat{}\ \langle v\rangle\uparrow$ and $s\square\hat{}\ \langle w\rangle\uparrow$ are mixed by b. Thus, by transitivity of mixing we obtain that $t\square$ and $t\square\hat{}\ \langle w\rangle\uparrow$ are mixed by b. Now (d) and (f) of canonical yield that $t\square$ and $t\square\hat{}\ m\uparrow$ are mixed by b for every $m \ll b$. Finally, since $a \ll b$, we can conclude that $t\square$ and $t\square\hat{}\ m\uparrow$ are mixed by a for every $m \ll a$. But this contradicts case (c) and (d) of Lemma 2.11.

Next, suppose that $min(x(0)) > min(y(0))$ and $min(x(0)) < max(y(0))$. Let v be the part of $y(0)$ below $min(x(0))$. We have that $s\square$ and $t\square\hat{}\ \langle v\rangle\uparrow$ are mixed by a. Since $s\square$ and $t\square$ are mixed by a, we get that $t\square$ and $t\square\hat{}\ \langle v\rangle\uparrow$ are mixed by a, a contradiction as above.

Finally, assume that $min(x(0)) = min(y(0))$. By Lemma 2.13 we have that $s\square\hat{}\ m\uparrow$ and $t\square\hat{}\ m\uparrow$ are mixed by a for all $m \ll a$. Moreover by (c) of Lemma 2.8 we have that $s\square\hat{}\ m\uparrow$ and $s\square\hat{}\ n\uparrow$ are mixed by a for every $m, n \ll a$, because $s\square$ is max-separated by a. By transitivity of mixing we get that $t\square\hat{}\ m\uparrow$ and $t\square\hat{}\ n\uparrow$ are mixed by a for every $m, n \ll a$.

This contradicts case (c) and (d) of Lemma 2.11. ⊣

LEMMA 2.31. *Let $s, t \ll a$. Suppose s and $t\square$ are mixed by a and s is min-max-separated by a. Then $t\square$ is not strongly separated by a.*

PROOF. Since s and $t\square$ are mixed by a, there exist $x, y \ll a$ such that $\Delta(s\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$. Assume to the contrary that $t\square$ is strongly separated by a. Two applications of Lemma 2.12 yield that $max(x(0)) > min(y(0))$ and $min(x(0)) < max(y(0))$. Now we distinguish three cases.

For the first case suppose that $min(x(0)) < min(y(0))$ and $max(x(0)) > min(y(0))$. Let v be the part of $x(0)$ below $min(y(0))$. Since $\Delta(s\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, we have that $s\,\hat{}\,\langle v\rangle\uparrow$ and $t\square$ are mixed by a. Moreover s and $t\square$ are mixed by a. Hence by transitivity of mixing we get that $s\,\hat{}\,\langle v\rangle\uparrow$ and s are mixed by a. But this contradicts case (c) of Lemma 2.11.

Next, suppose that $min(x(0)) > min(y(0))$ and $min(x(0)) < max(y(0))$. Let v be the part of $y(0)$ below $min(x(0))$. Since $\Delta(s\,\hat{}\,x) = \Delta(t\square\,\hat{}\,y)$, we have that s and $t\square\,\hat{}\,\langle v\rangle\uparrow$ are mixed by a. Moreover s and $t\square$ are mixed by a. Hence by transitivity of mixing we get that $t\square$ and $t\square\,\hat{}\,\langle v\rangle\uparrow$ are mixed by a, contradicting case (d) of Lemma 2.11.

Finally, assume that $min(x(0)) = min(y(0))$. By Lemma 2.13 we get that $s\,\hat{}\,m\uparrow$ and $t\square\,\hat{}\,m\uparrow$ are mixed by a for all $m \ll a$. Moreover by (d) of Lemma 2.8 we have that $s\,\hat{}\,m\uparrow$ and $s\,\hat{}\,n\uparrow$ are mixed by a for every $m, n \ll a$ with $min(m) = min(n)$, because s is min-max-separated by a. By transitivity of mixing we get that $t\square\,\hat{}\,m\uparrow$ and $t\square\,\hat{}\,n\uparrow$ are mixed by a for every $m, n \ll a$ with $min(m) = min(n)$. This contradicts case (d) of Lemma 2.11. ⊣

Now we define the parameter γ of the mapping Γ_γ which will canonize our given Δ.

DEFINITION. For given canonical a define $\gamma\colon (a)^{<\omega} \to \{sm, min\text{-}sep, max\text{-}sep, min\text{-}max, sss, vss\}$ as follows: Let $\gamma(s) = sm$ iff s is strongly mixed by a; moreover let $\gamma(s) = min\text{-}sep$ iff s is min-separated by a; let $\gamma(s) = max\text{-}sep$ iff s is max-separated by a; let $\gamma(s) = min\text{-}max$ iff s is min-max-separated by a; let $\gamma(s) = sss$ iff s is still strongly separated by a; finally, let $\gamma(s) = vss$ iff s is very strongly separated by a.

Recall that Γ_γ is defined as follows:

For $m \in [\omega]^{<\omega}$ let $sm(m) = \emptyset$, $min\text{-}sep(m) = \{min(m)\}$, $max\text{-}sep(m) = \{max(m)\}$, $min\text{-}max(m) = \{min(m), max(m)\}$ and $sss(m) = vss(m) = m$.

Let $x \in (a)^\omega$. Define $\Gamma_\gamma(x)$ as follows: Let $k(0) = 0$ and $\langle k(i)\colon 0 < i < N \leqslant \omega\rangle$ increasingly enumerate those k such that $\gamma(x \upharpoonright (k-1)) = vss$. Moreover let $k(N) = \omega$, if $N < \omega$. Now let $\Gamma_\gamma(x) = \langle \bigcup_{k(i)\leqslant j<k(i+1)} \gamma(x \upharpoonright j)(x(j))\colon i < N\rangle$.

Finally, we need three more definitions in order to give our last few lemmas.

DEFINITION. Let $x, y \ll a$ and $k \in \omega$.

If possible, choose $i > 0$ maximal such that $max(x(i-1)) < min(a(k))$, otherwise choose $i = 0$. Additionally, let v denote the part of $x(i)$ below $min(a(k))$. Now define $x_{k\downarrow}$ as follows: If $v = \emptyset$, let $x_{k\downarrow} = x \restriction i$, otherwise let $x_{k\downarrow} = x \restriction i ^\frown \langle v \rangle{\uparrow}$.

Next, choose i minimal such that $min(a(k)) \leqslant min(x(i))$. Additionally if $i > 0$, let v denote the part of $x(i-1)$ larger than or equal to $min(a(k))$, otherwise let $v = \emptyset$. Now define $x_{k\uparrow}$ as follows: If $v = \emptyset$, let $x_{k\uparrow} = x \upharpoonleft i$, otherwise let $x_{k\uparrow} = \langle v \rangle ^\frown x \upharpoonleft i$.

Finally, if possible, choose $0 < i < dom(\Gamma_\gamma(x))$ resp. $0 < j < dom(\Gamma_\gamma(y))$ maximal such that $max(\Gamma_\gamma(x)(i-1)) < min(a(k))$ resp. $max(\Gamma_\gamma(y)(j-1)) < min(a(k))$, otherwise choose $i = 0$ resp. $j = 0$. Additionally, let v resp. w denote the part of $\Gamma_\gamma(x)(i)$ resp. $\Gamma_\gamma(y)(j)$ below $min(a(k))$. Now we say that *$\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k* iff $\Gamma_\gamma(x) \restriction i = \Gamma_\gamma(y) \restriction j$ and $v = w$.

REMARK. By definition of $x_{k\downarrow}$ and $x_{k\uparrow}$ it follows that $x_{k\downarrow} {}^\frown x_{k\uparrow} = x$ for every $x \ll a$ and $k \in \omega$.

LEMMA 2.32. *Let $x, y \ll a$. Suppose that $x_{i\downarrow}$ and $y_{i\downarrow}$ are mixed by a for every $i < \omega$. Then $\Delta(x) = \Delta(y)$.*

PROOF. For every $i < \omega$ let $x_i, y_i \ll a$ be such that $\Delta(x_{i\downarrow} {}^\frown x_i) = \Delta(y_{i\downarrow} {}^\frown y_i)$. These sets exist, because $x_{i\downarrow}$ and $y_{i\downarrow}$ are mixed by a. Moreover by definition of $x_{k\downarrow}$ we obtain that $lim_{i<\omega}\, x_{i\downarrow} {}^\frown x_i = x$ and $lim_{i<\omega}\, y_{i\downarrow} {}^\frown y_i = y$. By (a) of canonical we have that $\Delta \restriction (a)^\omega$ is continuous. Hence we get that $\Delta(x) = lim_{i<\omega}\, \Delta(x_{i\downarrow} {}^\frown x_i)$ and $\Delta(y) = lim_{i<\omega}\, \Delta(y_{i\downarrow} {}^\frown y_i)$. Thus, $lim_{i<\omega}\, \Delta(x_{i\downarrow} {}^\frown x_i)$ and $lim_{i<\omega}\, \Delta(y_{i\downarrow} {}^\frown y_i)$ exist. Finally, since $\Delta(x_{i\downarrow} {}^\frown x_i) = \Delta(y_{i\downarrow} {}^\frown y_i)$ for every $i < \omega$, we get that $lim_{i<\omega}\, \Delta(x_{i\downarrow} {}^\frown x_i) = lim_{i<\omega}\, \Delta(y_{i\downarrow} {}^\frown y_i)$, so we are done. ⊣

LEMMA 2.33. *Let $x, y \ll a$ and $k \in \omega$. Suppose that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k. Then $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a.*

PROOF. We prove the assertion in the lemma by induction on k.

Suppose first that $k = 0$. By definition of $x_{k\downarrow}$ we have that $x_{0\downarrow} = \varnothing$ and $y_{0\downarrow} = \varnothing$. Thus, by definition of mixing we have that $x_{0\downarrow}$ and $y_{0\downarrow}$ are mixed by a.

Now assume that the assertion is true for some k. We show that it is also true for $k + 1$. For that purpose suppose that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to $k + 1$. Hence $\Gamma_\gamma(x)$ also corresponds with $\Gamma_\gamma(y)$ up to k. By inductional assumption we have that $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a. Additionally, assume without loss of generality that $x_{k+1\downarrow} \neq x_{k\downarrow}$ or $y_{k+1\downarrow} \neq y_{k\downarrow}$. We distinguish ten cases.

For the first case suppose that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are strongly mixed by a. We have that $sm(m) = \varnothing$ for every $m \ll a$. Since $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to $k + 1$, we have that either $x_{k+1\downarrow} = x_{k\downarrow}$ or $x_{k+1\downarrow} = x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\Box$ and that either $y_{k+1\downarrow} = y_{k\downarrow}$ or $y_{k+1\downarrow} = y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\Box$. By (a) of Lemma 2.8 we get that $x_{k\downarrow}$ and $x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\Box$ as well as $y_{k\downarrow}$ and $y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\Box$ are mixed by a. Moreover since $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a, by transitivity of mixing we obtain that $x_{k+1\downarrow}$ and $y_{k+1\downarrow}$ are mixed by a.

Next, assume that $x_{k\downarrow}$ is strongly mixed by a. Moreover suppose that $y_{k\downarrow}$ is either min-separated, min-max-separated or strongly separated by a. We have that $sm(m) = \varnothing$ as well as $min\text{-}sep(m) = \{min(m)\}$, $min\text{-}max(m) = \{min(m), max(m)\}$ and $sss(m) = vss(m) = m$ for every $m \ll a$. Since $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to $k + 1$, we must have that $x_{k+1\downarrow} = x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\Box$ and $y_{k+1\downarrow} = y_{k\downarrow}$. By (a) of Lemma 2.8 we get that $x_{k\downarrow}$ and $x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\Box$ are mixed by a. Therefore, since $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a, by transitivity of mixing we obtain that $x_{k+1\downarrow}$ and $y_{k+1\downarrow}$ are mixed by a.

Now assume that $x_{k\downarrow}$ is strongly mixed by a and $y_{k\downarrow}$ is max-separated by a. We have that $sm(m) = \varnothing$ and $max\text{-}sep(m) = \{max(m)\}$ for every $m \ll a$. Since $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to $k + 1$, we must have that either $x_{k+1\downarrow} = x_{k\downarrow}$ or $x_{k+1\downarrow} = x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\Box$ and that either $y_{k+1\downarrow} = y_{k\downarrow}$ or $y_{k+1\downarrow} = y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\uparrow$. By (a) of Lemma 2.8 we get that $x_{k\downarrow}$ and $x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\Box$ are mixed by a. Moreover by (c) of Lemma 2.8 we get that $y_{k\downarrow}$ and $y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\uparrow$ are mixed by a. Since $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a, by transitivity of mixing we obtain that $x_{k+1\downarrow}$ and $y_{k+1\downarrow}$ are mixed by a.

For the fourth case suppose that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are min-separated by a. We have that $min\text{-}sep(m) = \{min(m)\}$ for every $m \ll a$. Since $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to $k + 1$, we must have that $x_{k+1\downarrow} = x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\Box$ and $y_{k+1\downarrow} = y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\Box$. By Lemma 2.15 we get that $x_{k\downarrow}\,\hat{}$

$\langle a(k)\rangle$ and $y_{k\downarrow}{}^\frown \langle a(k)\rangle$ are mixed by a, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a. Moreover by (b) of Lemma 2.8 we get that $x_{k\downarrow}{}^\frown \langle a(k)\rangle$ and $x_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$ as well as $y_{k\downarrow}{}^\frown \langle a(k)\rangle$ and $y_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$ are mixed by a. Therefore, possibly by transitivity of mixing, we obtain that $x_{k+1\downarrow}$ and $y_{k+1\downarrow}$ are mixed by a.

Next, assume that $x_{k\downarrow}$ is min-separated by a and $y_{k\downarrow}$ is min-max-separated by a. We have that $min\text{-}sep(m) = \{min(m)\}$ and $min\text{-}max(m) = \{min(m), max(m)\}$ for every $m \ll a$. Since $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to $k + 1$, we must have that $x_{k+1\downarrow} = x_{k\downarrow}{}^\frown \langle a(k)\rangle\square$ and $y_{k+1\downarrow} = y_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$. By Lemma 2.27 we get that $x_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$ and $y_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$ are mixed by a, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a. Moreover by (b) of Lemma 2.8 we get that $x_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$ and $x_{k\downarrow}{}^\frown \langle a(k)\rangle$ are mixed by a. Therefore, possibly by transitivity of mixing, we obtain that $x_{k+1\downarrow}$ and $y_{k+1\downarrow}$ are mixed by a.

We observe that if $x_{k\downarrow}$ is min-separated by a, then by Lemma 2.29 $y_{k\downarrow}$ is neither max-separated nor strongly separated by a.

For the sixth case suppose that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are max-separated by a. We have that $max\text{-}sep(m) = \{max(m)\}$ for every $m \ll a$. Since $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to $k + 1$, we must have that either $((x_{k+1\downarrow} = x_{k\downarrow} \vee x_{k+1\downarrow} = x_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow) \wedge (y_{k+1\downarrow} = y_{k\downarrow} \vee y_{k+1\downarrow} = y_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow))$ or that $(x_{k+1\downarrow} = x_{k\downarrow}{}^\frown \langle a(k)\rangle \wedge y_{k+1\downarrow} = y_{k\downarrow}{}^\frown \langle a(k)\rangle)$. By (c) of Lemma 2.8 we get that $x_{k\downarrow}$ and $x_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$ as well as $y_{k\downarrow}$ and $y_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$ are mixed by a. Moreover by Lemma 2.17 we get that $x_{k\downarrow}{}^\frown \langle a(k)\rangle$ and $y_{k\downarrow}{}^\frown \langle a(k)\rangle$ are mixed by a, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a. Therefore, possibly by transitivity of mixing, we obtain that $x_{k+1\downarrow}$ and $y_{k+1\downarrow}$ are mixed by a.

We observe that if $x_{k\downarrow}$ is max-separated by a, then by Lemma 2.30 $y_{k\downarrow}$ is neither min-max-separated nor strongly separated by a.

For the seventh case assume that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are min-max-separated by a. We have that $min\text{-}max(m) = \{min(m), max(m)\}$ for every $m \ll a$. Since $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to $k + 1$, we must have that either $x_{k+1\downarrow} = x_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$ and $y_{k+1\downarrow} = y_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$ or that $x_{k+1\downarrow} = x_{k\downarrow}{}^\frown \langle a(k)\rangle$ and $y_{k+1\downarrow} = y_{k\downarrow}{}^\frown \langle a(k)\rangle$. By Lemma 2.19 we get that $x_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$ and $y_{k\downarrow}{}^\frown \langle a(k)\rangle\uparrow$ as well as $x_{k\downarrow}{}^\frown \langle a(k)\rangle$ and $y_{k\downarrow}{}^\frown \langle a(k)\rangle$ are mixed by a, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a. Therefore, $x_{k+1\downarrow}$ and $y_{k+1\downarrow}$ are mixed by a.

We observe that if $x_{k\downarrow}$ is min-max-separated by a, then by Lemma 2.31 $y_{k\downarrow}$ is not strongly separated by a.

For the eighth case suppose that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are still strongly separated by a. We have that $sss(m) = m$ for every $m \ll a$. Since $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to $k + 1$, we must have that $x_{k+1\downarrow} = x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\square$ and $y_{k+1\downarrow} = y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\square$. By (a) of Lemma 2.28 we get that $x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\square$ and $y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\square$ are mixed by a, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a. Therefore, $x_{k+1\downarrow}$ and $y_{k+1\downarrow}$ are mixed by a.

Next, assume that $x_{k\downarrow}$ is still strongly separated by a and $y_{k\downarrow}$ is very strongly separated by a. We have that $sss(m) = vss(m) = m$ for every $m \ll a$. Since $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to $k + 1$, we must have that $x_{k+1\downarrow} = x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\square$ and $y_{k+1\downarrow} = y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\uparrow$. By (b) of Lemma 2.28 we get that $x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\square$ and $y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\uparrow$ are mixed by a, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a. Therefore, $x_{k+1\downarrow}$ and $y_{k+1\downarrow}$ are mixed by a.

Finally, suppose that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are very strongly separated by a. We have that $vss(m) = m$ for every $m \ll a$. Since $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to $k + 1$, we must have that either $x_{k+1\downarrow} = x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\uparrow$ and $y_{k+1\downarrow} = y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\uparrow$ or that $x_{k+1\downarrow} = x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle$ and $y_{k+1\downarrow} = y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle$. By (c) of Lemma 2.28 we get that $x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\uparrow$ and $y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle\uparrow$ are mixed by a, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a. Moreover by Lemma 2.23 we get that $x_{k\downarrow}\,\hat{}\,\langle a(k)\rangle$ and $y_{k\downarrow}\,\hat{}\,\langle a(k)\rangle$ are mixed by a. Therefore, $x_{k+1\downarrow}$ and $y_{k+1\downarrow}$ are mixed by a.

Altogether, by symmetry we can conclude that in every case $x_{k+1\downarrow}$ and $y_{k+1\downarrow}$ are mixed by a. This completes the proof. ⊣

LEMMA 2.34. *Let $x, y \ll a$. Suppose that $\Gamma_\gamma(x) = \Gamma_\gamma(y)$. Then $\Delta(x) = \Delta(y)$.*

PROOF. First, we observe that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to i for every $i < \omega$, because $\Gamma_\gamma(x) = \Gamma_\gamma(y)$. Hence by Lemma 2.33 we get that $x_{i\downarrow}$ and $y_{i\downarrow}$ are mixed by a for all $i < \omega$. Thus, Lemma 2.32 yields that $\Delta(x) = \Delta(y)$. ⊣

LEMMA 2.35. *Let $x, y \ll a$. Suppose that $\Gamma_\gamma(x) \neq \Gamma_\gamma(y)$. Then $\Delta(x) \neq \Delta(y)$.*

PROOF. Since $\Gamma_\gamma(x) \neq \Gamma_\gamma(y)$, we can choose k maximal such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k. By Lemma 2.33 we get that $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a. We show that $\Delta(x) \neq \Delta(y)$. For that purpose we distinguish nine cases.

For the first case assume that $x_{k\downarrow}$ is strongly mixed by a. Moreover suppose that $y_{k\downarrow}$ is either min-separated, min-max-separated or strongly separated by a. We have that $sm(m) = \emptyset$ as well as $min\text{-}sep(m) = \{min(m)\}$, $min\text{-}max(m) = \{min(m), max(m)\}$ and $sss(m) = vss(m) = m$ for every $m \ll a$. Since k is chosen maximal such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k, we must have that either $x_{k+1\downarrow} = x_{k\downarrow}$ or $x_{k+1\downarrow} = x_{k\downarrow} {}^\frown \langle a(k)\rangle\square$ and that $y_{k+1\downarrow} = y_{k\downarrow} {}^\frown \langle a(k)\rangle\square$. This implies that $max(x_{k\uparrow}(0)) > min(y_{k\uparrow}(0))$. Thus, by Lemma 2.24 we obtain that $\Delta(x_{k\downarrow} {}^\frown x_{k\uparrow}) \neq \Delta(y_{k\downarrow} {}^\frown y_{k\uparrow})$, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a.

Next, assume that $x_{k\downarrow}$ is strongly mixed by a and $y_{k\downarrow}$ is max-separated by a. We have that $sm(m) = \emptyset$ and $max\text{-}sep(m) = \{max(m)\}$ for every $m \ll a$. Since k is chosen maximal such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k, we must have that either $x_{k+1\downarrow} = x_{k\downarrow}$ or $x_{k+1\downarrow} = x_{k\downarrow} {}^\frown \langle a(k)\rangle\square$ and that $y_{k+1\downarrow} = y_{k\downarrow} {}^\frown \langle a(k)\rangle$. This implies that $max(x_{k\uparrow}(0)) \geqslant max(y_{k\uparrow}(0))$. Thus, by Lemma 2.25 we obtain that $\Delta(x_{k\downarrow} {}^\frown x_{k\uparrow}) \neq \Delta(y_{k\downarrow} {}^\frown y_{k\uparrow})$, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a.

We observe that we cannot have that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are strongly mixed by a. This would contradict the choice of k, because $sm(m) = \emptyset$ for all $m \ll a$.

For the third case suppose that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are min-separated by a. We have that $min\text{-}sep(m) = \{min(m)\}$ for every $m \ll a$. Moreover we have chosen k maximal such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k. Therefore, by symmetry we must have without loss of generality that $x_{k+1\downarrow} = x_{k\downarrow}$ and $y_{k+1\downarrow} = y_{k\downarrow} {}^\frown \langle a(k)\rangle\square$. This implies that $min(x_{k\uparrow}(0)) > min(y_{k\uparrow}(0))$. Thus, by Lemma 2.14 we obtain that $\Delta(x_{k\downarrow} {}^\frown x_{k\uparrow}) \neq \Delta(y_{k\downarrow} {}^\frown y_{k\uparrow})$, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a.

Next, assume that $x_{k\downarrow}$ is min-separated by a and $y_{k\downarrow}$ is min-max-separated by a. We have that $min\text{-}sep(m) = \{min(m)\}$ and $min\text{-}max(m) = \{min(m), max(m)\}$ for every $m \ll a$. Since k is chosen maximal such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k, we must have that either $x_{k+1\downarrow} = x_{k\downarrow} {}^\frown \langle a(k)\rangle\square$ and $y_{k+1\downarrow} = y_{k\downarrow}$, $x_{k+1\downarrow} = x_{k\downarrow}$ and $y_{k+1\downarrow} = y_{k\downarrow} {}^\frown \langle a(k)\rangle\square$ or that $x_{k+1\downarrow} = x_{k\downarrow} {}^\frown \langle a(k)\rangle\square$ and $y_{k+1\downarrow} = y_{k\downarrow} {}^\frown \langle a(k)\rangle$. This implies that $min(x_{k\uparrow}(0)) \neq min(y_{k\uparrow}(0))$ or $max(x_{k\uparrow}(0)) \geqslant max(y_{k\uparrow}(0))$. Thus, by Lemma 2.26 we obtain that $\Delta(x_{k\downarrow} {}^\frown x_{k\uparrow}) \neq \Delta(y_{k\downarrow} {}^\frown y_{k\uparrow})$, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a.

We observe that if $x_{k\downarrow}$ is min-separated by a, then by Lemma 2.29 $y_{k\downarrow}$ is neither max-separated nor strongly separated by a.

For the fifth case suppose that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are max-separated by a. We have that $max\text{-}sep(m) = \{max(m)\}$ for every $m \ll a$. Moreover we have chosen k maximal such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k. Therefore, by symmetry we must have without loss of generality that either $x_{k+1\downarrow} = x_{k\downarrow}$ or $x_{k+1\downarrow} = x_{k\downarrow} {}^\frown \langle a(k)\rangle\uparrow$ and that $y_{k+1\downarrow} = y_{k\downarrow} {}^\frown \langle a(k)\rangle$. This implies that $max(x_{k\uparrow}(0)) > max(y_{k\uparrow}(0))$. Thus, by Lemma 2.16 we obtain that $\Delta(x_{k\downarrow} {}^\frown x_{k\uparrow}) \neq \Delta(y_{k\downarrow} {}^\frown y_{k\uparrow})$, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a.

We observe that if $x_{k\downarrow}$ is max-separated by a, then by Lemma 2.30 $y_{k\downarrow}$ is neither min-max-separated nor strongly separated by a.

For the sixth case assume that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are min-max-separated by a. We have that $min\text{-}max(m) = \{min(m), max(m)\}$ for every $m \ll a$. Moreover we have chosen k maximal such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k. Therefore, by symmetry we must have without loss of generality that either $x_{k+1\downarrow} = x_{k\downarrow}$ and $y_{k+1\downarrow} = y_{k\downarrow} {}^\frown \langle a(k)\rangle\square$ or that $x_{k+1\downarrow} = x_{k\downarrow} {}^\frown \langle a(k)\rangle\uparrow$ and $y_{k+1\downarrow} = y_{k\downarrow} {}^\frown \langle a(k)\rangle$. This implies that $min(x_{k\uparrow}(0)) > min(y_{k\uparrow}(0))$ or $max(x_{k\uparrow}(0)) > max(y_{k\uparrow}(0))$. Thus, by Lemma 2.18 we obtain that $\Delta(x_{k\downarrow} {}^\frown x_{k\uparrow}) \neq \Delta(y_{k\downarrow} {}^\frown y_{k\uparrow})$, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a.

We observe that if $x_{k\downarrow}$ is min-max-separated by a, then by Lemma 2.31 $y_{k\downarrow}$ is not strongly separated by a.

For the seventh case suppose that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are still strongly separated by a. We have that $sss(m) = m$ for every $m \ll a$. Moreover we have chosen k maximal such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k. Therefore, by symmetry we must have without loss of generality that $x_{k+1\downarrow} = x_{k\downarrow}$ and $y_{k+1\downarrow} = y_{k\downarrow} {}^\frown \langle a(k)\rangle\square$. This implies that neither $x_{k\uparrow}(0)$ is an initial segment of $y_{k\uparrow}(0)$ nor conversely. Thus, by Lemma 2.20 we obtain that $\Delta(x_{k\downarrow} {}^\frown x_{k\uparrow}) \neq \Delta(y_{k\downarrow} {}^\frown y_{k\uparrow})$, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a.

Next, assume that $x_{k\downarrow}$ is still strongly separated by a and $y_{k\downarrow}$ is very strongly separated by a. We have that $sss(m) = vss(m) = m$ for every $m \ll a$. Since k is chosen maximal such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k, we must have that either $x_{k+1\downarrow} = x_{k\downarrow} {}^\frown \langle a(k)\rangle\square$ and $y_{k+1\downarrow} = y_{k\downarrow}$, $x_{k+1\downarrow} = x_{k\downarrow}$ and $y_{k+1\downarrow} = y_{k\downarrow} {}^\frown \langle a(k)\rangle\square$ or that $x_{k+1\downarrow} = x_{k\downarrow} {}^\frown \langle a(k)\rangle\square$ and $y_{k+1\downarrow} = y_{k\downarrow} {}^\frown \langle a(k)\rangle$. The former two cases imply that neither $x_{k\uparrow}(0)$ is an initial segment of $y_{k\uparrow}(0)$ nor conversely. Thus, by Lemma 2.20 we obtain that $\Delta(x_{k\downarrow} {}^\frown x_{k\uparrow}) \neq \Delta(y_{k\downarrow} {}^\frown y_{k\uparrow})$, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by

a. In the latter case by (b) of Lemma 2.28 we get that $x_{k\downarrow} \hat{} \langle a(k)\rangle\downarrow$ and $y_{k\downarrow} \hat{} \langle a(k)\rangle$ are separated by a. Therefore, by definition of separation we obtain that $\Delta(x_{k+1\downarrow} \hat{} x_{k+1\uparrow}) \neq \Delta(y_{k+1\downarrow} \hat{} y_{k+1\uparrow})$.

Finally, suppose that both $x_{k\downarrow}$ and $y_{k\downarrow}$ are very strongly separated by a. We have that $vss(m) = m$ for every $m \ll a$. Moreover we have chosen k maximal such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k. Therefore, by symmetry we must have without loss of generality that either $x_{k+1\downarrow} = x_{k\downarrow} \hat{} \langle a(k)\rangle\downarrow$ and $y_{k+1\downarrow} = y_{k\downarrow}$ or that $x_{k+1\downarrow} = x_{k\downarrow} \hat{} \langle a(k)\rangle\uparrow$ and $y_{k+1\downarrow} = y_{k\downarrow} \hat{} \langle a(k)\rangle$. This implies that $x_{k\uparrow}(0) \neq y_{k\uparrow}(0)$. Thus, by Lemma 2.22 we obtain that $\Delta(x_{k\downarrow} \hat{} x_{k\uparrow}) \neq \Delta(y_{k\downarrow} \hat{} y_{k\uparrow})$, because $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a.

Altogether, by symmetry we can conclude that in every case $\Delta(x) \neq \Delta(y)$. This completes the proof. ⊣

REMARK. Both the following definition and Lemma 2.36 are necessary to guarantee that property (b) of Theorem 0.7 follows from our Main Theorem.

DEFINITION. Let $x, y \ll a$. We say that *$\Gamma_\gamma(x)$ is a proper initial segment of $\Gamma_\gamma(y)$* iff there exists $k \in \omega$ such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k, $x_{j\downarrow}$ is strongly mixed by a for every $j \geqslant k$ and there exists $l \geqslant k$ such that $y_{l\downarrow}$ is separated in some sense by a.

LEMMA 2.36. *There are no $x, y \ll a$ such that $\Gamma_\gamma(x)$ is a proper initial segment of $\Gamma_\gamma(y)$.*

PROOF. Assume to the contrary that there exist $x, y \ll a$ such that $\Gamma_\gamma(x)$ is a proper initial segment of $\Gamma_\gamma(y)$. According to the definition above there exists $k \in \omega$ such that $\Gamma_\gamma(x)$ corresponds with $\Gamma_\gamma(y)$ up to k, $x_{j\downarrow}$ is strongly mixed by a for every $j \geqslant k$ and there exists $l \geqslant k$ such that $y_{l\downarrow}$ is separated in some sense by a. By Lemma 2.33 we get that $x_{k\downarrow}$ and $y_{k\downarrow}$ are mixed by a. Hence by definition of being mixed there exist $x_0, y_0 \ll a$ such that $\Delta(x_{k\downarrow} \hat{} x_0) = \Delta(y_{k\downarrow} \hat{} y_0)$. Since $x_{j\downarrow}$ is strongly mixed by a for all $j \geqslant k$, by (e) of canonical we have that $(x_{k\downarrow} \hat{} x_0)_{j\downarrow}$ is also strongly mixed by a for every $j \geqslant k$. Equally, since there exists $l \geqslant k$ such that $y_{l\downarrow}$ is separated in some sense by a, by (e) of canonical there exists $i \geqslant k$ such that $(y_{k\downarrow} \hat{} y_0)_{i\downarrow}$ is also separated in some sense by *a*. Hence by definition of Γ_γ we have $\Gamma_\gamma(x_{k\downarrow} \hat{} x_0) \neq \Gamma_\gamma(y_{k\downarrow} \hat{} y_0)$. Since $\Delta(x_{k\downarrow} \hat{} x_0) = \Delta(y_{k\downarrow} \hat{} y_0)$, we get a contradiction to Lemma 2.35.

This completes the proof of the lemma and with it the proof of the Main Theorem. ⊣

A. COROLLARIES

Finally we show that both the theorem of Taylor (Theorem 0.4) and the theorem of Prömel-Voigt (Theorem 0.7) follow from our Main Theorem.

PROOF. *(ad Theorem 0.4)* Consider for a given $f\colon [\omega]^{<\omega} \to \omega$ the mapping $\Delta^*\colon \Omega^\omega \to \omega$ with $x \mapsto f(x(0))$. First we show that Δ^* is a Borel measurable mapping, which is an assumption of our Main Theorem. Note that this result refers to the metric topology on Ω^ω. Therewith the sets $\Pi_{i\in\omega} U_i \subseteq \Omega^\omega$ form a basis, where $U_i = [\omega]^\omega$ for all but finitely many $i \in \omega$. Since the inverse image $(\Delta^*)^{-1}$ of an arbitrary subset of ω is open, Δ^* is even continuous.

For all $k > 0$ we have that $x \upharpoonright k$ is strongly mixed by every $a \ll \omega_{max}$. This holds by definition of strongly mixing. Moreover $\emptyset$ cannot be still strongly separated by some a. Otherwise we would have that $\langle a(0)\rangle$ and $\langle a(0)\rangle\uparrow$ are mixed by a. By Corollary 2.21 we would get that $\langle a(0)\rangle\uparrow$ is still strongly separated by a, too. Because of our observation above we additionally have that $\langle a(0)\rangle$ is strongly mixed by a. Hence we would be in the case of Lemma 2.24. Since $\langle a(0)\rangle^\frown x \upharpoonright k$ is still strongly mixed by a for every k and $x \ll a$, we would get a contradiction.

Invoking our Main Theorem there exist $\gamma^* \in \{sm, min\text{-}sep, max\text{-}sep, min\text{-}max, vss\}$ and $a \in \Omega^\omega$ such that for all $x, y \ll a$

$$f(x(0)) = f(y(0)) \text{ iff } \gamma^*(x(0)) = \gamma^*(y(0)).$$

This is Theorem 0.4. ⊣

PROOF. *(ad Theorem 0.7)* Assuming a Borel function $\Delta\colon [\omega]^\omega \to \mathbb{R}$ we construct the mapping $\Delta^*\colon \Omega^\omega \to \mathbb{R}$ with $x \mapsto \Delta(\{min(x(i))\colon i \in \omega\})$. In order to apply our Main Theorem we have to prove that Δ^* is Borel. Let $g\colon \Omega^\omega \to [\omega]^\omega$ with $g(x) = \{min(x(i))\colon i \in \omega\}$. Since $\Delta^* = \Delta \circ g$ and Δ is Borel, it is enough to show that g is Borel. $\wp(\omega)$ can be identified with the Cantor space 2^ω as a topological space endowed with the product topology. Since $[\omega]^\omega \subseteq \wp(\omega)$, for every $I, J \in [\omega]^{<\omega}$ with $I \cap J = \emptyset$ the sets $U_{I,J} = \{X \in [\omega]^\omega\colon \forall i \in I\ \forall j \in J\ i \in X \wedge j \notin X\}$ form a basis for the topology on $[\omega]^\omega$. It is obvious that the (sub)basis is countable, so it suffices to show that $g^{-1}(U_{I,J})$ is Borel for each I, J.

We have that $g^{-1}(U_{I,\,J}) = \{x \in \Omega^{\omega}: \{min(x(i)): i \in \omega\} \in U_{I,\,J}\}$. The sets $\{\Pi_{i\in\omega} V_i: \forall i \in \omega\ V_i \subseteq [\omega]^{<\omega} \wedge V_i = [\omega]^{<\omega}$ *for all but finitely many* $i\} \cap \Omega^{\omega}$ form a basis for Ω^{ω}. Since only a finite number of pieces $x(i)$ consider the sets I, J, $g^{-1}(U_{I,\,J})$ is a union of open sets of Ω^{ω}. Hence $g^{-1}(U_{I,\,J})$ is open, too. Therewith g is continous. Since continuous mappings are Borel, g is also Borel.

For given a let $x \ll a$. Assume that for some k the set $x \restriction k$ is max-separated, min-max-separated or strongly separated by a. We have that $x \restriction k\ \hat{}\ \langle x(k)\rangle$ and $x \restriction k\ \hat{}\ \langle x(k) \cup x(k+1)\rangle$ are mixed by a, since $\langle x(k)\rangle$ and $\langle x(k) \cup x(k+1)\rangle$ have the same minimum and hence $\Delta^*(x \restriction k\ \hat{}\ \langle x(k)\rangle\ \hat{}\ y) = \Delta^*(x \restriction k\ \hat{}\ \langle x(k) \cup x(k+1)\rangle\ \hat{}\ y)$ for all $y \ll a$. But this contradicts the cases (b) - (d) of Lemma 2.11. Thus, for all k we neither have that $x \restriction k$ is max-separated, min-max-separated, still strongly separated nor very strongly separated by a.

Therewith our Main Theorem yields $\gamma^*: \Omega^{<\omega} \to \{sm, min\text{-}sep\}$ and $a \in \Omega^{\omega}$ such that for all x, $y \ll a$ it holds that $\Delta^*(x) = \Delta^*(y)$ iff $\Gamma_{\gamma^*}(x) = \Gamma_{\gamma^*}(y)$. Let $A = \{min(a(i)): i \in \omega\}$ and define for every $x \ll a$ the mapping $\Gamma: [A]^{\omega} \to [A]^{\leq\omega}$ by $\Gamma(\{min(x(i)): i \in \omega\}) := \Gamma_{\gamma^*}(x)$.

By definition of Γ_{γ^*} we get that $\Gamma(X) \subseteq X$ for all $X \in [A]^{\omega}$.

Additionally, Theorem 0.7 requires that no $\Gamma(X)$ is a proper initial segment of some $\Gamma(Y)$. This property directly follows from Lemma 2.36.

Finally, since both Δ^* and Γ_{γ^*} only depend on the minima of all pieces, for all X, $Y \in [A]^{\omega}$ it follows that $\Delta(X) = \Delta(Y)$ iff $\Gamma(X) = \Gamma(Y)$. ⊣

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