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Research Article

On the Homotopy Analysis Method for Fractional SEIR Epidemic Model

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Abstract: This study investigates the accuracy of solution for fractional-order an SEIR epidemic model by using the homotopy analysis method. The homotopy analysis method provides us with a simple way to adjust and control the convergence region of the series solution by introducing an auxiliary parameter. Mathematical modeling of the problem leads to a system of nonlinear fractional differential equations. Indeed, we find the analytical solution of the proposed model by Homotopy analysis method which is one of the best methods for finding the solution of the nonlinear problem. Numerical simulations are given to illustrate the validity of the proposed results.

Keywords: Epidemic models, fractional differential equations, homotopy analysis method, series solution

INTRODUCTION

Mathematical modeling in epidemiology provides understanding of the mechanisms that influence the spread of a disease and it suggests control strategies. One of the early models in epidemiology was introduced in Kermack and McKendrick (1991) to predict the spreading behavior of a disease. In this model, the total population is assumed to be constant and divided into three classes namely suspended, infectious and recovered. Over the years, more complex models have been derived (Hethcote, 2000). For some diseases, it is found that for a period of time, a part of the infectious class does not show the symptoms. For modeling such diseases SEIR models are used (El-Sheikh and El-Marouf, 2004).

In the SEIR model, the population is divided into four compartments: a susceptible compartment labeled S, in which all individuals are susceptible to the disease; an exposed compartment labeled E, in which all individuals are infected but not yet infectious; an infected compartment labeled I, in which all individuals are infected by the disease and have infectivity; and a removed compartment labeled R, in which all individuals are removed from the infected compartment. Let $S(t)$, $E(t)$, $I(t)$ and $R(t)$ denote the number of individuals in the compartments S, E, I and R at time t, respectively. The model consists of the following nonlinear differential system (El-Sheikh and El-Marouf, 2004):

$$
\begin{aligned}
S'(t) &= -\beta \frac{I(t)S(t)}{N(t)} - \mu S(t) + rN(t) + \delta R(t) \\
E'(t) &= \beta \frac{I(t)S(t)}{N(t)} - (\mu + \sigma + \kappa)E(t) \\
I'(t) &= \sigma E(t) - (\mu + \alpha + \gamma)I(t) \\
R'(t) &= \kappa E(t) + \gamma I(t) - \mu R(t) - \delta R(t) \\
0 &= S(t) + E(t) + I(t) + R(t) - N(t)
\end{aligned} \tag{1}
$$

subject to the initial condition:

$$S(0) = N_S, E(0) = N_E, I(0) = N_I, R(0) = N_R \tag{2}$$

where, $\beta, \mu, r, \sigma, \kappa, \gamma, \alpha, \delta$ and N_S, N_E, N_I, N_R are positive real numbers and $N(t)$ represents the total population at time t.

In the SEIR model (1), the biological meaning of the parameters are summarize as follows. β is the transmission coefficient of the disease; μ is the natural mortality rate; r is the birth rate; σ^{-1} is the incubation period; κ and γ are the recovery rate for both exposed and infected populations; α is the disease induced morality rate; δ^{-1} is the loss of immunity period and t is controls how long the model will run. Moreover, the force of infection is $\beta \frac{I(t)}{N(t)}$ and the incidence rate is $\beta \frac{I(t)S(t)}{N(t)}$.

The differential equations of system (1) describe the dynamical behaviors of every dynamic element for whole epidemic system (1) and the last algebraic equation describes the restriction of every dynamic element of system (1). That is, the differential and algebraic system (1) can describe the whole behavior of certain epidemic spreads in a certain area.

In SEIR model (1) and (2) considered here, we assume that immunity is permanent and that recovered

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individuals do not revert to the susceptible class. It is uniform birthrate. Also, we ignore any subdivisions of the population by age, sex, mobility, or other factors, although such distinctions are obviously of importance. A detailed history of mathematical epidemiology may be found in the classical books (Bailey, 1975; Murray, 1993; Anderson and May, 1998).

In terms of the dimensionless proportions of susceptible, exposed, infectious and recovered individuals it is assumed that $s(t) = \frac{S(t)}{N(t)}$, $e(t) = \frac{E(t)}{N(t)}$, $i(t) = \frac{I(t)}{N(t)}$ and $r(t) = \frac{R(t)}{N(t)}$. After some manipulations and replacing s by S, e by E, i by I and r by R, the SEIR model (1) and (2) can be written as:

$$\begin{aligned}
S'(t) &= (-\beta + \alpha)I(t)S(t) - rS(t) + \delta R(t) + r \\
E'(t) &= \beta S(t)I(t) - (\delta + \kappa + r)E(t) + \alpha I(t)E(t) \\
I'(t) &= \sigma E(t) - (\alpha + \gamma + r)I(t) + \alpha\big(I(t)\big)^2 \\
R'(t) &= \kappa E(t) + \gamma I(t) - rR(t) + \alpha I(t)R(t) - \delta R(t) \\
0 &= S(t) + E(t) + I(t) + R(t) - 1
\end{aligned} \tag{3}$$

subject to the initial condition:

$$S(0) = \frac{N_S}{N(0)}, E(0) = \frac{N_E}{N(0)}, I(0) = \frac{N_I}{N(0)}, R(0) = \frac{N_R}{N(0)} \tag{4}$$

Note that the total population size $N(t)$ does not appear in system (3), this is a direct result of the homogeneity of the system (1).

The differential equations with fractional order have recently proved to be valuable tools to the modeling of many real problems in different areas (Oldham and Spanier, 1974; Miller and Ross, 1993; Mainardi, 1997; Luchko and Gorenflo, 1998; Podlubny, 1999). This is because of the fact that the realistic modeling of a physical phenomenon does not depend only on the instant time, but also on the history of the previous time which can also be successfully achieved by using fractional calculus. For example, half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models. Furthermore, using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena (Luchko and Gorenflo, 1998; Podlubny, 1999). Lately, a large amount of studies developed concerning the application of fractional differential equations in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering (Miller and Ross, 1993; Mainardi, 1997; Podlubny, 1999). A review of some applications of fractional derivatives in continuum and statistical mechanics is given by Mainardi (1997).

In this study, we study the mathematical behavior of the solution of a fractional SEIR model as the order assumed that all newborns are susceptible and a of the fractional derivative changes by extend the classical SEIR model (3) to the following fractional SEIR model:

$$\begin{aligned}
&D_*^{\mu_1}S(t) = (-\beta + \alpha)S(t)I(t) - rS(t) + \delta R(t) + r \\
&D_*^{\mu_2}E(t) = \beta S(t)I(t) - (\delta + \kappa + r)E(t) + \alpha I(t)E(t) \\
&D_*^{\mu_3}I(t) = \sigma E(t) - (\alpha + \gamma + r)I(t) + \alpha\big(I(t)\big)^2 \\
&D_*^{\mu_4}R(t) = \kappa E(t) + \gamma I(t) - rR(t) \\
&+\alpha I(t)R(t) - \delta R(t) \\
&0 = S(t) + E(t) + I(t) + R(t) - 1
\end{aligned} \tag{5}$$

where, $D_*^{\mu_1}S(t)$, $D_*^{\mu_2}I(t)$, $D_*^{\mu_3}E(t)$ and $D_*^{\mu_3}R(t)$ are the derivative of $S(t)$, $E(t)$, $I(t)$ and $R(t)$, respectively, of order μ_i in the sense of Caputo and $0 < \mu_i \le 1, i = 1, 2, 3, 4$.

The Homotopy Analysis Method (HAM), which proposed by Liao (1992), is effectively and easily used to solve some classes of nonlinear problems without linearization, perturbation, or discretization. This method has been implemented in many branches of mathematics and engineering, such as unsteady boundary-layer flows over a stretching flat plate (Liao, 2006) and strongly nonlinear differential equation (Liao, 2010). For linear problems, its exact solution can be obtained by few terms of the homotopy analysis series. In the last years, extensive work has been done using HAM, which provides analytical approximations for linear and nonlinear equations. The reader is kindly requested to go through Liao (1992, 1998, 2003, 2004, 2006), El-Ajou *et al*. (2012) and Abu Arqub *et al*. (2013a) in order to know more details about HAM, including its history, its modification for use, its applications on the other problems and its characteristics. On the other hand, the numerical solvability of other version of differential problems can be found in Al-Smadi *et al*. (2013), Abu Arqub *et al*. (2013b) and Freihat and Al-Smadi (2013) and references therein.

This study is organized as follows. Firstly, we present some necessary definitions and preliminary results that will be utilized in our work. Then, the basic idea of the homotopy analysis method and the statement of the method for solving a fractional-order SEIR model by HAM are introduced. Base on the above, numerical results are given to illustrate the capability of the presented method and the convergence of the HAM series solution is analyzed. The conclusion remarks are given in the final part.

PRELIMINARIES AND MATERIALS

The material in this section is basic in some sense. For the reader's convenience, we present some necessary definitions from fractional calculus theory and preliminary results. For the concept of fractional derivative, we will adopt Caputo's definition, which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order, which is the case in most physical processes (Caputo, 1967; Oldham and Spanier, 1974; Miller and Ross, 1993; Mainardi, 1997; Luchko and Gorenflo, 1998; Podlubny, 1999).

Definition 1: A real function $f(x), x > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$ and it is said to be in the space C_μ^n if $f^{(n)}(x) \in C_\mu, n \in \mathbb{N}$.

Definition 2: The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f(x) \in C_\mu, \mu \geq -1$ is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, x > 0$$

$$J^0 f(x) = f(x)$$

where, $\alpha > 0$ and Γ is the well-known Gamma function.

Properties of the operator J^α can be found in Caputo (1967), Oldham and Spanier (1974), Miller and Ross (1993), Mainardi (1997), Luchko and Gorenflo (1998) and Podlubny (1999), we mention only the following: for $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0 \; and \; \gamma \geq -1$, we have:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) = J^\beta J^\alpha f(x)$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by Caputo in his work on the theory of viscoelasticity (Caputo, 1967).

Definition 3: The fractional derivative of $f \in C_{-1}^n$ in the Caputo sense is defined as:

$$D_*^\alpha f(x) = \begin{cases} J^{n-\alpha} D^n f(x), & n-1 < \alpha < n, x > 0 \\ \frac{d^n f(x)}{dx^n}, & \alpha = n \end{cases}$$

where $n \in \mathbb{N}$ and α is the order of the derivative.

Lemma 1: If $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and $f \in C_\mu^n, \mu \geq -1$, then:

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{x}{k!}, x > 0$$

$$D_*^\alpha J^\alpha f(x) = f(x)$$

For mathematical properties of fractional derivatives and integrals, one can consult the mentioned references.

THE HOMOTOPY ANALYSIS METHOD (HAM)

The principles of the HAM and its applicability for various kinds of differential equations are given in Liao (1992, 1998, 2003, 2004, 2006), El-Ajou *et al*. (2012) and Abu Arqub *et al*. (2013a). For convenience of the reader, we will present a review of the HAM and then we will implement the HAM to construct a symbolic approximate solution for the fractional SEIR model (5) and (4). To achieve our goal, we consider the nonlinear differential equation:

$$N_i[y_i(t)] = 0, t \geq 0, i = 1,2,\dots,n \tag{6}$$

where, N_i are a nonlinear differential operator and $y_i(t)$ are unknown function of the independent variable t.

Liao (1992, 1998, 2003, 2004) constructs the so-called zeroth-order deformation equation:

$$(1-q)\Lambda_i\,[\phi_i(t;q) - y_{i0}(t)] = q\hbar_i H_i(t) \; N_i[\phi_i(t;q)], i = 1,2,\dots,n \tag{7}$$

where, $q \in [0,1]$ is an embedding parameter, $\hbar_i \neq 0$ is an auxiliary parameter, $H_i(t) \neq 0$ are an auxiliary function, Λ_i are an auxiliary linear operator, N_i are a nonlinear differential operator, $\phi_i(t;q)$ are an unknown function and $y_{i0}(t)$ are an initial guess of $y_i(t)$, which satisfies the initial conditions. It should be emphasized that one has great freedom to choose the initial guess $y_{i0}(t)$, the auxiliary linear operator Λ_i, the auxiliary parameter $\hbar_i$ and the auxiliary function $H_i(t)$. According to the auxiliary linear operator and the suitable initial conditions, when $q = 0$, we have:

$$\phi_i(t;0) = y_{i0}(t), i = 1,2,\dots,n \tag{8}$$

and when $q = 1$, since $\hbar_i \neq 0$ and $H_i(t) \neq 0$, the zeroth-order deformation Eq. (7) is equivalent to Eq. (6), hence:

$$\phi_i(t;1) = y_{i0}(t), i = 1,2,\dots,n \tag{9}$$

Thus, according to Eq. (8) and (9), as q increasing from 0 to 1, the solution $\phi_i(t;q)$ various continuously from the initial approximation $y_{i0}(t)$ to the exact solution $y_i(t)$.

Define the so-called mth-order deformation derivatives:

$$y_{im}(t) = \frac{1}{m!}\frac{\partial^m \phi_i(t;q)}{\partial q^m}\bigg|_{q=0}, i = 1,2,\dots,n \tag{10}$$

Expanding $\phi_i(t;q)$ in a Taylor series with respect to the embedding parameter q, by using Eq. (8) and (10), we have:

$$\phi_i(t;q) = y_{i0}(t) + \sum_{m=1}^{\infty} y_{im}(t)q^m \tag{11}$$

Assume that the auxiliary parameter $\hbar_i$, the auxiliary function $H_i(t)$, the initial approximation $y_{i0}(t)$ and the auxiliary linear operator Λ_i are properly chosen so that the series (11) of $\phi_i(t;q)$ converges at $q = 1$. Then, we have under these assumptions the solution series $y_i(t) = y_{i0}(t) + \sum_{m=1}^{\infty} y_{im}(t)$.

According to Eq. (10), the governing equation can be deduced from the zeroth-order deformation Eq. (7). Define the vector $\vec{y}_{ij} = \{y_{i0}(t), y_{i1}(t), y_{i2}(t), \ldots, y_{ij}(t)\}, i = 1,2,3, \ldots, n$. Differentiating Eq. (7) m-times with respect to embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have, using Eq. (10), the so-called mth-order deformation equation:

$$\Lambda_i[y_{im}(t) - \chi_m y_{i(m-1)}(t)] = \hbar_i H_i(t) \mathcal{R}_{im}\left(\vec{y}_{i(m-1)}(t)\right), i = 1,2,\ldots,n \tag{12}$$

where,

$$\mathcal{R}_{im}\left(\vec{y}_{i(m-1)}(t)\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N_i[\phi_i(t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{13}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

For any given nonlinear operator N_i, the term $\mathcal{R}_{im}(\vec{y}_{i(m-1)})$ can be easily expressed by Eq. (13). Thus, we can gain $y_{i0}(t), y_{i1}(t), y_{i2}(t), \ldots, y_{ij}(t)$ by means of solving the linear high-order deformation Eq. (12) one after the other in order. The m^{th}-order approximation of $y_i(t)$ is given by $y_i(t) = \sum_{k=0}^{m-1} y_{ik}(t)$.

It should be emphasized that the so-called mth-order deformation Eq. (12) is a linear, which can be easily solved by symbolic computation software's such as Maple and Mathematica.

Solution of the fractional SEIR epidemic model: Now, we employ the algorithm of the HAM to find out series solutions for the fractional SEIR epidemic model. Let $q \in [0,1]$ be the so-called embedding parameter. The HAM is based on a kind of continuous mappings $S(t) \to \varphi_1(t;q)$, $E(t) \to \varphi_2(t;q)$, $I(t) \to \varphi_3(t;q), R(t) \to \varphi_4(t;q)$ such that, as the embedding parameter q increases from 0 to 1, $\varphi_i(t;q), i = 1,2,3$ varies from the initial approximation to the exact solution. To ensure this, choose such auxiliary linear operators Λ_i to be $D_*^{\mu_i}$, where $0 < \mu_i \leq 1, i = 1,2,3,4$.

We define the nonlinear operators:

$$N_1[\varphi_1(t;q)] = D_*^{\mu_1}[\varphi_1(t;q)] + (\beta - \alpha)\varphi_1(t;q)\varphi_3(t;q) + r\varphi_1(t;q) - \delta\varphi_4(t;q) - r$$
$$N_2[\varphi_2(t;q)] = D_*^{\mu_2}[\varphi_2(t;q)] - \beta\varphi_1(t;q)\varphi_3(t;q) + (\delta + \kappa + r)\varphi_2(t;q) - \alpha\varphi_2(t;q)\varphi_3(t;q)$$
$$N_3[\varphi_3(t;q)] = D_*^{\mu_3}[\varphi_3(t;q)] - \sigma\varphi_2(t;q) + (\alpha + \gamma + r)\varphi_3(t;q) - \alpha(\varphi_3(t;q))^2$$
$$N_4[\varphi_4(t;q)] = D_*^{\mu_4}[\varphi_4(t;q)] - \kappa\varphi_2(t;q) - \gamma\varphi_3(t;q) + r\varphi_4(t;q) - \alpha\varphi_3(t;q)\varphi_4(t;q) + \delta\varphi_4(t;q)$$

Let $\hbar_i \neq 0$ and $H_i(t) \neq 0, i = 1,2,3$, denote the so-called auxiliary parameter and auxiliary function, respectively. Using the embedding parameter q, we construct a family of equations:

$$(1 - q)\Lambda_1[\varphi_1(t;q) - S_0(t)] = q\hbar_1 H_1(t) N_1[\varphi_1(t;q)]$$
$$(1 - q)\Lambda_2[\varphi_2(t;q) - E_0(t)] = q\hbar_2 H_2(t) N_2[\varphi_2(t;q)]$$
$$(1 - q)\Lambda_3[\varphi_3(t;q) - I_0(t)] = q\hbar_3 H_3(t) N_3[\varphi_3(t;q)]$$
$$(1 - q)\Lambda_4[\varphi_4(t;q) - R_0(t)] = q\hbar_4 H_4(t) N_4[\varphi_4(t;q)]$$

subject to the initial conditions:

$$\varphi_1(0;q) = S_0(0), \varphi_2(0;q) = E_0(0), \varphi_3(0;q) = I_0(0), \varphi_4(0;q) = R_0(0)$$

By Taylor's theorem, we expand $\varphi_i(t;q), i = 1,2,3,4$ by a power series of the embedding parameter q as follows:

$$\varphi_1(t;q) = S_0(t) + \sum_{m=1}^{\infty} S_m(t) q^m, \varphi_2(t;q) = E_0(t) + \sum_{m=1}^{\infty} E_m(t) q^m$$
$$\varphi_3(t;q) = I_0(t) + \sum_{m=1}^{\infty} I_m(t) q^m, \varphi_4(t;q) = R_0(t) + \sum_{m=1}^{\infty} R_m(t) q^m$$

where,

$$S_m(t)=\frac{1}{m!}\frac{\partial^m\varphi_1(t;q)}{\partial q^m}\bigg|_{q=0}, E_m(t)=\frac{1}{m!}\frac{\partial^m\varphi_2(t;q)}{\partial q^m}\bigg|_{q=0}, I_m(t)=\frac{1}{m!}\frac{\partial^m\varphi_3(t;q)}{\partial q^m}\bigg|_{q=0}$$

and

$$R_m(t)=\frac{1}{m!}\frac{\partial^m\varphi_4(t;q)}{\partial q^m}\bigg|_{q=0}$$

Then at $q=1$, the series becomes:

$$\begin{aligned}S(t)&=S_0(t)+\textstyle\sum_{m=1}^{\infty}S_m(t)\\E(t)&=E_0(t)+\textstyle\sum_{m=1}^{\infty}E_m(t)\\I(t)&=I_0(t)+\textstyle\sum_{m=1}^{\infty}I_m(t)\\R(t)&=R_0(t)+\textstyle\sum_{m=1}^{\infty}R_m(t)\end{aligned} \tag{14}$$

From the so-called m^{th}-order deformation Eq. (12) and (13), we have:

$$\begin{aligned}\Lambda_1\,[S_m(t)-\chi_m S_{m-1}(t)]&=\hbar_1 H_1(t)\mathcal{R}S_m\left(\vec{S}_{m-1}(t)\right), m=1,2,\dots,n\\\Lambda_2\,[E_m(t)-\chi_m E_{m-1}(t)]&=\hbar_1 H_1(t)\mathcal{R}E_m\left(\vec{E}_{m-1}(t)\right), m=1,2,\dots,n\\\Lambda_3\,[I_m(t)-\chi_m I_{m-1}(t)]&=\hbar_2 H_2(t)\mathcal{R}I_m\left(\vec{I}_{m-1}(t)\right), m=1,2,\dots,n\\\Lambda_4\,[R_m(t)-\chi_m R_{m-1}(t)]&=\hbar_3 H_3(t)\mathcal{R}R_m\left(\vec{R}_{m-1}(t)\right), m=1,2,\dots,n\end{aligned} \tag{15}$$

with initial conditions:

$$S_m(0)=0, E_m(0)=0, I_m(0)=0, R_m(0)=0$$

where,

$$\begin{aligned}\mathcal{R}S_m\left(\vec{S}_{m-1}(t)\right)&=D_*^{\mu_1}S_{m-1}(t)+(\beta-\alpha)\textstyle\sum_{i=1}^{m-1}S_i(t)I_{m-1-i}(t)+rS_{m-1}(t)-\delta R_{m-1}-r\\\mathcal{R}E_m\left(\vec{E}_{m-1}(t)\right)&=D_*^{\mu_2}E_{m-1}(t)-\beta\textstyle\sum_{i=1}^{m-1}S_i(t)I_{m-1-i}(t)+(\delta+\kappa+r)E_{m-1}(t)-\alpha\textstyle\sum_{i=1}^{m-1}E_i(t)I_{m-1-i}(t)\\\mathcal{R}I_m\left(\vec{I}_{m-1}(t)\right)&=D_*^{\mu_3}I_{m-1}(t)-\sigma E_{m-1}(t)+(\alpha+\gamma+r)I_{m-1}(t)-\alpha\textstyle\sum_{i=1}^{m-1}I_i(t)I_{m-1-i}(t)\\\mathcal{R}S_m\left(\vec{S}_{m-1}(t)\right)&=D_*^{\mu_4}R_{m-1}(t)-\kappa E_{m-1}(t)-\gamma I_{m-1}(t)+(r+\delta)R_{m-1}-\alpha\textstyle\sum_{i=1}^{m-1}I_i(t)R_{m-1-i}(t)\end{aligned}$$

For simplicity, we can choose the auxiliary functions as $H_i(t)=1,\ i=1,2,3,4$ and take $\Lambda_i=D_*^{\mu_i},\ i=1,2,3,4$, then the right inverse of $D_*^{\mu_i}$ will be J^{μ_i}; the Riemann-Liouville fractional integral operator. Hence, the m^{th}-order deformation Eq. (15) for $m\geq 1$ becomes:

$$\begin{aligned}S_m(t)&=\chi_m S_{m-1}(t)+\hbar_1\,J^{\mu_1}\left[\mathcal{R}S_m\left(\vec{S}_{m-1}(t)\right)\right]\\E_m(t)&=\chi_m E_{m-1}(t)+\hbar_2\,J^{\mu_2}\left[\mathcal{R}E_m\left(\vec{E}_{m-1}(t)\right)\right]\\I_m(t)&=\chi_m I_{m-1}(t)+\hbar_3\,J^{\mu_3}\left[\mathcal{R}I_m\left(\vec{I}_{m-1}(t)\right)\right]\\R_m(t)&=\chi_m R_{m-1}(t)+\hbar_4\,J^{\mu_4}\left[\mathcal{R}R_m\left(\vec{R}_{m-1}(t)\right)\right]\end{aligned}$$

If we choose $S_0(t)=S(0)=N_S$, $E_0(t)=E(0)=N_E$, $I_0(t)=I(0)=N_I$ and $R_0(t)=R(0)=N_R$ as initial guess approximations of $S(t)$, $I(t)$ and $R(t)$, respectively, then two terms approximations for $S(t)$, $E(t)$, $I(t)$ and $R(t)$ are calculated and presented below:

$$S_1=\hbar t\left(-0.0000714-\tfrac{N_R}{365}+0.0000714N_S+1.9455159918287936N_I N_S\right)$$

$S_2 = \hbar t(-0.0000714 - \frac{N_R}{365} + 0.0000714 N_S + 1.9455159918287936 N_I N_S) + \hbar t(-0.0000714 - 2.54898 \times 10^{-9}\hbar t - 0.0002741682191780822\hbar t N_R + \hbar t N_E(2.543835616438356 \times 10^{-7} - 0.4863789979571984 N_S) + 2.54898 \times 10^{-9}\hbar t N_S + 1.8925071905814252\hbar t N_I^2 N_S + \hbar t N_I(0.00020451768183143813 - 0.0026650776600394433 N_R + N_S(0.19476901059496624 + \frac{1.9455159918287936 t^{-\mu_1}}{(2.-3.\mu_1+\mu_1^2)\Gamma[1-\mu_1]})) - \frac{0.0000714\hbar t^{1-\mu_1}}{(2.-3.\mu_1+\mu_1^2)\Gamma[1-\mu_1]} - \frac{0.0027397260273972603\hbar t^{1-\mu_1} N_R}{(2.-3.\mu_1+\mu_1^2)\Gamma[1-\mu_1]} + \frac{0.0000714\hbar t^{1-\mu_1} N_S}{(2.-3.\mu_1+\mu_1^2)\Gamma[1-\mu_1]})$

$E_1 = \hbar t(0.0029968260273972604 N_E - 0.000009300000000000001 N_I N_E - 1.9455252918287937 N_I N_S)$
$E_2 = \hbar t(0.0029968260273972604 N_E - 0.000009300000000000001 N_I N_E - 1.9455252918287937 N_I N_S) + \hbar^2 t^2(N_I^2(8.649000000000003 \times 10^{-11} N_E - 1.8925071904949353 N_S) + N_E(0.000004490483119242823 + 0.000002325 N_E + 0.48638132295719844 N_S + \frac{0.0029968260273972604 t^{-\mu_2}}{(-2.+\mu_2)(-1.+\mu_2)\Gamma[1-\mu_2]}) + N_I(0.00006945525291828793 + 0.0026651031394914985 N_R - 0.19761568679707905 N_S + N_E(-9.582457370547946 \times 10^{-7} - \frac{0.000009300000000000001 t^{-\mu_2}}{(-2.+\mu_2)(-1.+\mu_2)\Gamma[1-\mu_2]}) - \frac{1.9455252918287937 t^{-\mu_2} N_S}{(-2.+\mu_2)(-1.+\mu_2)\Gamma[1-\mu_2]}))$

$I_1 = \hbar t(0.2000807 N_I - 0.000009300000000000001 N_I^2 - 0.5 N_E)$
$I_2 = \hbar t(0.2000807 N_I - 0.000009300000000000001 N_I^2 - 0.5 N_E) + \hbar^2 t^2(8.649000000000003 \times 10^{-11} N_I^3 + N_I(0.020016143256245 + 0.000006975 N_E + 0.48638132295719844\, N_S - \frac{0.2000807 t^{-\mu_3}}{(1.-1.\mu_3)(-2.+\mu_3)\Gamma[1-\mu_3]}) + N_I^2(-0.000002791125765 + \frac{0.000009300000000000001 t^{-\mu_3}}{(1.-1.\mu_3)(-2.+\mu_3)\Gamma[1-\mu_3]}) + N_E(-0.050769381506849315 + \frac{0.5 t^{-\mu_3}}{(1.-1.\mu_3)(-2.+\mu_3)\Gamma[1-\mu_3]}))$

$R_1 = \hbar t(-0.2 N_I - 0.0001857 N_E + 0.2000714 N_R - 0.000009300000000000001 N_I N_R)$
$R_2 = \hbar t(-0.2 N_I - 0.0001857 N_E + 0.2000714 N_R - 0.000009300000000000001 N_I N_R) + \hbar^2 t^2(N_I^2(0.00000186 + 8.649000000000003 \times 10^{-11} N_R) + N_R(0.02001428254898 - \frac{0.2000714 t^{-\mu_4}}{(1.-1.\mu_4)(-2.+\mu_4)\Gamma[1-\mu_4]}) + N_E(0.04998114511521336 + 0.000002325 N_R + \frac{0.0001857 t^{-\mu_4}}{(1.-1.\mu_4)(-2.+\mu_4)\Gamma[1-\mu_4]}) + N_I(-0.04001521000000001 + 1.72701 \times 10^{-9} N_E + 0.00018064202334630 35 N_S + N_R(-0.000002791039275 + \frac{0.000009300000000000001 t^{-\mu_4}}{(1.-1.\mu_4)(-2.+\mu_4)\Gamma[1-\mu_4]}) + \frac{0.2 t^{-\mu_4}}{(1.-1.\mu_4)(-2.+\mu_4)\Gamma[1-\mu_4]}))$

Finally, we approximate the solution $S(t)$, $E(t)$, $I(t)$ and $R(t)$ of the model (5) and (4) by the k^{th}-truncated series:

$$\psi_{S,k}(t) = \sum_{m=0}^{k-1} S_m(t), \psi_{E,k}(t) = \sum_{m=0}^{k-1} E_m(t)$$
$$\psi_{I,k}(t) = \sum_{m=0}^{k-1} I_m(t), \psi_{R,k}(t) = \sum_{m=0}^{k-1} R_m(t) \quad (16)$$

RESULTS AND DISCUSSION

The HAM provides an analytical approximate solution in terms of an infinite power series. However, there is a practical need to evaluate this solution and to obtain numerical values from the infinite power series. The consequent series truncation and the practical procedure conducted to accomplish this task.

To consider the behavior of solution for different values of $\mu_i, i = 1, 2, 3,\ 4$ we will take advantage of the explicit formula (16) available for $0 < \mu_i \leq 1, i = 1,2,3,4$ and consider the following two special cases (Through this study, we set $\hbar_1 = \hbar_2 = \hbar_3 = \hbar_4 = \hbar$):

Case I: We will examine the classical SEIR model (5) and (4) by setting $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$. The partial sums (16) are determined and in particular seventh approximations are calculated for $S(t)$, $E(t)$, $I(t)$ and $R(t)$, respectively:

$\psi_{S,7}(t) = \sum_{m=0}^{6} S_m(t) = 0.9999 + 0.00009615002413776647\hbar t + 0.0014898750603444162\hbar^2 t + 0.0028195000804592217\,\hbar^3 t + 0.002489475060344416\hbar^4 t + 0.0010957500241377664\hbar^5 t + 0.00019452500402296108\hbar^6 t + 0.00029252503148930007\hbar^2 t^2 + 0.0007802882883202528\hbar^3 t^2 + 0.0008779738622957152\hbar^4 t^2 + 0.0004683058956014234\hbar^5 t^2 + 0.0000975716403103722\hbar^6 t^2 + \cdots + 1.648857241150932 \times 10^{-7}\hbar^6 t^6$

$\psi_{E,7}(t) = \sum_{m=0}^{6} E_m(t) = -\ 0.0011671984435797666\hbar t - 0.002917996108949 4163\hbar^2 t$
$-0.0038906614785992 22\hbar^3 t - 0.0029179961089494167\hbar^4 t - 0.0011671984435797666\hbar^5 t$
$-0.00019453307392996124\hbar^6 t - 0.00029643464214835135\hbar^2 t^2 - 0.00079065444131907 98\hbar^3 t^2$
$-0.0008895956385073109\hbar^4 t^2 - 0.00047448990214553355\hbar^5 t^2 - 0.00009885785088472 94\hbar^6 t^2$
$- \cdots - 1.65640084434164 \times 10^{-7}\hbar^6 t^6$

$\psi_{I,7}(t) = \sum_{m=0}^{6} I_m(t) = 0.0001 + 0.00012004841944200001\hbar t + 0.000300121048605\hbar^2 t$
$+0.0004001613981399999\hbar^3 t + 0.000300121048605\hbar^4 t + 0.0001200484194419 9999\hbar^5 t$
$+0.000020008069906999998\hbar^6 t + 0.0007595232417030528\hbar^2 t^2 + 0.0020253953112081406\hbar^3 t^2$
$+0.0022785697251091583\hbar^4 t^2 + 0.0012152371867248841\hbar^5 t^2 + 0.00025317441390101757\hbar^6 t^2$
$+ \cdots + 1.61104959300818 \times 10^{-7}\hbar^6 t^6$

$\psi_{R,7}(t) = \sum_{m=0}^{6} R_m(t) = -\ 0.00012\hbar t - 0.00030000000000000003\hbar^2 t - 0.00040000000000000001\hbar^3 t$
$-0.00030000000000000001\hbar^4 t - 0.00012000000000000003\hbar^5 t - 0.000020000000000000005\hbar^6 t$
$-0.00005975187878228405\hbar^2 t^2 - 0.0001593383434194241\hbar^3 t^2 - 0.00017925563634685213\hbar^4 t^2$
$-0.00009560300605165447\hbar^5 t^2 - 0.000019917292927428017\hbar^6 t^2$
$- \cdots - 2.339584820979872 \times 10^{-8}\hbar^6 t^6$

Case II: In this case, we will examine the fractional SEIR model (5) and (4) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$. The partial sums (16) are determined and in particular seventh approximations are calculated for $S(t), I(t), E(t)$ and $R(t)$, respectively:

$\psi_{S,7}(t) = \sum_{m=0}^{6} S_m(t) = 0.9999 + 0.000096150024137766 47\hbar t + 0.0013149788073698984\hbar^2 t^{5/4}$
$+0.0021209767682762045\hbar^3 t^{3/2} + 0.0015478350338328718\hbar^4 t^{7/4}$
$+0.00029252503148930007\hbar^2 t^2 + 0.000547875012068883\hbar^5 t^2 + \cdots + 1.648857241150932$
$\times 10^{-7}\hbar^6 t^6$

$\psi_{E,7}(t) = \sum_{m=0}^{6} E_m(t) = -\ 0.0011671984435797666\hbar t - 0.0025754528989627365\hbar^2 t^{5/4}$
$-0.00292676090578159\hbar^3 t^{3/2} - 0.0018142686697150703\hbar^4 t^{7/4}$
$-0.00029643464214835135\hbar^2 t^2 - 0.0005835992217898832\hbar^5 t^2 - \cdots - 1.65640084434164 \times$
$10^{-7}\hbar^6 t^6$

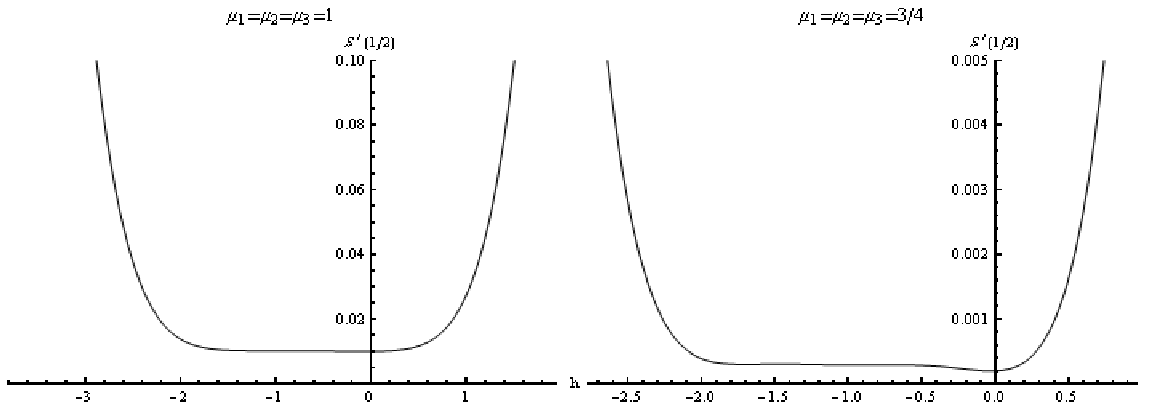
$\psi_{I,7}(t) = \sum_{m=0}^{6} I_m(t) = 0.0001 + 0.00012004841944200001\hbar t + 0.0002648898750409137 3\hbar^2 t^{5/4}$
$+0.0003010225234246593\hbar^3 t^{3/2} + 0.0001866007339544140 2\hbar^4 t^{7/4} +$
$0.0007595232417030528\hbar^2 t^2 + 0.000060024209720999 98\hbar^5 t^2 + \cdots + 1.61104959300818 \times$
$10^{-7}\hbar^6 t^6$

$\psi_{R,7}(t) =$
$\sum_{m=0}^{6} R_m(t) = -\ 0.00012\hbar t - 0.0002647830363170 01\hbar^2 t^{5/4} - 0.00030090111122547003\hbar^3 t^{3/2}$
$-0.00018652547179388867\hbar^4 t^{7/4} - 0.00005975187878228405\hbar^2 t^2 - 0.00006\hbar^5 t^2 - \cdots -$
$2.339584820979872 \times 10^{-8}\hbar^6 t^6$

Convergence of the series solution: The HAM yields rapidly convergent series solution by using a few iterations. For the convergence of the HAM, the reader is referred to Liao (1992).

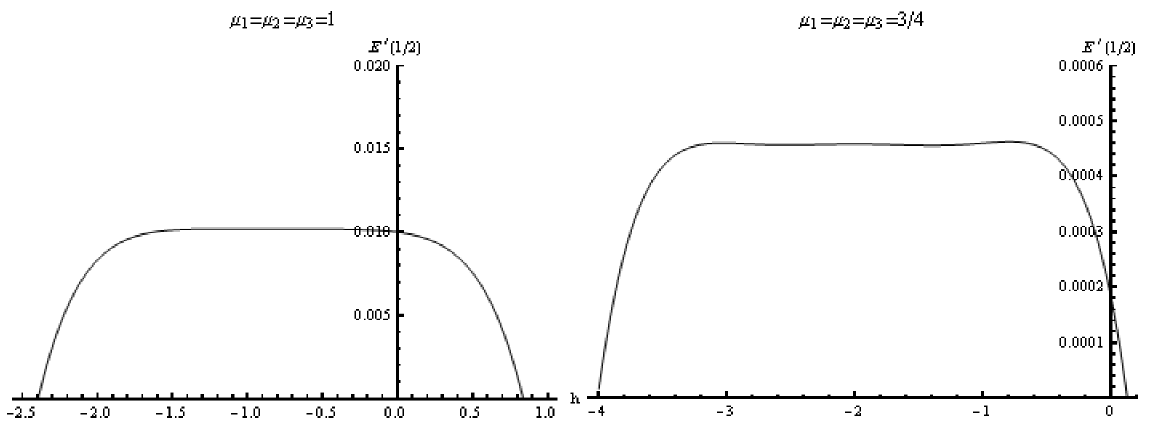
According to Oldhamand Spanier (1974) it is to be noted that the series solution contain the auxiliary parameters $\hbar$ which provides a simple way to adjust and control the convergence of the series solution. In fact, it is very important to ensure that the series Eq. (14) are convergent. To this end, we have plotted $\hbar_1$-curves of $S'\left(\frac{1}{2}\right)$, $E'\left(\frac{1}{2}\right)$, $E'\left(\frac{1}{2}\right)$ and $R'\left(\frac{1}{2}\right)$ by seventh order approximation of the HAM in Fig. 1 to 4, respectively, for $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$ and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$.

According to these $\hbar$-curves, it is easy to discover the valid region of $\hbar$ which corresponds to the line segment nearly parallel to the horizontal axis. These valid regions have been listed in Table 1 to 3. Furthermore, these valid regions ensure us the convergence of the obtained series.



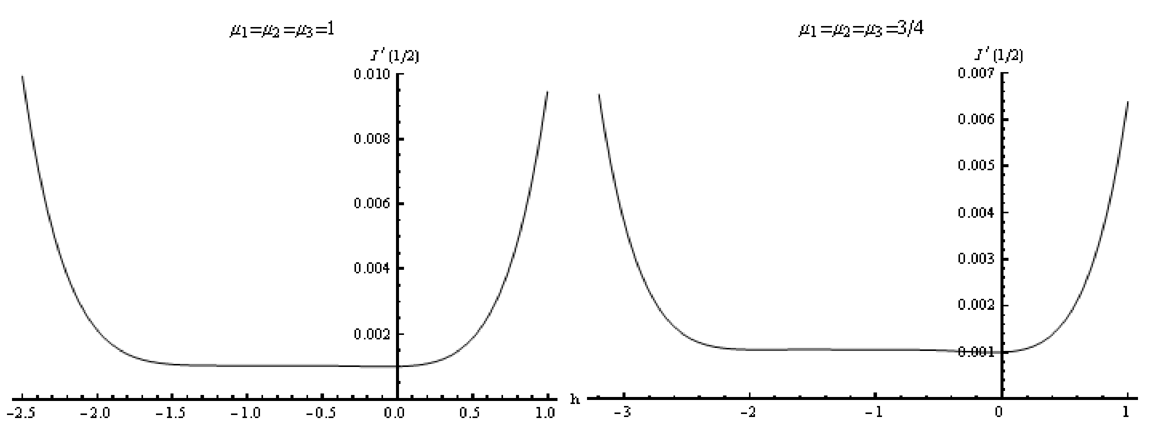
(a) (b)

Fig. 1: The $\hbar$-curves of $S'\left(\frac{1}{2}\right)$ obtained by the seventh order approximation of the HAM, (a) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$, (b) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$



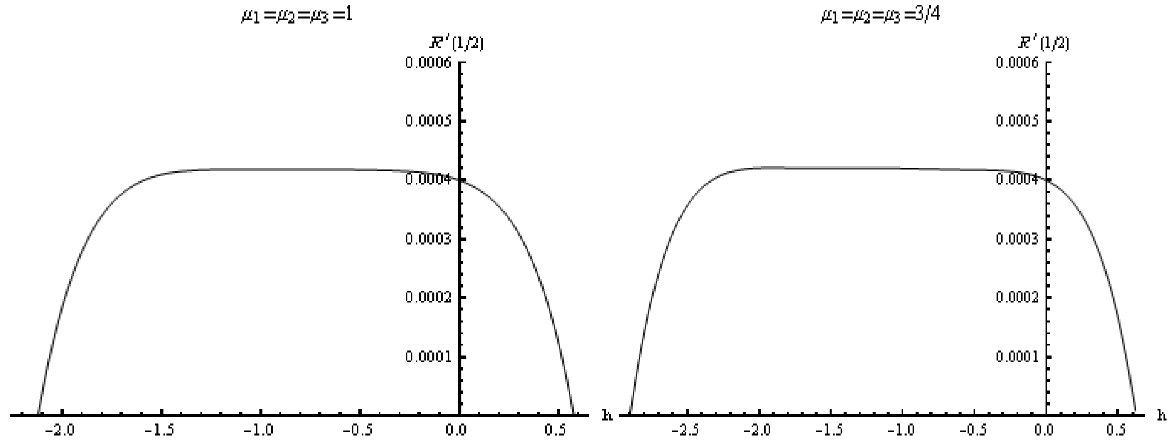
(a) (b)

Fig. 2: The $\hbar$-curves of $E'\left(\frac{1}{2}\right)$ obtained by the seventh order approximation of the HAM, (a) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$, (b) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$



(a) (b)

Fig. 3: The $\hbar$-curves of $I'\left(\frac{1}{2}\right)$ obtained by the seventh order approximation of the HAM, (a) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$, (b) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$



(a) (b)

Fig. 4: The $\hbar$-curves of $R'\left(\frac{1}{2}\right)$ obtained by the seventh order approximation of the HAM, (a) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$, (b) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$

To determine the optimal values of $\hbar$ in the interval $[0, 1]$, an error analysis is performed. We substitute the approximations $\psi_{S,7}(t)$, $\psi_{E,7}(t)$, $\psi_{I,7}(t)$ and $\psi_{R,7}(t)$ in Case 1 and Case 2 into system (5) and obtain the residual functions; ER_S, ER_E ER_I and ER_R as follows:

$$ER_S(t,\hbar) = D_*^{\mu_1}\left[\psi_{S,k}(t)\right]$$
$$+(\beta-\alpha)\psi_{I,k}(t)\psi_{S,k}(t)$$
$$+r\psi_{S,k}(t) - \delta\psi_{R,k}(t) - r$$
$$ER_E(t,\hbar) = D_*^{\mu_2}\left[\psi_{E,k}(t)\right] - \beta\psi_{S,k}(t)\psi_{I,k}(t)$$
$$+(\delta+\kappa+r)\psi_{E,k}(t) - \alpha\psi_{I,k}(t)\psi_{E,k}(t)$$
$$ER_I(t,\hbar) = D_*^{\mu_3}\left[\psi_{I,k}(t)\right] - \sigma\psi_{E,k}(t)$$
$$+(\alpha+\gamma+r)\psi_{I,k}(t) - \alpha\left(\psi_{I,k}(t)\right)^2$$
$$ER_R(t,\hbar) = D_*^{\mu_4}\left[\psi_{R,k}(t)\right] - \kappa\psi_{E,k}(t) - \gamma\psi_{I,k}(t)$$
$$+r\psi_{R,k}(t) - \alpha\psi_{I,k}(t)\psi_{R,k}(t) + \delta\psi_{R,k}(t)$$

Following Liao (2004, 2010), we define the square residual error for approximation solutions on the interval [0,1] as:

$$SER_S(\hbar) = \int_0^1 [ER_S(t,\hbar)]^2\, dt$$
$$SER_E(\hbar) = \int_0^1 [ER_E(t,\hbar)]^2\, dt$$
$$SER_I(\hbar) = \int_0^1 [ER_I(t,\hbar)]^2\, dt$$
$$SER_R(\hbar) = \int_0^1 [ER_R(t,\hbar)]^2\, dt$$

By using the first derivative test, we determine the values of $\hbar$ that minimize SER_S, SER_E, SER_I and SER_R.

In Liao (2006, 2010) and Podlubny (1999) several methods have been introduced to find the optimal value of $\hbar$. In Table 1 to 3, the optimal values of $\hbar$ for the two previous cases are tabulated.

Table 1: The parameters value of the fractional SEIR model

Parameter	Biological meaning
$N_S = 0.9999$	Initial population of N_S, who are susceptible
$N_E = 0$	Initial population of N_E, who are exposed
$N_I = 0.0001$	Initial population of N_I, who are infected
$N_R = 0$	Initial population of N_R, who are removed
$\alpha = 9.3 \times 10^{-6}$	Flu induced mortality rate per day
$\beta^{-1} = 1.945525$	Transmission coefficient
$\gamma^{-1} = 5$	Mean recovery time for clinically ill (days)
$\delta^{-1} = 365$	Duration of immunity loss (days)
$\kappa = 1.857 \times 10^{-4}$	Recovery rate of latents per day
$\sigma^{-1} = 2$	Mean duration of latency (days)
$r = 7.14 \times 10^{-5}$	Birth rate per day

Table 2: The valid region of $\hbar$ derived from Fig. 1 to 3

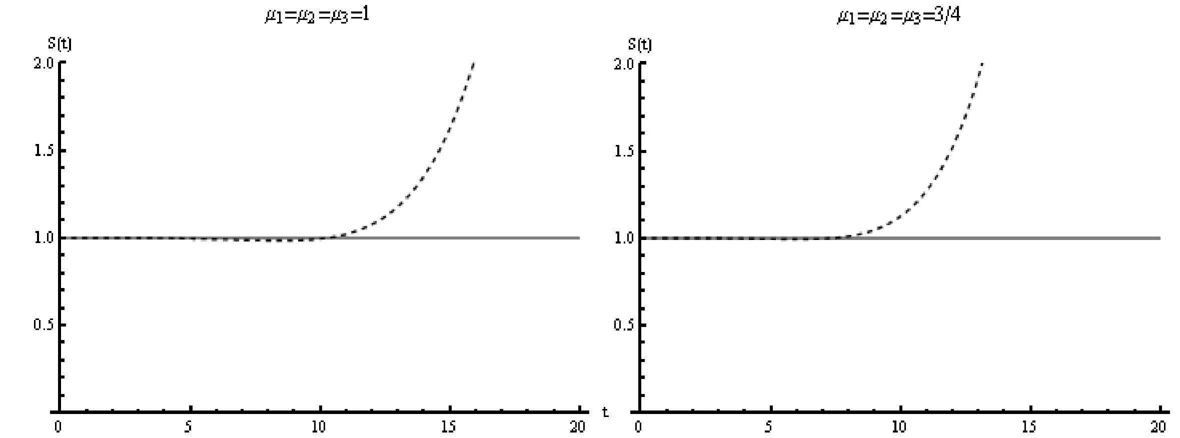
Components	$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$	$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$
S(t)	$-1.4 < \hbar < 0$	$-1.8 < \hbar < -0.4$
E(t)	$-1.4 < \hbar < 0$	$-2.8 < \hbar < -0.8$
I(t)	$-1.4 < \hbar < 0$	$-2 < \hbar < -0.4$
R(t)	$-1.3 < \hbar < -0.3$	$-2 < \hbar < -0.4$

Table 3: The optimal values of $\hbar$

Components	$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$	$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$
S(t)	−0.0494757	−0.0575899
E(t)	−0.9734870	−1.4314757
I(t)	−0.9809824	−1.3968149
R(t)	−0.9669356	−1.4246121

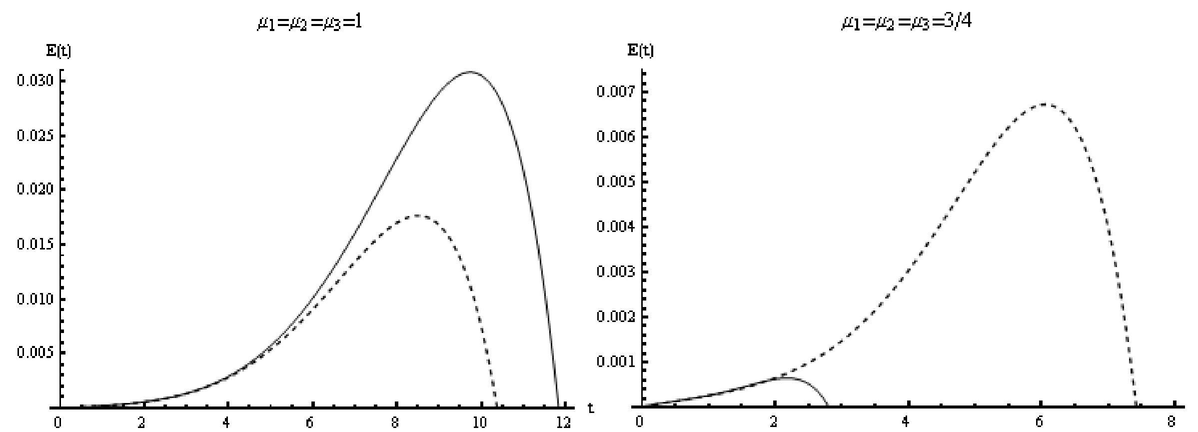
These results are plotted in Fig. 5 to 8 at the optimal points of the valid region together with $\hbar_1 = -1$ when $\mu_i = 1$ and $\mu_i = 0.75$, $i = 1,2,3,4$ for the four components $S(t)$, $E(t)$, $I(t)$ and $R(t)$, respectively. As the plots show while the number of susceptible increases, the population of who are infective decreases in the period of the epidemic. Meanwhile, the number of immune population increases, but the size of the population over the period of the epidemic is constant.

In Table 4 to 7, the absolute residual errors ER_S, ER_I, ER_I and ER_R have been calculated for various t in [0,1] when $\mu_i = 1$ and $\mu_i = 0.75$, $i = 1,2,3,4$. From the



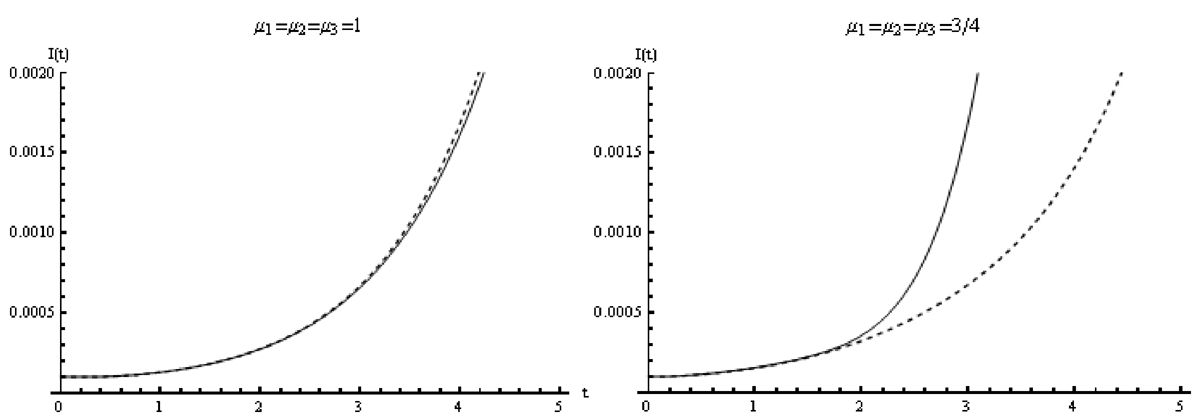
(a) (b)

Fig. 5: The HAM solution of $S(t)$, (a) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$; dotted line: $\hbar = -1$ and solid line: $\hbar = -0.0494757$, (b) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$; dotted line: $\hbar = -1$ and solid line: $\hbar = -0.0575899$



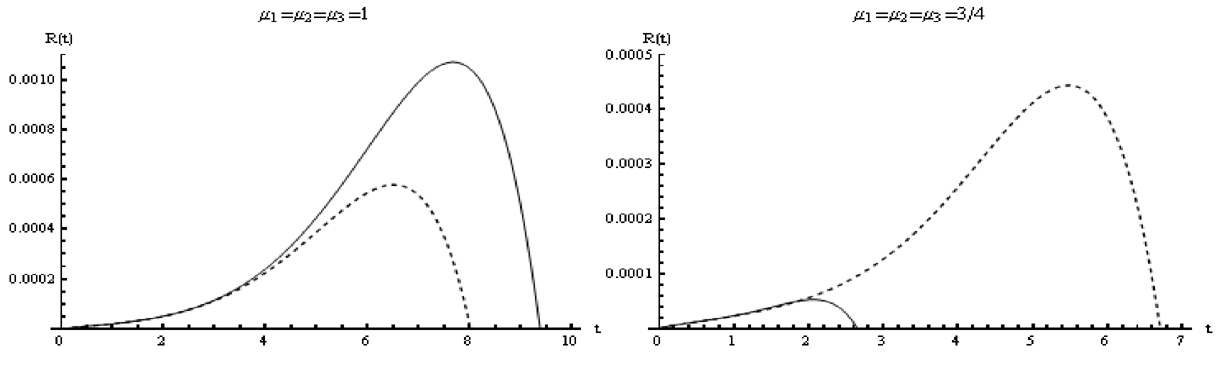
(a) (b)

Fig. 6: The HAM solution of $E(t)$, (a) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$; dotted line: $\hbar = -1$ and solid line: $\hbar = -0.9734870$, (b) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$; dotted line: $\hbar = -1$ and solid line: $\hbar = -1.4314757$



(a) (b)

Fig. 7: The HAM solution of $I(t)$, (a) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$; dotted line: $\hbar = -1$ and solid line: $\hbar = -0.9809824$, (b) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$; dotted line: $\hbar = -1$ and solid line: $\hbar = -1.3968149$



(a) (b)

Fig. 8: The HAM solution of $R(t)$, (a) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$; dotted line: $\hbar = -1$ and solid line: $\hbar = -0.9669356$, (b) when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$; dotted line: $\hbar = -1$ and solid line: $\hbar = -1.4246121$

Table 4: The values of $S(t)$, $E(t)$ and the residual errors ER_S, ER_E when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$

t	S(t)	ER_S	E(t)	ER_E
0.1	0.99989986249807	1.92220×10^{-4}	1.92884×10^{-5}	1.40248×10^{-9}
0.2	0.99989973710390	1.91410×10^{-4}	3.83750×10^{-5}	2.90607×10^{-9}
0.3	0.99989962339223	1.90660×10^{-4}	5.74476×10^{-5}	4.85031×10^{-9}
0.4	0.99989952093845	1.89968×10^{-4}	7.66901×10^{-5}	7.85562×10^{-9}
0.5	0.99989942931858	1.89334×10^{-4}	9.62842×10^{-5}	1.31434×10^{-8}
0.6	0.99989934810928	1.88759×10^{-4}	1.16411×10^{-4}	2.30482×10^{-8}
0.7	0.99989927688782	1.88242×10^{-4}	1.37254×10^{-4}	4.17226×10^{-8}
0.8	0.99989921523213	1.87782×10^{-4}	1.58997×10^{-4}	7.60338×10^{-8}
0.9	0.99989916272072	1.87381×10^{-4}	1.81829×10^{-4}	1.36652×10^{-7}
1.0	0.99989911893277	1.87037×10^{-4}	2.05946×10^{-4}	2.39332×10^{-7}

Table 5: The values of I (t), R (t) and the residual errors ER_I, ER_R when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$

t	I (t)	ER_I	R (t)	ER_R
0.1	9.84993×10^{-5}	3.65147×10^{-11}	1.96376×10^{-6}	1.79238×10^{-12}
0.2	9.79774×10^{-5}	1.60227×10^{-11}	3.86902×10^{-6}	1.52461×10^{-11}
0.3	9.84095×10^{-5}	4.34068×10^{-10}	5.73607×10^{-6}	6.57278×10^{-11}
0.4	9.97807×10^{-5}	9.73564×10^{-10}	7.58438×10^{-6}	2.03159×10^{-10}
0.5	1.02085×10^{-4}	1.88232×10^{-9}	9.43283×10^{-6}	5.28086×10^{-10}
0.6	1.05326×10^{-4}	3.11369×10^{-9}	1.12999×10^{-5}	1.23943×10^{-9}
0.7	1.09517×10^{-4}	4.25860×10^{-9}	1.32038×10^{-5}	2.70392×10^{-9}
0.8	1.14677×10^{-4}	4.25960×10^{-9}	1.51628×10^{-5}	5.54724×10^{-9}
0.9	1.20838×10^{-4}	1.05099×10^{-9}	1.7195×10^{-5}	1.07668×10^{-8}
1.0	1.28039×10^{-4}	8.87475×10^{-9}	1.93189×10^{-5}	1.98665×10^{-8}

Table 6: The values of S (t), E (t) and the residual errors ER_S, ER_E when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$

t	S (t)	ER_S	E (t)	ER_E
0.1	0.99989968795633	1.91631×10^{-4}	3.71386×10^{-5}	6.11293×10^{-7}
0.2	0.99989947546650	1.90671×10^{-4}	6.25660×10^{-5}	2.08795×10^{-7}
0.3	0.99989931925567	1.89878×10^{-4}	8.58507×10^{-5}	5.51398×10^{-7}
0.4	0.99989920682504	1.89203×10^{-4}	1.08408×10^{-4}	4.99197×10^{-7}
0.5	0.99989913132213	1.88630×10^{-4}	1.30880×10^{-4}	2.86759×10^{-7}
0.6	0.99989908818573	1.88151×10^{-4}	1.53729×10^{-4}	1.31014×10^{-6}
0.7	0.99989907403991	1.87761×10^{-4}	1.77383×10^{-4}	1.95521×10^{-6}
0.8	0.99989908621065	1.87457×10^{-4}	1.58997×10^{-4}	1.66344×10^{-6}
0.9	0.99989912247782	1.87237×10^{-4}	1.81829×10^{-4}	7.03184×10^{-8}
1.0	0.99989918093382	1.87097×10^{-4}	2.05946×10^{-4}	2.39332×10^{-6}

Table 7: The values of I (t), R (t) and the residual errors ER_I, ER_R, when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.75$

t	I (t)	ER_I	R (t)	ER_R
0.1	9.84581×10^{-5}	3.49205×10^{-7}	3.72807×10^{-6}	7.44281×10^{-8}
0.2	1.00054×10^{-4}	1.26430×10^{-7}	6.17930×10^{-6}	2.46439×10^{-8}
0.3	1.03084×10^{-4}	3.40459×10^{-7}	8.36708×10^{-6}	7.07102×10^{-8}
0.4	107298×10^{-4}	4.39583×10^{-7}	1.04413×10^{-5}	7.07102×10^{-8}
0.5	1.12630×10^{-4}	1.21694×10^{-7}	1.24682×10^{-5}	2.51506×10^{-8}
0.6	1.19048×10^{-4}	4.17937×10^{-7}	1.44951×10^{-5}	1.52519×10^{-7}
0.7	1.26515×10^{-4}	8.83303×10^{-7}	1.65665×10^{-5}	2.38572×10^{-7}
0.8	1.34987×10^{-4}	9.66765×10^{-7}	1.87273×10^{-5}	2.13403×10^{-7}
0.9	1.44417×10^{-4}	4.17180×10^{-7}	210215×10^{-5}	2.87728×10^{-8}
1.0	1.54771×10^{-4}	9.01092×10^{-7}	2.34895×10^{-5}	3.27176×10^{-7}

tables, it can be seen that the HAM provides us with the accurate approximate solution for the fractional SEIR model (5) and (4).

CONCLUSION

The analytical approximation solutions of the epidemiological model are reliable and confirm the power and ability of the HAM as an easy device for computing the solution of nonlinear problems. In this study, a fractional-order differential SEIR model is studied and its approximate solution is presented using HAM. The present scheme shows importance of choice of convergence control parameter $\hbar$ to guarantee the convergence of the solutions. Moreover, higher accuracy can be achieved using HAM by evaluating more components of the solution. In the near future, we intend to make more researches as a continuation to this study. One of these researches is: application of HAM to solve age-structured SEIR epidemic model.

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