A linear kernel for planar red-blue dominating set[*]

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Abstract

In the RED-BLUE DOMINATING SET problem, we are given a bipartite graph $G = (V_B \cup V_R, E)$ and an integer k, and asked whether G has a subset $D \subseteq V_B$ of at most k 'blue' vertices such that each 'red' vertex from V_R is adjacent to a vertex in D. We provide the first explicit linear kernel for this problem on planar graphs.

Keywords: parameterized complexity, planar graphs, linear kernels, red-blue domination.

1 Introduction

Motivation. The field of parameterized complexity (see [6, 7, 15]) deals with algorithms for decision problems whose instances consist of a pair (x, k), where k is known as the *parameter*. A fundamental concept in this area is that of *kernelization*. A kernelization algorithm, or *kernel*, for a parameterized problem takes an instance (x, k) of the problem and, in time polynomial in $|x| + k$, outputs an equivalent instance (x', k') such that $|x'|, k' \leq g(k)$ for some function g. The function g is called the *size* of the kernel and may be viewed as a measure of the "compressibility" of a problem using polynomial-time preprocessing rules. A natural problem in this context is to find polynomial or linear kernels for problems that admit such kernelization algorithms.

A notorious result in this area is the linear kernel for DOMINATING SET on planar graphs by Alber *et al.* [2], which gave rise to an explosion of (meta-)results on linear kernels on planar graphs [11] and other sparse graph classes [3, 8, 12]. Although of great theoretical importance, these meta-theorems have two important drawbacks from a practical point of view. On the one hand, these results rely on a problem property called *Finite Integer Integer*, which guarantees the *existence* of a linear kernel, but nowadays it is still not clear how and when such a kernel can be effectively *constructed*. On the other hand, at the price of generality one cannot hope that general results of this type may directly provide explicit reduction rules and small constants for particular graph problems. Summarizing, as mentioned explicitly by Bodlaender *et al.* [3], these meta-theorems provide simple criteria to decide whether a problem admits a linear kernel on a graph class, but finding linear kernels with reasonably small constant factors for concrete problems remains a worthy investigation topic.

Our result. In this article we follow this research avenue and focus on the RED-BLUE DOMINATING SET problem (RBDS for short) on planar graphs. In the RED-BLUE DOMINATING SET problem, we are given a bipartite graph $G = (V_B \cup V_R, E)$ and an integer k, and asked whether G has a subset $D \subseteq V_B$ of at most k 'blue' vertices such that each 'red' vertex from V_R

is adjacent to a vertex in D. This problem appeared in the context of the European railroad network [16]. From a (classical) complexity point of view, finding a red-blue dominating set of minimum size is NP-complete on planar graphs [1]. From a parameterized complexity perspective, RBDS parameterized by the size of the solution is $W[2]$-complete on general graphs and FPT on planar graphs [6]. It is worth mentioning that RBDS plays an important role in the theory of non-existence of polynomial kernels for parameterized problems [5].

The fact that RBDS involves a *coloring* of the vertices of the input graph makes it unclear how to make the problem fit into the general frameworks of [3, 8, 11, 12]. In this article we provide the first explicit (and quite simple) polynomial-time data reduction rules for RED-BLUE DOMINATING SET on planar graphs, which lead to a linear kernel for the problem.

Theorem 1 RED-BLUE DOMINATING SET *parameterized by the solution size has a linear kernel on planar graphs. More precisely, there exists a poly-time algorithm that for each positive planar instance* (G, k) *returns an equivalence instance* (G', k) *such that* $|V(G')| \leq 47 \cdot k$.

This result complements several explicit linear kernels on planar graphs for other domination problems such as DOMINATING SET [2], EDGE DOMINATING SET [11], EFFICIENT DOMINATING SET [11], CONNECTED DOMINATING SET [10, 14], or TOTAL DOMINATING SET [9]. It is worth mentioning that our constant is considerable smaller that most of the constants provided by these results. Since one can easily reduce the FACE COVER problem on a planar graph to RBDS (without changing the parameter)[1], the result of Theorem 1 also provides a linear *bikernel* for FACE COVER (i.e., a polynomial-time algorithm that given an input of FACE COVER, outputs an equivalent instance of RBDS with a graph whose size is linear in k). To the best of our knowledge, the best existing kernel for FACE COVER is quadratic [13]. Our techniques are much inspired from those of Alber *et al.* [2] for DOMINATING SET, although our reduction rules and analysis are slightly simpler.

Organization of the paper. We first describe in Section 2 our reduction rules for RED-BLUE DOMINATING SET when the input graph is embedded in the plane, and in Section 3 we prove that the size of a reduced plane YES-instance is linear in the size of the desired red-blue dominating set, thus proving Theorem 1.

2 Reduction rules

In this section we propose reduction rules for RED-BLUE DOMINATING SET, which are largely inspired from the rules that yielded the first linear kernel for DOMINATING SET on planar graphs [2]. The idea is to replace the neighborhood of some blue vertices by appropriate gadgets. We would like to point out that our rules have also some points in common with the ones for the current best kernel for DOMINATING SET [4]. In Subsection 2.1 we present three easy elementary rules that turn out to be helpful in simplifying the instance, and then in Subsections 2.2 and 2.3 we present the rules for a single vertex and a pair of vertices, respectively.

2.1 Elementary rules

The following simple rules enable us to simplify an instance of RBDS. For simplicity, we will use the shorthand rbds to denote a red-blue dominating set in a graph.

Rule 1 *If G is not bipartite, remove edges between two vertices of the same color.*

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[1]Just consider the *radial graph* corresponding to the input graph G and its dual G^*, and color the vertices of G (resp. G^*) as red (resp. blue).

Rule 2 *Remove blue vertices whose neighborhood is included into the neighborhood of another blue vertex.*

Rule 3 *Remove red vertices whose neighborhood includes the neighborhood of another red vertex.*

Lemma 1 *Let $G = (V_B \cup V_R, E)$ be a graph. If G' is the graph obtained from G by the application of Rule 1, 2, or 3, then there is a rbds in G of size at most k if and only if there is one in G'.*

Proof. For Rule 1, since blue vertices do not need to be dominated, edges between blue vertices can be safely remove. Similarly, since red vertices cannot dominate, edges between red vertices can also be removed.

For Rule 2, if $N(b) \subseteq N(b')$ for two blue vertices b and b', then any solution containing b can be transformed to a solution containing b' in which the set of dominated red vertices may have only increased.

For Rule 3, if $N(r') \subseteq N(r)$ for two red vertices r and r', then any blue vertex dominating r' dominates also r. $\qquad\square$

Because of Rule 1, we can indeed assume that the graph G is bipartite, with bipartition $V_B \cup V_R$.

2.2 Rule for a single vertex

We present a rule for reducing the size of the neighborhood of a blue vertex. For this we need the definition of neighborhood and private neighborhood.

Definition 1 *Let $G = (V_B \cup V_R, E)$ be a graph. The* neighborhood *of a vertex $v \in V_B \cup V_R$ is the set $N(v) = \{u : \{v, u\} \in E\}$. The* private neighborhood *of a blue vertex b is the set $P(b) = \{r \in N(b) : N(N(r)) \subseteq N(b)\}$.*

Let us remark that for (classical) DOMINATING SET, each neighborhood is split into three subsets [2]. The third one corresponds to our private neighborhood, but since non-private neighbors can be used to dominate the private ones, an intermediary set is necessary for (classical) DOMINATING SET. In our problem it does not occur because non-private vertices are red and thus cannot belong to a rbds. This is one of the reasons why our rules are simpler.

Rule 4 *Let $v \in V_B$ be a blue vertex. If $P(v) \neq \emptyset$:*

- *remove $P(v)$ from G,*

- *add a new red vertex r and the edge $\{v, r\}$.*

Note that if Rule 2 is applied to the vertices in $N(N(b))$, then for $r \in P(b)$ it holds that $N(r) = \{b\}$. Moreover, if Rule 3 is applied to the vertices in $N(b)$, then it holds that $|P(b)| \leq 1$. That is, Rule 4 can also be obtained by first applying Rule 2 on $N(N(b))$ and then Rule 3 on $N(b)$. However, for the sake of the analysis it will be simpler to keep Rule 4 as a separate rule. We provide the proof of the next lemma for completeness.

Lemma 2 *Let $G = (V_B \cup V_R, E)$ be a graph and let $v \in V_B$. If G' is the graph obtained from G by the application of Rule 4 on a vertex v, then there is a rbds in G of size at most k if and only if there is one in G'.*

Proof. Let D be a rbds in G with $|D| \leq k$. Since $P(v)$ needs to be dominated, necessarily $v \in D$. Hence D is also a rbds of G'. Conversely, let D' be a rbds in G' with $|D| \leq k$. Since the new vertex r needs to be dominated, necessarily $v \in D$. Hence D' is also a rbds of G. $\qquad\square$

2.3 Rule for a pair of vertices

We now provide a rule for reducing the size of the neighborhood of a pair of blue vertices. For this, we first define the neighborhood and the private neighborhood of a pair of blue vertices.

Definition 2 *Let $G = (V_B \cup V_R, E)$ be a graph. The* neighborhood *of a blue pair of vertices $v, w \in V_B$ is the set $N(v, w) = N(v) \cup N(w)$. The* private neighborhood *of a blue pair of vertices $v, w \in V_B$ is the set $P(v, w) = \{r \in N(w, v) : N(N(r)) \subseteq N(v, w)\}$.*

We would like to note that the definition of private neighborhood is similar to that of the third subset of neighbors defined for (classical) DOMINATING SET [2].

Rule 5 *Let b, c be two distinct blue vertices. If $|P(b, c)| > 1$ and there is no blue vertex $d \neq b, c$ which dominates $P(b, c)$:*

1. if $P(b, c) \nsubseteq N(b)$ and $P(b, c) \nsubseteq N(c)$:

- *remove $P(b, c)$ from G,*
- *add two new red vertices r_b, r_c and the edges $\{b, r_b\}, \{c, r_c\}$;*

2. if $P(b, c) \subseteq N(b)$ and $P(b, c) \subseteq N(c)$:

- *remove $P(b, c)$ from G,*
- *add a new red vertex r and the edges $\{b, r\}, \{c, r\}$;*

3. if $P(b, c) \subseteq N(b)$ and $P(b, c) \nsubseteq N(c)$:

- *remove $P(b, c)$ from G,*
- *add a new red vertex r and the edge $\{b, r\}$;*

4. if $P(b, c) \nsubseteq N(b)$ and $P(b, c) \subseteq N(c)$:

- *symmetrically to Case 3.*

Lemma 3 *Let $G = (V_B \cup V_R, E)$ be a graph and let $b, c \in V_B$. If G' is the graph obtained from G by the application of Rule 5 on b, c, then there is a rbds in G of size at most k if and only if there is one in G'.*

Proof. We distinguish the four possible cases of Rule 5:

1. Let D be a rbds in G. Since there is no single vertex which dominates $P(b, c)$, we need at least two vertices in order to dominate $P(b, c)$. By definition of the private neighborhood, we can assume that $b, c \in D$. Hence D is a rbds in G'. Conversely, let D' be a rbds in G'. Since r_b, r_c need to be dominated, we have that $b, c \in D'$. Hence D' is a rbds in G.

2. Let D be a rbds in G. By definition of the private neighborhood, we have that $b \in D$ or $c \in D$. Hence D is a rbds in G'. Conversely, let D' be a rbds in G'. Since r needs to be dominated, we have that $b \in D'$ or $c \in D'$. Hence D' is a rbds in G.

3. Let D be a rbds in G. By definition of the private neighborhood, we can assume that $b \in D$. Hence D is a rbds in G'. Conversely, let D' be a rbds in G'. Since r needs to be dominated, we have that $b \in D'$. Hence D' is a rbds in G.

4. Symmetrically to Case 3. $\qquad\square$

3 Analysis of the kernel size

We will show that a graph *reduced* under our rules (that is, a graph for which none of the rules can be applied anymore) has size linear in $|D|$, the size of a solution. To this aim we assume that the graph is *plane* (that is, given with a fixed embedding) and we will define a notion of region adapted to our definition of neighborhood. Then we will show that, given a solution D, there is a maximal region decomposition $\Re$ such that:

- $\Re$ has $O(|D|)$ regions,
- $\Re$ covers all vertices but $O(|D|)$ of them,
- each region of $\Re$ has size $O(1)$.

The three following propositions treat respectively each of the above claims.

Definition 3 *Let $G = (V_B \cup V_R, E)$ be a plane graph and let $v, w \in V_B$. A region $R(v, w)$ between v and w is a closed subset of the plane such that:*

- *the boundary of $R(v, w)$ is formed by two simple paths connecting v and w, each of them having at most 4 edges;*
- *all vertices (strictly) inside $R(v, w)$ belong to $N(v, w)$ or $N(N(v, w))$.*

We denote by $\partial R(v, w)$ the boundary of $R(v, w)$ and by $V(R(v, w))$ the set of vertices in the region (that is, vertices strictly inside, on the boundary, and the two extremities v, w).

We say that an edge crosses a region R if it sits strictly inside R (except for its endpoints which can be on ∂R). Similarly a region (resp. a path) crosses a region R if there is an edge in the region (resp. the path) which crosses R.

Definition 4 *Let $G = (V_B \cup V_R, E)$ be a plane graph and let $D \subseteq V_B$. A D-decomposition of G is a set of regions $\Re$ between pairs of vertices in D such that:*

- *any region between v, w does not contain vertices in $D \setminus \{v, w\}$;*
- *any two regions have only the boundary in common.*

We note $V(\Re) = \bigcup_{R \in \Re} V(R)$. A D-decomposition is maximal *if there is no region $R \notin \Re$ such that $\Re \cup \{R\}$ is a D-decomposition with $V(\Re) \subsetneq V(\Re \cup \{R\})$.*

Proposition 1 *Let G be a reduced plane graph and let D be a rbds in G. There is a maximal D-decomposition of G such that $|\Re| \leq 3 \cdot |D| - 6$.*

Proof. The proof strongly follows the one of Alber *et al.* [2, Lemma 5 and Proposition 1]. Even if our definition of region is different, we shall show that the same algorithm can be used to construct such a D-decomposition.

We consider the algorithm which, for each vertex u, adds greedily to the decomposition $\Re$ a region R between any two vertices $v, w \in D$, containing u, not containing any vertex of $D \setminus \{v, w\}$, not crossing any region of $\Re$, and of maximal size; if it exists. By definition, $\Re \cup \{R\}$ is a region decomposition, and by greediness it is maximal.

We will now prove that for each pair of regions $R_1(v, w), R_2(v, w)$ between an identical pair of vertices v, w, there is a vertex of D in both open sets defined by the complement of the two regions in the plane. This property allows to apply [2, Lemma 5], implying that the constructed decomposition has at most $3 \cdot |D| - 6$ regions.

Indeed, let R_1, R_2 be two regions and let O be one of the open sets. Let us assume, for the sake of contradiction, that there is no vertex of D in O. We distinguish two cases.

- If O does not contain any blue vertex, then the red vertices in O (if any) must be dominated by v or w. Hence $R_1 \cup R_2 \cup O$ is a larger region which must have been chosen by the algorithm. We have a contradiction with the maximality of the regions R_1 and R_2.

- If O contains at least one blue vertex $b \notin D$, then, since Rule 2 has been applied, b is a neighbor of red vertices r and r' respectively dominated by v and w. The path $\{v, r, b, r', w\}$ is a degenerated region which has not been chosen by the algorithm. We have a contradiction with the maximality of the decomposition $\Re$. $\square$

Proposition 2 *Let $G = (V_B \cup V_R, E)$ be a reduced plane graph and let D be a rbds in G. If $\Re$ is a maximal D-decomposition, then $|V \setminus (V(\Re) \cup D)| \leq 2 \cdot |D|$.*

Proof. The proof again follows that of Alber *et al.* [2, Lemma 6 and Proposition 2], where similar arguments are used to bound the number of vertices which are not included in a maximal region decomposition. We bound separately the number of vertices in V_R and V_B which do not belong to $V(\Re)$. Since $N(D)$ covers V_R, it holds that $V_R = \bigcup_{v \in D} N(v)$, which we rewrite for convenience as $V_R = \bigcup_{v \in D} P(v) \cup (N(v) \setminus P(v))$.

We first bound $P(v)$ for all $v \in D$. By Rule 4, $|P(v)| \leq 1$, so $|\bigcup_{v \in D} P(v)| \leq |D|$.

We now show that $N(v) \setminus P(v) \subseteq V(\Re)$ for all $v \in D$. Let $u \in N(v) \setminus P(v)$. By definition of $P(v)$, there is a blue vertex $b \in N(u)$ and another red vertex $r \in N(b) \setminus N(v)$. We distinguish two cases.

- If $b \in D$, the (degenerated) region defined by the path $\{v, u, b\}$ crosses $\Re$ (since $\Re$ is maximal). Hence $u \in V(\Re)$.

- If $b \notin D$, then there is a blue vertex $w \in D \cap N(r)$ which dominates r. The (degenerated) region defined by the path $\{v, u, b, r, w\}$ crosses $\Re$ (since $\Re$ is maximal). We distinguish three cases.

    - If any of the edges $\{v, u\}$ or $\{u, b\}$ crosses $\Re$, then $u \in V(\Re)$.
    - Otherwise, if edge $\{b, r\}$ crosses a region $R(x, y) \in \Re$, then b is on $\partial R(x, y)$, as otherwise edge $\{u, v\}$ would cross $R(x, y)$. Let r' be a vertex on $\partial R(x, y)$ such that $r' \in N(b) \cap N(x)$. Then the (degenerated) region defined by the path $\{v, u, b, r', x\}$ could be added to $\Re$, which contradicts the maximality of $\Re$.
    - Finally, necessarily edge $\{r, w\}$ crosses a region $R(w, x) \in \Re$. Then r is on $\partial R(w, x)$, as otherwise edge $\{b, r\}$ would cross $R(w, x)$. If $r \in N(w)$, edge $\{r, w\}$ would not cross $R(w, x)$, so $r \in N(x)$. Then the (degenerated) region defined by the path $\{v, u, b, r, x\}$ could be added to $\Re$, which contradicts again the maximality of $\Re$.

So $\bigcup_{v \in D} N(v) \setminus P(v) \subseteq V(\Re)$, as we want to prove.

We finally show that $V_B \setminus D \subseteq V(\Re)$. Let $b \in V_B \setminus D$. Since G is reduced, by Rule 2 b is neither isolated, nor pendant, nor neighbor of a private vertex of $v \in D$. Hence b is neighbor of two red vertices r' and r'' dominated respectively by v and w. We consider the (degenerated) region $\{v, r', b, r'', w\}$, and with an argument similar to the previous one, we obtain a contradiction. So $V_B \setminus D \subseteq V(\Re))$.

Therefore, vertices not belonging to the decomposition are blue vertices in D and red vertices in $\bigcup_{v \in D} P(v)$, that is, at most $2 \cdot |D|$ vertices overall. $\square$

Proposition 3 *Let $G = (V_B \cup V_R, E)$ be a reduced plane graph, let D be a rbds in G, and let $v, w \in D$. A region R between v and w contains at most 15 vertices distinct from v, w.*

Proof. We bound separately the number of (red) non-private neighbors, (red) private neighbors of v and w, and blue vertices in the region. We distinguish the cases where Rule 5 is applied on v, w, and the case where there is a single blue vertex dominating $P(v, w)$. It will become clear from the proof that the worst bound is given by the case where ∂R contains 8 vertices, which will be henceforth denoted by $v, r_v, b, r_w, w, r'_w, b', r'_v$. In order to bound the total number of vertices, we need the following two simple facts.

Fact 1 *If there is a blue vertex b in the region R, then there is a path contained in R connecting v and w through b.*

Proof. Since Rule 2 has been applied, $N(b)$ is neither included in $N(v)$ nor in $N(w)$. Since all red vertices in R belong to $N(v, w)$, vertex b has a neighbor belonging to $N(v) \setminus N(w)$ and another one belonging to $N(w) \setminus N(v)$, which implies the existence of the desired path. □

Fact 2 *Given two pairs of red vertices in ∂R, there are at most 3 blue vertices in R with incomparable neighborhoods which are neighbors of at least one red vertex from each pair, and such that the edges joining them to the red vertices are drawn entirely in R.*

Proof. If all blue vertices have degree 2, it can be easily checked (see left-hand side of Figure 1) that there can be at most 3 blue vertices respecting planarity and the constraints of the statement of the fact. On the other hand, if there is a blue vertex adjacent to at least 3 red vertices (see right-hand side of Figure 1), there can be at most 2 such blue vertices. □

Figure 1: Examples in Fact 2. Blue (resp. red) vertices are depicted with ■ (resp. ●).

By definition the vertices in $N(v, w) \setminus P(v, w)$ sit on ∂R. Hence $|N(v, w) \setminus P(v, w)| \leq 4$.

The number of (red) private neighbors depends on whether Rule 5 has been applied to the pair v, w or not:

- Assume first that Rule 5 has been applied. Therefore $P(v, w)$ consists only of newly added vertices, and therefore $|P(v, w)| \leq 2$ (see Figure 2(a)).

- Otherwise, the vertices in $P(v, w)$ are dominated by a vertex u, so for any vertex $r \in P(v, w)$ it holds that $r \in N(u) \cap (N(v) \cup N(w))$. Since G is reduced, by Rule 3 all red vertices have incomparable neighborhoods (with respect to inclusion). If there is another blue vertex in R distinct from v, b, w, b', by Fact 1 this vertex is contained in a path from v to w, which disconnects u from either b or b'. This path separates R into two subregions, and since u dominates $P(v, w)$, all vertices in $P(v, w)$ are in the subregion containing u. This argument applies to any blue vertex in R distinct from v, b, w, b'. Therefore, in order to bound the number of private neighbors, without loss of generality we can assume that the blue vertices (in the subregion containing u) are exactly u, v, b, w, b', but note that the red vertices in the border of this subregion may also be counted as private (and in that case, they are necessarily dominated by u). Out of the possible neighborhoods of a red vertex $r \in P(v, w) \cap V(R)$, the reader can check that, while preserving planarity and the incomparability of neighborhoods (Rule 3), there can be at most 4 private vertices, and this case is attained with the neighborhoods $\{u, v, b\}, \{u, v, b'\}, \{u, w, b\}$, and $\{u, w, b'\}$. Hence $|P(v, w)| \leq 4$ (see Figure 2(b-c-d), where the considered subregion containing u is the darker one).
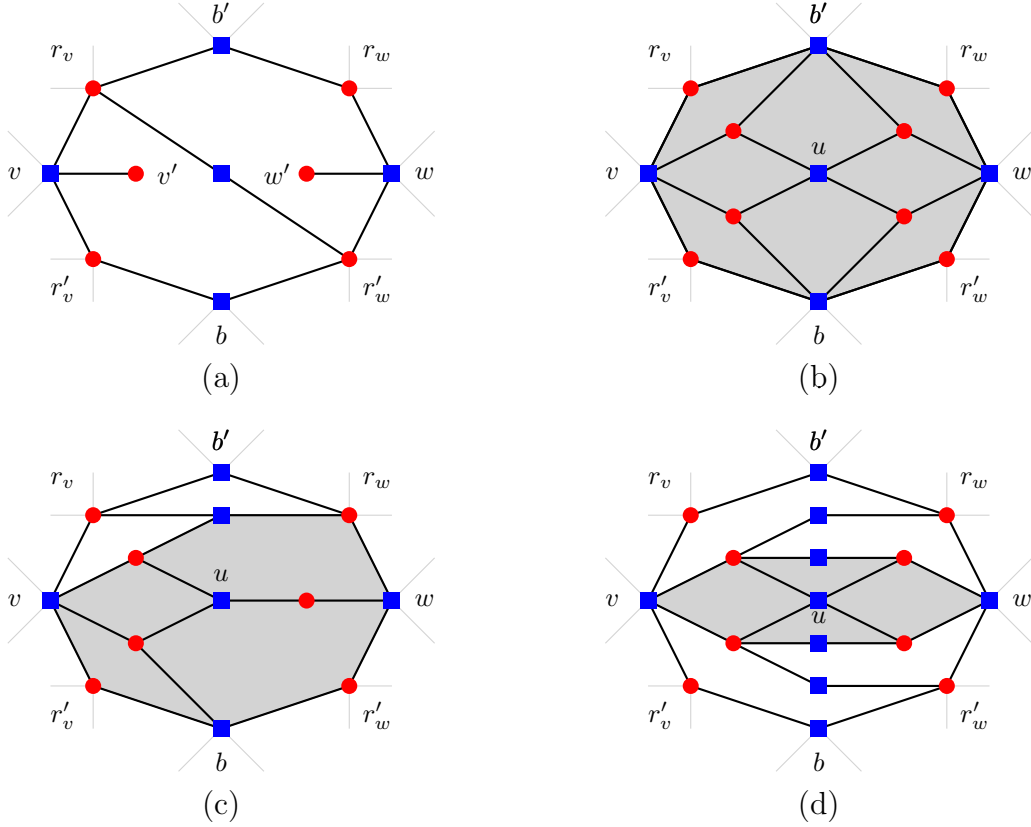
Figure 2: Examples in the proof of Proposition 3. Blue (resp. red) vertices are depicted with ■ (resp. ●). In (a), Rule 5 has been applied on the pair v, w. In (b)-(c)-(d), the red vertices in $P(v, w)$ are dominated by a blue vertex u, which is contained in the darker subregion.

It just remains to bound the number of blue vertices. Since G is reduced by Rule 2, blue vertices have incomparable neighborhoods, in particular with $N(v)$ and $N(w)$, so for any blue vertex b it holds that $N(b) \cap N(v) \neq \emptyset$ and $N(b) \cap N(w) \neq \emptyset$. In the sequel we will bound the number of blue vertices by using Fact 2 for appropriately chosen quadruples of red vertices. We distinguish whether Rule 5 has been applied to the pair v, w or not:

- Assume that Rule 5 has been applied. Since G is reduced, by Rule 2 all blue vertices have incomparable neighborhoods. Recall that R contains 6 red vertices $r_v, r'_v, v' \in N(v)$ and $r_w, r'_w, w' \in N(w)$, where v', w' are the newly added vertices. Note that the neighborhoods of the new vertices are included in $\{v, w\}$. So the neighborhood of a blue vertex in R can contain at most 4 red vertices $r_v, r'_v \in N(v)$ and $r_w, r'_w \in N(w)$. We now apply Fact 2 on the red quadruple r_v, r'_v, r_w, r'_w (note that b and b' are necessarily neighbors of r_v, r_w and $r_{v'}, r_{w'}$, respectively), yielding that $|V_B \cap V(R)| \leq 3$ (see again Figure 2(a)).

- Otherwise, the vertices in $P(v, w)$ can be dominated by a vertex u. If $P(v, w) = \emptyset$, then we can apply Fact 2 on the 4 red vertices in ∂R, and deduce that there can be at most 3 blue vertices in R distinct from b, b'. Otherwise, according to Fact 1, the region R can be split into at most 3 subregions by at most 2 paths (since u dominates $P(v, w)$; see Figure 2(b) (resp. (c), (d)) for an example with 0 (resp. 1, 2) separating paths). Note that, by Rule 2 and Fact 1, the subregion containing u (the darker one in Figure 2) cannot contain any blue vertex strictly inside. We can now apply Fact 2 to each of the (at most) 2 subregions not containing u (the white subregions in Figure 2(c-d)), and deduce that $|V_B \cap V(R)| \leq 3 + 3 + 1$, where we have also counted the blue vertex u.

Thus, the region R contains at most $4 + \max(2 + 5, 4 + 7) = 15$ vertices distinct from v, w. $\qquad\square$

We are finally ready to piece everything together and prove Theorem 1.

Proof of Theorem 1. Let G be the plane input graph and let G' be the reduced graph obtained from G. According to Lemmas 1, 2, and 3, G admits a rbds with size at most k if and only if G' admits one. It is easy to see that the same time analysis of [2] implies that our reduction rules can be applied in time $O(|V(G)|^3)$. According to Propositions 1, 2, and 3, if G' admits a rbds with size at most k, then G' has size at most $15 \cdot (3k - 6) + 2k \leq 47k$. $\qquad\square$

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