ON WEAKLY SS-QUASINORMAL SUBGROUPS OF FINITE GROUPS

Tao Zhao[1]
Gangfu Lu
Chengjun Lv

School of Science
Shandong University of Technology
Zibo, Shandong 255049
P.R. China

Abstract. Let H be a subgroup of a finite group G, then we say that H is weakly SS-quasinormal in G, if there exists a normal subgroup T of G such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G. In this paper, we investigate the influence of some weakly SS-quasinormal subgroups on the structure of G. Some new criterias about the p-nilpotency and supersolubility of a finite group were obtained. We also generalized some known results about formations.

Keywords: s-permutable subgroup, SS-quasinormal subgroup, weakly SS-quasinormal subgroup, p-nilpotent group, formation.

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1. Introduction

All groups considered in this paper will be finite and we use conventional notions and notation, as in D. Gorenstein [5]. Let $\mathcal{F}$ denote a formation, we use $\mathcal{U}$ to stand for the class of all supersoluble groups. Let H be a subgroup of G, $T \leq G$ is said to be a supplement of H in G if $HT = G$. A subgroup H of G is said to be $\mathcal{F}$-supplemented in G if there exists a subgroup $L \in \mathcal{F}$ such that $G = HL$. In this case, we say that L is an $\mathcal{F}$-supplement of H in G.

Recall that a subgroup H of G is said to be s-permutable [11] (or s-quasinormal [3]) in G, if H permutes with every Sylow subgroup P of G. Following Wang in [18], a subgroup H is c-normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G. By assuming that some subgroups of G satisfying the s-permutability or c-normality, many interesting results have been derived (see for example, [16], [1], [15], [12], [20], [22], [2], [10]). As a development, recently in [7], the concept of S-embedded subgroup

[1]Corresponding author. E-mail: zht198109@163.com

was introduced: a subgroup H is said to be S-embedded in G if there exists a normal subgroup N such that HN is s-permutable in G and $H \cap N \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H. In [7], the authors obtained that:

Theorem C and D. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that for every non-cyclic Sylow subgroup P of E (or $F^*(E)$, respectively), every maximal subgroup of P or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) is S-embedded in G. Then $G \in \mathcal{F}$.*

These two theorems generalized a lot of meaningful results. As another generalizations of the s-permutability, in [14] the authors introduced that: a subgroup H of G is said to be an SS-quasinormal subgroup (Supplement-Sylow-quasinormal subgroup) of G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. In this paper, we integrated the above concepts and introduce that:

Definition 1.1 A subgroup H of a group G is said to be weakly SS-quasinormal in G, if there exists a normal subgroup T of G such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G.

Remark. Obviously, every S-embedded subgroup and SS-quasinormal subgroup of G is weakly SS-quasinormal in G. In general, a weakly SS-quasinormal subgroup of G need not be S-embedded or SS-quasinormal in G. For instance:

Example 1. Let $G = S_5$ be the symmetric group of degree 5, $H = S_4$ and P a Sylow 5-subgroup of G. Since P is a supplement of H to G and H permutes with P, H is SS-quasinormal and thus weakly SS-quasinormal in G. But neither H nor $H \cap A_5 = A_4$ is s-permutable in G, because they are not subnormal subgroups of G. Since the only normal subgroups of G are A_5 and G itself, $H = S_4$ is not S-embedded in G.

Example 2. Let $G = S_5$, $K = \langle(12)\rangle$ and $T = A_5$. Since $T \trianglelefteq G$ is a complement of K, K is weakly SS-quasinormal in G. But the only supplement of K to G are A_5 and G itself and $K\langle(12345)\rangle \neq \langle(12345)\rangle K$, thus we know K is not SS-quasinormal in G.

In this paper, we investigate the influence of some weakly SS-quasinormal subgroups on the structure of a finite group G. Our main result is:

Main results. Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups $\mathcal{U}$. Then a group $G \in \mathcal{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and for every non-cyclic Sylow subgroup P of E (or $F^*(E)$, respectively), every maximal subgroup of P not having a supersoluble supplement in G or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) without a supersoluble supplement in G is weakly SS-quasinormal in G.

2. Preliminaries

We list here some basic results which will be used in the sequel.

Lemma 2.1 ([11]) *Suppose that H is an s-permutable subgroup of G, $K \leq G$ and $N \trianglelefteq G$. Then the following statements hold:*

(1) *If $K \leq G$, then $H \cap K$ is s-permutable in K.*

(2) *HN and $H \cap N$ are s-permutable in G, HN/N is s-permutable in G/N.*

(3) *H is subnormal in G.*

(4) *If H is a p-group for some prime p, then $N_G(H) \geq O^p(G)$.*

Lemma 2.2 ([14, Lemma 2.1]) *Suppose that H is SS-quasinormal in a group G, $K \leq G$ and N is a normal subgroup of G. Then*

(1) *If $H \leq K$, then H is SS-quasinormal in K.*

(2) *HN/N is SS-quasinormal in G/N.*

Lemma 2.3 ([14, Lemma 2.2]) *Let P be a p-subgroup of G, where p is a prime. Then the following statements are equivalent:*

(1) *P is s-permutable in G.*

(2) *$P \leq O_p(G)$ and P is SS-quasinormal in G.*

Lemma 2.4 ([14, Lemma 2.5]) *If a p-subgroup P of G is SS-quasinormal in G, then P permutes with every Sylow q-subgroup of G with $q \neq p$.*

Now, we can prove that:

Lemma 2.5 *Suppose that H is weakly SS-quasinormal in a group G, $K \leq G$ and $N \trianglelefteq G$.*

(1) *If $H \leq K$, then H is weakly SS-quasinormal in K.*

(2) *If $N \leq H$, then H/N is weakly SS-quasinormal in G/N.*

(3) *Let π be a set of primes, H a π-subgroup and N a normal π'-subgroup of G. Then HN/N is weakly SS-quasinormal in G/N.*

(4) *If $K \trianglelefteq G$ and $H \leq K$, then G has a normal subgroup L contained in K such that HL is s-permutable and $H \cap L$ is SS-quasinormal in G.*

(5) *If $H \leq O_p(G)$, then H is S-embedded in G.*

Proof. By hypothesis, there exists a normal subgroup T of G such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G.

(1) First, we have $K \cap T \trianglelefteq K$. By Lemma 2.1(1) and Lemma 2.2(1), we can see that $H(K \cap T) = K \cap HT$ is s-permutable and $H \cap (K \cap T) = H \cap T$ is SS-quasinormal in K, respectively. Hence H is weakly SS-quasinormal in K.

(2) Clearly, we have $TN/N \trianglelefteq G/N$, $(H/N)(TN/N) = HT/N$ is s-permutable in G/N and $(H/N) \cap (TN/N) = (H \cap TN)/N = (H \cap T)N/N$. By Lemma 2.2(2), $(H \cap T)N/N$ is SS-quasinormal in G/N. Hence H/N is weakly SS-quasinormal in G/N.

(3) It is easy to see that $TN/N \trianglelefteq G/N$, $(HN/N)(TN/N) = HTN/N$ is s-permutable in G/N. Since H is a π-group and N a π'-group,

$$|H \cap TN| = \frac{|H| \cdot |TN|_\pi}{|HTN|_\pi} = \frac{|H| \cdot |T|_\pi}{|HT|_\pi} = |H \cap T|_\pi = |H \cap T|.$$

This implies that $H \cap TN = H \cap T$, so $(HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap TN)N/N = (H \cap T)N/N$ which is SS-quasinormal in G/N by Lemma 2.2(2). Hence HN/N is weakly SS-quasinormal in G/N.

(4) Let $L = K \cap T$, then it is easy to see that $L \trianglelefteq G$, $HL = K \cap HT$ is s-permutable in G and $H \cap L = H \cap T$ is SS-quasinormal in G.

(5) From Lemma 2.3, it is clear. ∎

Lemma 2.6 *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.*

(1) *If N is normal in G of order p, then N lies in $Z(G)$.*
(2) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent.*
(3) *If M is a subgroup of G with index p, then M is normal in G.*

Lemma 2.7 ([4, A, Lemma 1.2]) *Let U, V and W be subgroups of a group G. Then the following statements are equivalent:*

(1) *$U \cap VW = (U \cap V)(U \cap W)$;*
(2) *$UV \cap UW = U(V \cap W)$.*

Lemma 2.8 ([17, Lemma 2.20]) *Let A be a p'-group of automorphisms of the p-group P of odd order. Assume that every subgroup of P with prime order is A-invariant, then A is cyclic.*

3. Main results

Theorem 3.1 *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P not having a p-nilpotent supplement in G, or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) without a p-nilpotent supplement in G is weakly SS-quasinormal in G, then G is p-nilpotent.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. We treat with the following two cases:

Case 1. Every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_{\infty}(G)$) without a p-nilpotent supplement in G is weakly SS-quasinormal in G.

Let K be a proper subgroup of G and $P_0 \in Syl_p(K)$. Then there exists some $x \in G$ such that $P_0^x \leq P$. Since K is p-nilpotent if and only if K^x is p-nilpotent. Without loss of generality, we may assume that $P_0 \leq P$. Since $Z_{\infty}(G) \cap K \leq Z_{\infty}(K)$, by hypothesis and Lemma 2.5(1) every cyclic subgroup H of P_0 with prime order or order 4 (if P_0 is a non-abelian 2-group and $H \nsubseteq Z_{\infty}(K)$) is weakly SS-quasinormal in K. Thus K is p-nilpotent by induction. Therefore, G is a minimal non-p-nilpotent group. Then we have: (i) $G = [P]Q$, where P is a normal Sylow p-subgroup and Q a non-normal cyclic Sylow q-subgroup of G; (ii) $P/\Phi(P)$ is a chief factor of G; (iii) the exponent of P is p or 4.

Let $X/\Phi(P)$ be a minimal subgroup of $P/\Phi(P)$, then there exists $x \in X \backslash \Phi(P)$ such that $X/\Phi(P) = \langle x \rangle \Phi(P)/\Phi(P)$ and $|\langle x \rangle| = p$ or 4. If $\langle x \rangle$ has a p-nilpotent supplement B in G, then $B\Phi(P)/\Phi(P)$ is a p-nilpotent supplement of $X/\Phi(P)$ in $G/\Phi(P)$ and $|G/\Phi(P) : B\Phi(P)/\Phi(P)| \leq p$. Thus we have $B\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ and the normal p-complement of $B\Phi(P)/\Phi(P)$ is also a normal p-complement of $G/\Phi(P)$. Since the class of all p-nilpotent groups formed a saturated formation and $\Phi(P) \leq \Phi(G)$, G is p-nilpotent, which is a contradiction. Therefore, by hypothesis either $\langle x \rangle \subseteq Z_{\infty}(G)$ or $\langle x \rangle$ is weakly SS-quasinormal in G. In the former case, we have $P \cap Z_{\infty}(G) \nsubseteq \Phi(P)$. Then by the fact that $P/\Phi(P)$ is a chief factor of G, we have $(P \cap Z_{\infty}(G))\Phi(P) = P$ and hence $P \leq Z_{\infty}(G)$. In this case, it is easy to see that G is nilpotent, which is a contradiction. Next, we suppose that $\langle x \rangle$ is weakly SS-quasinormal in G. By Lemma 2.5(4), there are some s-permutable subgroup C and normal subgroup T of G such that $\langle x \rangle T = C \leq P$ and $\langle x \rangle \cap T$ is SS-quasinormal in G. If $X/\Phi(P)$ is s-permutable in $G/\Phi(P)$, then we can easily deduce that $X/\Phi(P) \trianglelefteq G/\Phi(P)$ since $P/\Phi(P)$ is a chief factor of G. Therefore, $P/\Phi(P) = X/\Phi(P)$ is a cyclic group. Hence P is cyclic and G is p-nilpotent. This contradiction shows that $X/\Phi(P)$ is not s-permutable in $G/\Phi(P)$. Therefore, $\langle x \rangle$ is not s-permutable in G. Since $\langle x \rangle \cap T$ is s-permutable in G by Lemma 2.3, we have $1 < T < P$. Hence $T\Phi(P) \neq P$, which implies that $T \leq \Phi(P)$. But then $X/\Phi(P) = \langle x \rangle \Phi(P)/\Phi(P) = \langle x \rangle T\Phi(P)/\Phi(P) = C\Phi(P)/\Phi(P)$ is an s-permutable subgroup of $G/\Phi(P)$, a contradiction.

Case 2. Every maximal subgroup of P not having a p-nilpotent supplement in G is weakly SS-quasinormal in G.

In this case, we break the proof into the following six steps:

(1) P is not cyclic and every maximal subgroup of P has no p-nilpotent supplement in G.

By Lemma 2.6(2), we may assume that P is not cyclic. Suppose that H is a maximal subgroup of P which has a p-nilpotent supplement T in G, we prove that G is p-nilpotent. If not, let K be a non-p-nilpotent subgroup of G which contains P and is such that every proper subgroup of K is p-nilpotent. Then by [8, IV, Theorem 5.4], K is a minimal non-nilpotent group and the following hold: (i) $K = [P]K_q$, where P is a normal Sylow p-subgroup and K_q a cyclic Sylow q-subgroup of K; (ii) $P/\Phi(P)$ is a chief factor of K.

Since $G = HT$, $K = K \cap HT = H(K \cap T)$. The facts $K \cap T \leq T$ is p-nilpotent but K is not p-nilpotent implies that $L = K \cap T$ is a proper subgroup of K. Hence L is nilpotent. Let $L = L_p \times L_q$. Obviously, L_q is also a Sylow q-subgroup of K. Since $P = HL_p$, L_p is not contained in $\Phi = \Phi(P)$. Now we consider the factor group K/Φ. The fact $L_q \leq N_K(L_p)$ implies that $L_q\Phi/\Phi \leq N_{K/\Phi}(L_p\Phi/\Phi)$. On the other hand, since P/Φ is an elementary abelian group, we have $L_p\Phi/\Phi \trianglelefteq P/\Phi$. Hence $L_p\Phi/\Phi \trianglelefteq \langle L_q\Phi/\Phi, P/\Phi\rangle = K/\Phi$. Since $L_p\Phi/\Phi \neq 1$ and P/Φ is a chief factor of K, $L_p\Phi/\Phi = P/\Phi$. It follows that $L_p = P$. Consequently, we get that $L = K$. This contradiction completes the proof of (1).

(2) G is not a non-abelian simple group.

Assume that G is a non-abelian simple group. Let P_1 be a maximal subgroup of P, by (1) we know P_1 is weakly SS-quasinormal in G. Then there exists a normal subgroup T of G such that P_1T is s-permutable and $P_1 \cap T$ is SS-quasinormal in G. Note that $T = 1$ or G since G is a simple group. If $T = 1$, then $P_1 = P_1T$ is s-permutable in G. Hence P_1 is a proper subnormal subgroup of G, a contradiction. Thus $T = G$ and therefore $P_1 = P_1 \cap T$ is SS-quasinormal in G. Then there exists some supplement B of P_1 such that P_1 permutes with every Sylow subgroup of B. From $G = P_1B$, we know $|B : P_1 \cap B|_p = |G : P_1|_p = p$. Hence $P_1 \cap B$ is of index p in B_p, a Sylow p-subgroup of B containing $P_1 \cap B$. Thus $S \nsubseteq P_1$ for all $S \in Syl_p(B)$ and $P_1S = SP_1$ is a Sylow p-subgroup of G. By comparison of orders, we know that $S \cap P_1 = B \cap P_1$ holds for each $S \in Syl_p(B)$. So $B \cap P_1 = \bigcap_{b\in B}(S^b \cap P_1) \leq \bigcap_{b\in B} S^b = O_p(B)$. Since $|O_p(B) : B \cap P_1| = p$ or 1, $|B/O_p(B)|_p = p$ or 1. Then by Lemma 2.6(2), $B/O_p(B)$ is p-nilpotent, and so B is p-soluble. Hence B has a Hall p'-subgroup K. It is clear that K is also a Hall p'-subgroup of G. Thus, P_1 permutes with every Sylow subgroup of K and so P_1K is a subgroup of G. Since $|G : P_1K| = p$, P_1K is normal in G by Lemma 2.6(3), which is a contradiction. Therefore, G is not a non-abelian simple group.

(3) G has a unique minimal normal subgroup N, G/N is p-nilpotent and $\Phi(G)$=1.

Let N be a minimal normal subgroup of G. By Lemma 2.6(2), we may assume that PN/N is non-cyclic and $|PN/N| \geq p^2$. Let M/N be a maximal subgroup of PN/N, then $M = P_1N$ for some maximal subgroup P_1 of P and $P \cap N = P_1 \cap N \in Syl_p(N)$. By (1), we know P_1 is weakly SS-quasinormal in G. Then there exists a normal subgroup T of G such that P_1T is s-permutable and $P_1 \cap T$ is SS-quasinormal in G. Clearly, TN/N is a normal subgroup of G/N and $P_1N/N \cdot TN/N = P_1TN/N$ is s-permutable in G/N. Moreover, since $P_1 \cap N$ is a Sylow

p-subgroup of N, $|(P_1 \cap N)(T \cap N)|_p = |P_1 \cap N| = |N|_p = |N \cap P_1T|_p$. Since P_1 is a p-group,

$$|N \cap P_1T|_{p'} = \frac{|N|_{p'} \cdot |P_1T|_{p'}}{|NP_1T|_{p'}} = \frac{|N|_{p'} \cdot |T|_{p'}}{|NT|_{p'}} = |N \cap T|_{p'} = |(P_1 \cap N)(T \cap N)|_{p'}.$$

This implies that $(N \cap P_1)(N \cap T) = N \cap P_1T$. Thus by Lemma 2.7, we have $P_1N \cap TN = (P_1 \cap T)N$. It follows from Lemma 2.2(2) that $P_1N/N \cap TN/N = (P_1 \cap T)N/N$ is SS-quasinormal in G/N. Hence M/N is weakly SS-quasinormal in G/N. Therefore, G/N satisfies the hypothesis and so it is p-nilpotent by the minimal choice of G. Since the class of all p-nilpotent groups formed a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$.

(4) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ and $G/O_{p'}(G)$ is p-nilpotent by (3). Hence G is p-nilpotent, a contradiction.

(5) $O_p(G) = 1$ and N is not p-nilpotent.

If $O_p(G) \neq 1$, then $N \leq O_p(G)$. Since $\Phi(G) = 1$, G has a maximal subgroup M such that $G = [N]M$. Since $O_p(G) \leq F(G) \leq C_G(N)$ and $C_G(N) \cap M \trianglelefteq G$, the uniqueness of N yields that $N = O_p(G)$. Since $P = N(P \cap M)$ and $N \cap M = 1$, $P \cap M$ is a Sylow p-subgroup of M and there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$ and $P = NP_1$. By (1), P_1 is weakly SS-quasinormal in G. Then G has a normal subgroup T such that P_1T is s-permutable in G, and there exists some supplement B of $P_1 \cap T$ to G such that $(P_1 \cap T)B_q = B_q(P_1 \cap T)$ for any $B_q \in Syl_q(B)$. If $T = 1$, then $P_1 = P_1T$ is s-permutable in G. It follows that $P_1 \leq O_p(G) = N$ and so $P = P_1N = N$ is a minimal normal subgroup of G. Since $N_G(P_1) \geq O^p(G)$ by Lemma 2.1(4) and $P_1 \trianglelefteq P$, P_1 is a proper normal subgroup of G contained in $P = O_p(G)$, a contradiction. Thus we have $T \neq 1$ and then $N \leq T$. In this case, $N \cap P_1 = N \cap P_1 \cap T = N \cap (P_1 \cap T)B_q \trianglelefteq (P_1 \cap T)B_q$ for any $B_q \in Syl_q(B)$ with $q \neq p$. Hence $B_q \leq N_G(N \cap P_1)$ holds for any $q \neq p$. Since $N \cap P_1 \trianglelefteq P$, it is normal in G. Thus $N \cap P_1 = 1$ and $|N| = p$. By Lemma 2.6(1), $N \leq Z(G)$. Since G/N is p-nilpotent, G is also p-nilpotent, a contradiction.

If N is p-nilpotent, then $N_{p'}$ *char* $N \trianglelefteq G$, so $N_{p'} \leq O_{p'}(G) = 1$ by (4). Thus N is a p-group and hence $N \leq O_p(G) = 1$, a contradiction.

(6) The final contradiction.

Since N is not soluble, $N = S_1 \times S_2 \times \cdots \times S_k$, where S_i are isomorphic non-abelian simple groups. Let $S_p \in Syl_p(S_1)$, we now prove that $S_p \leq P_1$ for some maximal subgroup P_1 of P. If $P \cap S_1 < P$, it is clear. If $P \leq S_1$, then by hypothesis and Lemma 2.5(1), we know that every maximal subgroup of P is weakly SS-quasinormal in S_1. With a similar argument as in (2), we can get a contradiction. Thus $S_p \leq P_1$, where P_1 is a maximal subgroup of P. Then there exists a normal subgroup T of G such that P_1T is s-permutable in G, and there is a supplement B of $P_1 \cap T$ to G such that $P_1 \cap T$ permutes with every Sylow subgroup of B.

If $T = 1$, then P_1 is s-permutable in G and so $O_p(G) \neq 1$, which contradicts with (5). Thus $T \neq 1$ and so $N \leq T$. If $P_1 \cap T = 1$, then $|T|_p \leq p$. Hence T is p-nilpotent by Lemma 2.6(2), N is also p-nilpotent. This contradiction shows that $P_1 \cap T \neq 1$. Let B_q be a Sylow q-subgroup of B, where $q \neq p$. Then

$$|B_q \cap P_1T| = \frac{|B_q| \cdot |P_1T|_q}{|B_qP_1T|_q} = \frac{|B_q| \cdot |T|_q}{|B_qT|_q} = |B_q \cap T| = |(B_q \cap P_1)(B_q \cap T)|.$$

This implies that $B_q \cap P_1T = (B_q \cap P_1)(B_q \cap T)$. Thus by Lemma 2.7, we have $B_qP_1 \cap B_qT = B_q(P_1 \cap T)$. Therefore, $N \cap P_1B_q = N \cap (P_1B_q \cap TB_q) = N \cap (P_1 \cap T)B_q$. Then we can conclude that $S_1 \cap (P_1 \cap T) = S_1 \cap P_1 = S_p$ is a Sylow p-subgroup of S_1. This means that for any prime q ($\neq p$), $S_1 \cap (P_1 \cap T)B_q$ is a Hall $\{p, q\}$-subgroup of S_1. Since N is non-abelian, $p = 2$. Then for any prime divisor q ($q \neq 2$) of $|S_1|$, the non-abelian simple group S_1 has a Hall $\{2, q\}$-subgroup, which contradicts with [13, Lemma 2.6]. This contradiction completes the proof of the theorem. ∎

Since a supersoluble group G is p-nilpotent for the minimal prime divisor p of $|G|$, every subgroup H of G not having a p-nilpotent supplement in G also has no supersoluble supplement in G. Thus, from Theorem 3.1 we can easily deduce that:

Corollary 3.2 *Let P be a Sylow p-subgroup of a group G, where $p = min\pi(G)$. If every maximal subgroup of P not having a supersoluble supplement in G, or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) without a supersoluble supplement in G is weakly SS-quasinormal in G, then G is p-nilpotent.*

Next, by using the weakly SS-quasinormal properties of some subgroups, we give out some new criteria for the supersolubility of a group G.

Theorem 3.3 *Let $\mathcal{F}$ be a saturated formation containing the class of all supersoluble groups $\mathcal{U}$. Then a group $G \in \mathcal{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and for every non-cyclic Sylow subgroup P of E, every maximal subgroup of P having no supersoluble supplement in G or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) without a supersoluble supplement in G is weakly SS-quasinormal in G.*

Proof. We need to prove only the sufficiency. Suppose that the result is false and consider a counterexample (G, E) for which $|G||E|$ is minimal. Let p be the smallest prime divisor of $|E|$ and P a Sylow p-subgroup of E. Then

(1) E is p-nilpotent.

We may assume that P is not cyclic. Since $Z_\infty(G) \cap E \leq Z_\infty(E)$ and every subgroup of E having no supersoluble supplement in E also has no supersoluble supplement in G. By hypothesis and Lemma 2.5(1), we know that every maximal subgroup of P having no supersoluble supplement in E or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nleq Z_\infty(E)$) without a supersoluble supplement in E is weakly SS-quasinormal in E. Thus Corollary 3.2 implies that E is p-nilpotent.

(2) $E = P$ is not cyclic.

By (1), E is p-nilpotent. Suppose that $P < E$ and let T be a non-trivial normal p-complement of E. Then $T \trianglelefteq G$ and from Lemma 2.5(3), we can easily deduce that the hypothesis holds for G/T (with respect to E/T). Hence $G/T \in \mathcal{F}$ by the choice of G. This implies that the hypothesis is still true for (G, T). Thus $T = E$ by the choice of (G, E), a contradiction. Hence $P = E$. Since $G/E \in \mathcal{F}$, by [17, Lemma 2.16] we may suppose that E is not cyclic.

(3) Every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nleq Z_\infty(G)$) having no supersoluble supplement in G is weakly SS-quasinormal in G.

Suppose that every maximal subgroup of P having no supersoluble supplement in G is weakly SS-quasinormal in G. We first prove that $P = G^{\mathcal{F}}$ is a minimal normal subgroup of G. Indeed, let N be a minimal normal subgroup of G contained in P. By Lemma 2.5, the hypothesis holds for G/N and so $G/N \in \mathcal{F}$ by the minimal choice of G. This implies that N is the only minimal normal subgroup of G contained in P and $N \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $G = [N]M$. Then $P = P \cap NM = N(P \cap M)$. Since $P \leq F(G) \leq C_G(N)$, $P \cap M$ is normal in G and hence $P \cap M = 1$. It follows that $P = N = G^{\mathcal{F}}$ is a minimal normal subgroup of G and $|P| > p$. Let P_1 be a maximal subgroup of $P = N$. If P_1 has a supersoluble supplement K in G, then $PK = G$ and $1 \neq P \cap K \trianglelefteq G$. Thus $P \cap K = P$ and so $G = K$ is supersoluble, which is a contradiction. Next, we assume that every maximal subgroup P_1 of P is weakly SS-quasinormal in G. Then by hypothesis and Lemma 2.5(4), G has a normal subgroup T contained in P such that P_1T is s-permutable and $P_1 \cap T$ is SS-quasinormal in G. Since P is a minimal normal subgroup of G, $T = 1$ or $T = P$. If $T = 1$, then $P_1 = P_1T$ is s-permutable in G. If $T = P$, then $P_1 = P_1 \cap T$ is SS-quasinormal in G. By Lemma 2.3, we can also conclude that P_1 is s-permutable in G. By Lemma 2.1(4), we have $O^p(G) \leq N_G(P_1)$. This means that for any maximal subgroup P_1 of P, we have $|G : N_G(P_1)| = p^a$ for some integer a. Let $M_1, M_2, \cdots, M_t$ be the set of all maximal subgroups of P. Then p divides t, which contradicts with [8, III, 8.5(d)].

(4) The final contradiction.

By (3), we know that every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) having no supersoluble supplement in G is weakly SS-quasinormal in G. By the choice of (G, E), we know $P = G^{\mathcal{F}}$. Let M be any maximal subgroup of G not containing P, then $M/M \cap P \cong G/P \in \mathcal{F}$. Hence by Lemma 2.5, the hypothesis holds for M. Then the minimal choice of G implies that $M \in \mathcal{F}$. This shows that every maximal subgroup of G not containing P belongs to $\mathcal{F}$. Thus by [6, Theorem 3.4.2], the following statements hold:

(i) $P/\Phi(P)$ is a G-chief factor of P;

(ii) P is a group of exponent p or exponent 4 (if $p = 2$ and P is non-abelian);

(iii) if P is abelian, then $\Phi(P) = 1$.

If every minimal subgroup of $P/\Phi(P)$ is s-permutable in $G/\Phi(P)$, then every maximal subgroup of $P/\Phi(P)$ is s-permutable in $G/\Phi(P)$. With a similar argument as in the proof of Theorem 3.1 case 1, we can get a contradiction. Now we choose $X/\Phi(P)$ to be a minimal subgroup of $P/\Phi(P)$ which is not s-permutable in $G/\Phi(P)$. Pick an $x \in X \backslash \Phi(P)$ and let $L = \langle x \rangle$, then $|L| = p$ or 4. If L has a supersoluble supplement K in G, then $P = P \cap LK = L(P \cap K)$ and $(P \cap K)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. If $P \cap K \leq \Phi(P)$, then $P = L$ is cyclic, which contradicts (2). If $P \cap K = P$, then $G = K$ is supersoluble, which is a contradiction too. Thus by hypothesis either $L \subseteq Z_\infty(G)$ or L is weakly SS-quasinormal in G. If $L \subseteq Z_\infty(G)$, then $P \cap Z_\infty(G) \nsubseteq \Phi(P)$ and so $(P \cap Z_\infty(G))\Phi(P) = P$, i.e., $P \leq Z_\infty(G)$. Therefore, from (ii) we obtain that $|P/\Phi(P)| = p$ and so P is a cyclic group, which contradicts with (2). Now suppose that L is weakly SS-quasinormal in G. Since $X/\Phi(P)$ is not s-permutable in $G/\Phi(P)$, L is not s-permutable in G. Thus by Lemma 2.5(4), there exists a non-identity normal subgroup T of G contained in P such that LT is s-permutable and $L \cap T$ is SS-quasinormal in G. Since L is not s-permutable in G, by Lemma 2.3 it is clear that $T \neq P$. Hence $T\Phi(P) \neq P$, which implies that $T \leq \Phi(P)$. But then $X/\Phi(P) = L\Phi(P)/\Phi(P) = LT\Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$. This final contradiction completes the proof of the theorem. ∎

From Theorem 3.3, we know that:

Corollary 3.4 *A finite group G is supersoluble if and only if G has a normal subgroup E such that G/E is supersoluble and for every non-cyclic Sylow subgroup P of E, at least one of the following holds:*

(1) *Every maximal subgroup of P either has a supersoluble supplement in G or is weakly SS-quasinormal in G.*

(2) *Every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) either has a supersoluble supplement in G or is weakly SS-quasinormal in G.*

Theorem 3.5 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Then a group $G \in \mathcal{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and for every non-cyclic Sylow subgroup P of the generalized Fitting subgroup $F^*(E)$ of E, every maximal subgroup of P not having a supersoluble supplement in G or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) without a supersoluble supplement in G is weakly SS-quasinormal in G.*

Proof. The necessity is evident and we need to prove only the sufficiency. Assume that the theorem is false and let (G, E) be a counterexample with $|G||E|$ minimal. Let $F = F(E)$, $p = min\pi(F(E))$ and $P \in Syl_p(F(E))$. Then

(1) Each subgroup of $F^*(E)$ has no supersoluble supplement in G.

If some subgroup H of $F^*(E)$ has a supersoluble supplement in G, then $F^*(E)$ also has a supersoluble supplement in G and so $G/F^*(E)$ is supersoluble. Thereby, $G/F^*(E)$ belongs to $\mathcal{F}$. Then Theorem 3.3 implies that $G \in \mathcal{F}$. Thus, we may assume that none of the subgroup of $F^*(E)$ has a supersoluble supplement in G.

(2) E is soluble, $F^*(E) = F$ and $C_E(F) = C_E(F^*) \leq F$.

By (1) and Lemma 2.5(2), we know that for every non-cyclic Sylow subgroup P of $F^*(E)$, every maximal subgroup of P or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(E)$) is S-embedded in E. Thus, we can deduce that E is soluble by [7, Theorem B]. Hence $F^*(E) = F(E) = F$. By [9, X, Theorem 13.11], we have $C_E(F) = C_E(F^*) \leq F$.

(3) Let $V/P = F(E/P)$ and $Q \in Syl_q(V)$, where $q||V/P|$. Then $q \neq p$ and either $Q \leq F$ or $p > q$ and $C_Q(P) = 1$.

Since V/P is nilpotent and $QP/P \in Syl_q(V/P)$, QP/P is a characteristic subgroup of V/P and so $QP \trianglelefteq E$. Thus $q \neq p$. By Theorem 3.3, we know QP is supersoluble. Assume that $q > p$, then Q is normal in QP and so $Q \leq F = F(E)$. If $p > q$, then $p > 2$ and F is a q'-group since p is the smallest prime divisor of $|F|$. Now let U be a Sylow r-subgroup of F, where $r \neq p$. Then $r \neq q$ and so $[U, Q] \leq P$. Assume that for some $x \in Q$ we have $x \in C_E(P)$. Since V/P is nilpotent, by [5, V, Theorem 3.6] we know $[U, \langle x\rangle] = [U, \langle x\rangle, \langle x\rangle] = 1$ and so $x \in C_E(F)$. Since $C_E(F) \leq F$ by (2), $C_Q(P) = 1$.

(4) $p > 2$.

Assume that $p = 2$, then by (3) we have $F^*(E/P) = F(E/P) = F/P$. Since $(G/P)/(E/P) \cong G/E \in \mathcal{F}$, by Lemma 2.5(3) we know that the hypothesis is still true for $(G/P, E/P)$. Therefore, $G/P \in \mathcal{F}$ by induction. Hence $G \in \mathcal{F}$ by Theorem 3.3, a contradiction.

(5) The final contradiction.

Let $V/P = F(E/P)$ and $Q \in Syl_q(V)$, where $q||V/P|$. Then by (3), either $Q \leq F$ or $p > q$ and $C_Q(P) = 1$. In the second case, Q is cyclic by (4) and Lemma 2.8. Hence every Sylow subgroup of $F^*(E/P) = F(E/P)$ either is cyclic or has the form QP/P, where Q is a Sylow subgroup of $F^*(E) = F$. Thus by Lemma 2.5(3), we know for each non-cyclic Sylow subgroup RP/P of $F^*(E/P)$, every maximal subgroup of RP/P or every cyclic subgroup HP/P of RP/P with prime order or order 4 (if R is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) is weakly SS-quasinormal in G/P. Therefore, $G/P \in \mathcal{F}$ by induction. It is clear that G/P satisfies the hypothesis of the theorem. Since $F^*(P) = P$, by Theorem 3.3 we have $G \in \mathcal{F}$, as required. ∎

From our Theorem 3.5, we can conclude that:

Corollary 3.6 *A finite group G is supersoluble if and only if G has a normal subgroup E such that G/E is supersoluble and for every non-cyclic Sylow subgroup P of $F^*(E)$, at least one of the following holds:*

(1) *Every maximal subgroup of P either has a supersoluble supplement in G or is weakly SS-quasinormal in G.*

(2) *Every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) either has a supersoluble supplement in G or is weakly SS-quasinormal in G.*

Remarks. Since all normal, quasinormal, s-permutable, c-normal, SS-quasinormal, nearly s-normal [19] and S-embedded subgroups of G are all weakly SS-quasinormal in G, our theorems 3.3 and 3.5 generalized many meaningful results.

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