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An explicit solution for the logistic map

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Abstract

An explicit functional integral solution for the logistic map is presented. Furthermore, the discrete nature of this equation is used to explicitly calculate the corresponding Radon–Nikodym derivatives. This enables us to represent the solution as a multidimensional integral. © 1999 Published by Elsevier Science B.V. All rights reserved.

The *logistic map*, $x_{n+1} = qx_n(1 - x_n)$, is one of the most famous recurrent equations in multidisciplinary science. Its importance for understanding of chaos was recognized about 40 years ago and it was studied numerically in great detail in many studies. Its only analytic solution known is written in terms of infinite-dimensional matrices [1,4].

In this paper we present a new analytic solution for the logistic equation. In order to reach this goal we exploit a known functional integral solution of a special nonlinear equation [2].

We first use a discrete analog of the Green function, the Green sequence, g_n. Following a standard procedure one can rewrite the original equation as

$$x_{n+1} - qx_n = -qx_n^2 \quad \text{with} \quad x_0 = x \tag{1}$$

and consider the RHS as an inhomogeneity. Then the solution of an homogeneous part reads

$$g_{n+1} - qg_n = 0, \quad g_0 = 1, \quad \Rightarrow \quad g_n = q^n ,$$

and, thus Eq. (1) becomes

$$x_n = xq^n - \sum_{k=0}^{n-1} q^{n-k} x_k^2 \quad \text{or} \quad xq^n = x_n + \sum_{k=0}^{n-1} q^{n-k} x_k^2 \, . \tag{2}$$

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Eq. (2) is a quasi-linear operator equation, $b = a + L(a \cdot a) \equiv a + Aa$, its solution can be written as [2]

$$a = \int_X u p_1^{-f}(u + Bu) p_2^B(u) \mu(\mathrm{d}u)\,, \tag{3}$$

where

$$f \equiv \int_X u p_3^A(u)(p_1^{-b}(u + Au) - 1)\mu(\mathrm{d}u)\,,$$

$$Bu \equiv -2 \int_X L(u \cdot z) p_3^A(z) \mu(\mathrm{d}z)\,.$$

In our case $b = xq^n$, $Aa = \sum_{k=0}^{n-1} q^{n-k} x_k^2$ and $L(u \cdot z) = \sum_{k=0}^{n-1} q^{n-k} u_k z_k$. As a measure μ one can take the Gauss measure with a vanishing mean value. Notations $p_{1,2,3}$ stand for Radon–Nikodym derivatives [3]: $p_1^c(x) = \mathrm{d}\mu_c/\mathrm{d}\mu(x)$ under the shift $x \to y = x + c$ by element $c \in X$, $p_2^L(x) = \mathrm{d}\mu_L/\mathrm{d}\mu(x)$ under linear transformation $x \to y = x + Lx$ and $p_3^A(x) = \mathrm{d}\mu_A/\mathrm{d}\mu(x)$ under nonlinear transformation $x \to y = x + Ax = x + L(x \cdot x)$. The operation $x_1 \cdot x_2 = x$, $x_1, x_2, x \in X$ is just a usual product for scalar functions x_1 and x_2. If x_1 and x_2 are vectors then x is a vector with components that are products of corresponding components of x_1 and x_2.

Being of a discrete nature Eq. (2) is a very special case of an operator equation. Therefore, the functional Gauss measure degenerates to a product of usual ones

$$\mu(\mathrm{d}u) = \prod_{j=0}^{n} \left\{ \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u_j^2\right) \mathrm{d}u_j \right\}\,. \tag{4}$$

This observation simplifies drastically all the expressions for Radon–Nikodym derivatives. Namely, using standard definitions [3] one gets

$$p_1^{-f}(u + Bu) = \exp\left\{ \sum_{k=0}^{n} f_k (u + Bu)_k - \frac{1}{2} \sum_{k=0}^{n} f_k^2 \right\} \tag{5}$$

and

$$p_1^{-y}(v + Av) = \exp\left\{ x \sum_{j=0}^{n} q^j \left(v_j + \sum_{l=0}^{j-1} q^{j-l} v_l^2 \right) - \frac{1}{2} x^2 \sum_{j=0}^{n} q^{2j} \right\}\,, \tag{6}$$

$$p_2^B(u) = \exp\left\{ -\frac{1}{2} \sum_{k=0}^{n} (Bu)_k^2 - \sum_{k=0}^{n} (Bu)_k u_k + \operatorname{Tr} B \right\}\,, \tag{7}$$

$$p_3^A(v) = \exp\left\{ -\sum_{j=0}^{n} v_j \sum_{l=0}^{j-1} q^{j-l} v_l^2 - \frac{1}{2} \sum_{j=0}^{n} \left(\sum_{l=0}^{j-1} q^{j-l} v_l^2 \right)^2 \right\}\,. \tag{8}$$

Using Eqs. (8) and (6) one writes down

$$(Bu)_k = -2 \int L(u \cdot z) p_3^A(z) \mu(\mathrm{d}z) \tag{9}$$

and

$$f_k = \int v_k p_3^A(v)[p_1^{-y}(v + Av) - 1]\mu(\mathrm{d}v) . \tag{10}$$

Finally, Eqs. (9), (10) and (7) enable us to rewrite Eq. (3) as

$$x_n = \int u_n p_1^{-f}(u + Bu) p_2^B(u)\mu(\mathrm{d}\, u) . \tag{11}$$

Thus, the solution of the logistic map is written by means of usual (while stubborn) multiple integrals.

Considering that Ref. [2] is hardly available for a general reader and has a disturbing misprint we decided to give here the derivation of its main result, Eq. (3).

The Gauss measure with a vanishing mean value, μ, has two basic properties: $\int_X \mu(\mathrm{d}\,x) = 1$ and $\int_X Lx\, \mu(\mathrm{d}\,x) = 0$, where L is a linear mapping. The definitions of $p_{1,2,3}$ were given in the body of this paper.

Consider the equation

$$y = x + L(x \cdot x) = x + Ax , \tag{12}$$

where $x \in X$ – unknown and $y \in X$ – a known element from a functional space X.

Let us take two elements x_1 and x_2 and calculate appropriate y_1 and y_2, i.e.

$$y_1 = x_1 + L(x_1 \cdot x_1), \qquad y_2 = x_2 + L(x_2 \cdot x_2) .$$

Combining these equalities one gets

$$y_1 + y_2 = x_1 + x_2 + L(x_1 \cdot x_1 + x_2 \cdot x_2) ,$$

or

$$y_1 + y_2 = x_1 + x_2 + L((x_1 + x_2) \cdot (x_1 + x_2)) - 2L(x_1 \cdot x_2) .$$

Let $x = Ry$ be a solution of Eq. (12). Then

$$Ry_1 + Ry_2 = y_1 + y_2 - L((x_1 + x_2) \cdot (x_1 + x_2)) + 2L(x_1 \cdot x_2) .$$

Inserting the sum $x_1 + x_2$ instead of x into Eq. (12)

$$R(y_1 + y_2) = y_1 + y_2 - L((x_1 + x_2) \cdot (x_1 + x_2))$$

and combining the last two equations we obtain

$$R(y_1 + y_2) = Ry_1 + Ry_2 - 2L(Ry_1 \cdot Ry_2)$$

and, therefore,

$$\int_X R(y_1 + y_2)\mu(\mathrm{d}y_2) = Ry_1 + \int_X Ry_2\mu(\mathrm{d}y_2) - 2\int_X L(Ry_1 \cdot Ry_2)\mu(\mathrm{d}y_2) . \tag{13}$$

We now transform the left-hand side of Eq. (13) by shifting the integration variable by $-y_1$ and then performing a nonlinear transform:

$$\int_X R(y_1 + y_2)\mu(\mathrm{d}y_2) = \int_X R(z) p_1^{-y_1}(z)\mu(\mathrm{d}z) = \int_X u p_1^{-y_1}(u + Au) p_3^A(u)\mu(\mathrm{d}u) .$$

In a similar manner one applies a nonlinear variable transform $x+Lx$ to the right-hand side of Eq. (13):

$$Ry_1 + \int_X Ry_2\mu(\mathrm{d}y_2) - 2\int_X L(Ry_1 \cdot Ry_2)\mu(\mathrm{d}y_2)$$

$$= Ry_1 + \int_X up_3^A(u)\mu(\mathrm{d}u) - 2\int_X L(Ry_1 \cdot u)p_3^A(u)\mu(\mathrm{d}u)\,.$$

Thus Eq. (13) takes the form of linear operator equation:

$$(I+B)a = f\,, \tag{14}$$

where the operator B and the vector f have been defined above and $a \equiv Ry_1$.

Now it is a matter of direct verification to prove that a defined by Eq. (3) obeys Eq. (14).

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