The Asymptotic Justification of a Nonlocal 1-D Model Arising in Porous Catalyst Theory*

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An asymptotic model of isothermal catalyst is obtained from a well-known model of porous catalyst for appropriate, realistic limiting values of some nondimensional parameters. In this limit, the original model is a singularly perturbed m-D reaction–diffusion system. The asymptotic model consists of an ordinary differential equation coupled with a semilinear parabolic equation on a semi-infinite one-dimensional interval.

1. INTRODUCTION

This paper deals with a well-known model of porous catalyst that after suitable nondimensionalization [1, Vol. I] may be written as

$$\partial u/\partial t = \Delta u - \phi^2 f(u, v) \quad \text{in } \Omega, \qquad \partial u/\partial n = \sigma(1-u) \quad \text{at } \partial\Omega, \tag{1.1}$$

$$L^{-1}\,\partial v/\partial t = \Delta v + \beta\phi^2 f(u, v) \quad \text{in } \Omega, \qquad \partial v/\partial n = \nu(1-v) \quad \text{at } \partial\Omega, \tag{1.2}$$

for $t > 0$, with appropriate initial conditions

$$u = u_0 > 0, \qquad v = v_0 > 0 \qquad \text{in } \Omega, \qquad \text{at} \quad t = 0. \tag{1.3}$$

Here $u > 0$ and $v > 0$ are the reactant concentration and the temperature respectively, Δ is the Laplacian operator, n is the outward unit normal to the smooth boundary of the bounded domain $\Omega \subset \mathbb{R}^m$ (with $m \geqslant 1$) and the parameters ϕ^2 (*Damköhler number*), L (*Lewis number*), β (*Prater number*), σ, and ν (*material and thermal Biot numbers*) are strictly positive. The nonlinearity f accounts for the reaction rate and is usually of one of the following forms, that are associated with the so-called *Arrhenius* and *Langmuir–Hinshelwood* kinetic laws [1],

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$$f(u, v) = u^p \exp(\gamma - \gamma/v) \tag{1.4}$$

$$f(u, v) = u^p (u + k)^{-q} \exp(\gamma - \gamma/v) \tag{1.5}$$

$$f(u, v) = u^p [u + k \exp(\gamma_a - \gamma_a/v)]^{-q} \exp(\gamma - \gamma/v) \tag{1.6}$$

where the *reaction orders* p and q and the *activation energies* γ and γ_a are strictly positive. The reaction orders may be non-integers.

Porous catalysts usually exhibit a large thermal conductivity and consequently β is usually very small, the ratio σ/ν is large and ν is either small or of order unity, depending on the size of the catalyst (see [1]). In addition, L and ϕ^2 vary in a wide range (from small to large values). Then, the limit

$$\beta \to 0, \qquad \sigma/\nu \to \infty \tag{1.7}$$

is realistic, and leads to simpler submodels than (1.1)–(1.3). If, in addition, ν is small and ϕ^2 remains bounded, then the following simpler sub-model is obtained from (1.1)–(1.3)

$$\partial u/\partial t = \Delta u - \phi^2 f(u, v), \qquad \partial u/\partial n = \sigma(1 - u) \qquad \text{at } \partial\Omega, \tag{1.8}$$

$$(V_\Omega/\nu L)\, dV/dt = S_\Omega(1 - V) + (\beta\phi^2/\nu) \int_\Omega f(u, v)\, dx, \tag{1.9}$$

with appropriate initial conditions, where V_Ω and S_Ω are the measures of the domain Ω and of its boundary respectively, i.e.,

$$V_\Omega = \int_\Omega dx, \qquad S_\Omega = \int_{\partial\Omega} ds \quad \text{if} \quad m \geqslant 2, \qquad S_\Omega = 2 \quad \text{if} \quad m = 1, \tag{1.10}$$

and V is the spatial average of the temperature v, i.e.,

$$V = V_\Omega^{-1} \int_\Omega v(x, t)\, dx. \tag{1.11}$$

The model (1.8)–(1.9) was obtained in [2] by means of formal, singular perturbation techniques. For a rigorous derivation of a slightly different model (namely, the boundary conditions in (1.1) and (1.8) being replaced by new ones of the Dirichlet type) see [3]. For the rigorous derivation of related simplified sub-models of general reaction-diffusion systems, see [4–7]. As a by-product of the results below, a fairly direct derivation of (1.8)–(1.9) could be readily obtained by means of the ideas in this paper; but for the sake of brevity we shall omit that derivation. The steady states of (1.8)–(1.9) and their linear stability were analyzed in [2] for the particular case when f is as given in (1.4) with $p = 1$; some global stability properties for more general, smooth nonlinearities were obtained in [8], and the steady states for some non-Lipschitzian nonlinearities were analyzed [9].

If ϕ^2 is large and ν is small then the limit (1.7) is much more interesting because the original model (1.1)–(1.2) is *singularly perturbed.* We shall consider the case when $\phi^2 \to \infty$ but $\beta\phi$ is appropriately small. In this limit, the following sub-model of (1.1)–(1.3) applies

$$\partial \bar{U}/\partial \tau = \partial^2 \bar{U}/\partial \xi^2 - f(\bar{U}, V) \quad \text{in} \quad -\infty < \xi < 0, \tag{1.12}$$

$$u \to 0 \quad \text{as} \quad \xi \to -\infty, \qquad \partial u/\partial \xi = (\sigma/\phi)(1-u) \quad \text{at} \quad \xi = 0, \tag{1.13}$$

$$(V_\Omega \phi^2/(S_\Omega \nu L))\, dV/d\tau = 1 - V + (\beta\phi/\nu) \int_{-\infty}^{0} f(\bar{U}, V)\, d\xi, \tag{1.14}$$

with appropriate initial conditions, where V_Ω, S_Ω and V are given again by (1.10)–(1.11) and $\bar{U}$ is appropriatetely close to u. The new rescaled variables τ and ξ are

$$\tau = \phi^2 t, \qquad \xi = \phi\eta, \tag{1.15}$$

where η is a co-ordinate along the outward unit normal to $\partial\Omega$. Let us now briefly explain (in loose, physical terms, but following the main ideas in the derivation below) where this model comes from. Since ϕ^2 is large, the chemical reaction is very strong and, after some time, the reactant is consumed and u becomes very small in Ω except in a thin *boundary layer* near the boundary of Ω. Since, in addition, $\beta\phi$ and ν are small, the temperature v becomes spatially constant (in first approximation) after some time. Finally, if $f_u > 0$, after some time, the reactant concentration in the boundary layer depends only on time and on the distance to the boundary of Ω (and not on transversal co-ordinates along $\partial\Omega$ if the spatial dimension m is greather than one) in first approximation. Then (1.12)–(1.13) gives the evolution of u in the boundary layer, and (1.14) provides the spatially averaged temperature in first approximation.

Notice that the sub-model (1.12)–(1.14) consists of a 1-D semilinear PDE coupled with an ODE and thus is much simpler than the original model (1.1)–(1.2); in particular, the sub-model is independent of the shape of the domain Ω (it depends only on the overall quantities V_Ω and S_Ω). A formal derivation of this sub-model, based on singular perturbation techniques, was given in [2], along with the analysis of the steady states, their linear stability and local Hopf bifurcation, for the particular case when the nonlinearity f is as given in (1.4), with $p = 1$.

If ϕ^2 is large but ν is no longer small, then the temperature does not become spatially constant after some time and a third sub-model is obtained that consists of the m-D heat equation with appropriate nonlinear boundary conditions, coupled with infinitely many 1-D semilinear equations (one for each point of $\partial\Omega$). This non-standard sub-model was derived in [10] via formal, singular perturbation techniques and will be rigorously justified elsewhere [11]. Besides its intrinsic mathematical interest, this sub-model exhibits a large variety of codimension two and three bifurcations that predict interesting dynamic behaviors (see [10]).

The main object of this paper is to provide a rigorous derivation of (1.12)–(1.14). More precisely, we shall prove that, after some time T, (i) u is very small except in a thin boundary layer and v is spatially constant in first approximation, and (ii) the concentration in the boundary layer and the averaged temperature satisfy (1.12)–(1.14), in first approximation, *uniformly* in $t \geqslant T$.

Let us now state precisely the assumptions to be made below. We shall consider the limit

$$\phi \to \infty, \qquad \beta\phi\sigma/(\phi+\sigma) \to 0, \qquad \nu \to 0 \qquad \text{and} \qquad \sigma^{-1} = O(1). \quad (1.16)$$

The domain Ω and the nonlinearity f will be assumed to be such that

(H.1) $\Omega \subset \mathbb{R}^m$ ($m \geqslant 1$) is a *bounded* domain, with a *connected*, $C^{4+\alpha}$ (for some $\alpha > 0$) boundary if $m \geqslant 2$. Notice that then Ω satisfies *uniformly* the interior and exterior sphere conditions: there are two constants, $\rho_1 > 0$ and $\rho_2 > 0$, such that for every point x of $\partial\Omega$, two hyperspheres, of radii ρ_1 and ρ_2, S_1 and S_2, are tangent to $\partial\Omega$ at x and satisfy $S_1 \subset \Omega$ and $\bar{S}_2 \cap \bar{\Omega} = \{x\}$ (overbars stand for the closure).

(H.2) The C^1-function f: $[0, \infty[\times [0, \infty[\to \mathbb{R}$ is such that $f(0, v) = 0$ for all $v \geqslant 0$ and $f(u\, v) > 0$ for all $u > 0$ and all $v \geqslant 0$.

(H.3) There is a continuous, increasing function, g_1: $[0, \infty[\to \mathbb{R}$ such that

$$f(u, v) \leqslant g_1(u) \qquad \text{if} \quad u \geqslant 0 \qquad \text{and} \qquad v \geqslant 0.$$

(H.4) There are two strictly positive constants, k_1 and k_2, and a positive, continuous, decreasing function, g_2: $[0, \infty[\to \mathbb{R}$, such that

$$k_2 u \leqslant f(u, v) \leqslant k_1 u \qquad \text{if} \quad 0 \leqslant u \leqslant 2 \qquad \text{and} \qquad v \geqslant 1/2,$$

$$u g_2(u) \leqslant f(u, v) \qquad \text{if} \quad u \geqslant 0 \qquad \text{and} \qquad v \geqslant 1/2,$$

(H.5) There are three constants, $k_3 > 0$, $k_4 > 0$ and $k_5 > 0$, such that

$$k_3 \leqslant f_u(u, v) \leqslant k_4, \quad |f_v(u, v)| \leqslant k_5 u \text{ if } 0 \leqslant u \leqslant \sigma/(\sigma + \phi\sqrt{k_2/2m}) \text{ and } v \geqslant 1.$$

In addition, the initial conditions (1.3) will be assumed to be such that

(H.6) $\|u_0\|_{C(\bar{\Omega})} = O(1)$ and $\|v_0\|_{C(\bar{\Omega})} = O(1)$ in the limit (1.16).

Notice that the non-linearities (1.4)–(1.6) satisfy (**H**.2) for $p \geqslant 1$ and all (positive) values of the remaining parameters; our results below do not apply (and are not straightforwardly extended) to non-Lipschitzian non-linearities, such as (1.4)–(1.6) if $p < 1$, that are also of practical interest. Assumption (**H**.4) is satisfied by (1.4)–(1.6) only if $p = 1$. Now the restriction is purely technical; if the inequalities in (H.4) are replaced by $k_2 u^p \leqslant f(u, v) \leqslant k_1 u^p$ and $u^p g_2 \leqslant f(u, v)$ with $p > 1$, then our results below still apply after some (unfortunately, not always obvious) changes, but we do not pursue this extension for the sake of brevity. The first inequality in assumption (**H**.5) (namely, $f_u(u, v) \geqslant k_3$) is essential in our derivation below; although we have some reasons to believe that the model (1.4)–(1.6) should still apply without this restriction, we do not see how to eliminate it completely (we are only able to replace it by $k_3 u^{p-1} \leqslant f_u(u, v)$ with $p > 1$, but even this small extension requires additional technicalities that are again omitted for the sake of brevity). The remaining restrictions in (**H**.5) are clearly satisfied by the nonlinearities (1.4)–(1.6) for all (positive) values of the parameters.

To end up this section let us state the main result of this paper, which is proved at the end of Section 2.

THEOREM 1.1. *Under the assumptions* (**H**.1)–(H.6), *there are two constants,* $\lambda > 0$ *and* $\varepsilon > 0$, *and for each solution of* (1.1)–(1.3) *there is a solution of*

$$\partial \bar{U}/\partial t = \partial^2 \bar{U}/\partial \eta^2 - \phi^2 f(\bar{U}, V) \qquad \text{in} \quad -\infty < \eta < 0, \tag{1.17}$$

$$\bar{U} = 0 \qquad \text{at} \quad \eta = -\infty, \qquad \partial \bar{U}/\partial \eta = \sigma(1 - \bar{U}) \qquad \text{at} \quad \eta = 0, \tag{1.18}$$

$$(V_\Omega / S_\Omega L)\, dV/dt = \nu(1 - V) + \beta \phi^2 \int_{-\infty}^{0} f(\bar{U}, V)\, d\eta + \psi(t), \tag{1.19}$$

and a constant $\bar{T} > 0$ *such that*

(i) λ *depends only on the domain* Ω, ε *depends only on* Ω *and on the quantities*

$$\phi,\ \sigma,\ L,\ \beta \text{ and } \nu, \tag{1.20}$$

and $\bar{T}$ *depends only on* Ω, *on the quantities,* (1.20) *and on*

$$\|u_0\|_{C(\bar{\Omega})}, \quad \|v_0\|_{C(\bar{\Omega})}. \tag{1.21}$$

(ii) *ε and $\bar{T}$ are such that*

$$\varepsilon = O(\beta\phi\sigma/(\phi+\sigma)),$$

$$\bar{T} = O(\phi^{-1}+L^{-1})\log\left(\frac{1}{\varepsilon+\nu}\right) + O(L^{-1})\log\left(2+\frac{\phi}{\sigma}+\frac{\varepsilon}{\nu(\varepsilon+\nu)}\right)$$
$$+ O((\nu L)^{-1})\log\left(2+\frac{\nu}{\varepsilon}+\frac{\beta\phi^2}{\nu}\right), \tag{1.22}$$

in the limit (1.16).

(iii) *For all $t \geqslant \bar{T}$ we have*

$$|\bar{U}(-d(x), t) - u(x, t)| \leqslant [\sigma(\varepsilon+\nu)/(\phi+\sigma)] \exp[-\lambda\phi d(x)]$$
$$\text{if } d(x) < \rho_1/2, \tag{1.23}$$

$$|V(t) - v(x, t)| \leqslant \varepsilon + \nu \quad \text{if } x \in \Omega, \qquad |\psi(t)| \leqslant (\varepsilon+\nu)^2 + \varepsilon/\phi^2, \tag{1.24}$$

where $d(x)$ is the distance from x to $\partial\Omega$ and ρ_1 is as defined in assumption (H.1).

2. MATHEMATICAL DERIVATION OF THE APPROXIMATE MODEL

Under the assumptions (H.1)–(H.2), the parabolic problem (1.1)–(1.3) is readily seen to have a unique classical solution in a maximal time interval, $0 \leqslant t < T$, that satisfies

$$u > 0 \quad \text{and} \quad v > 0 \quad \text{for all} \quad (x, t) \in \bar{\Omega} \times [0, T[, \tag{2.1}$$

$$\|u(\cdot, t)\|_{C(\bar{\Omega})} + \|v(\cdot, t)\|_{C(\bar{\Omega})} \to \infty \quad \text{as} \quad t \nearrow T \quad \text{if} \quad T < \infty. \tag{2.2}$$

If, in addition, (H.3) holds then $T = \infty$ and every solution of (1.1)–(1.3) is uniformly bounded in $0 \leqslant t < \infty$.

In order to derive the asymptotic model (1.12)–(1.14) we shall first obtain, in Section 2.1, some estimates on related linear elliptic problems and on the solution of (1.1)–(1.3). Then, the asymptotic model will be derived in Section 2.2, under the assumptions (H.1)–(H.5). Finally, the asymptotic model will be analyzed in Section 3 and some concluding remarks will be drawn in Section 4.

In order to avoid too clumsy expresions, we shall only give the orders of magnitude (in the limit (1.16)) of the several constants that appear in this section.

2.1. *Some Preliminary Estimates*

Let us first prove some results concerning two singularly perturbed, linear elliptic problems, that will be systematically used in the sequel.

LEMMA 2.1. *Let the domain $\Omega \subset \mathbb{R}^m$ be such that assumption* (H.1) (*at the end of Section* 1) *holds, and let u and v be the unique solutions of*

$$\Delta u = \Lambda^2 u \quad \text{in } \Omega, \qquad \partial u/\partial n = \sigma(1-u) \quad \text{at } \partial\Omega, \tag{2.3}$$

$$\Delta v + \varepsilon \Lambda^2 u = 0 \quad \text{in } \Omega, \qquad \partial v/\partial n = \nu(1-v) \quad \text{at } \partial\Omega, \tag{2.4}$$

where Λ, ε, σ and ν are positive and $\sigma > \nu$. As $\Lambda \to \infty$, the following estimates hold

$$[\sigma/(\sigma+\delta_1)] \exp[-\delta_1 d(x)] \leqslant u(x) \leqslant [\sigma/(\sigma+\delta_2)][\cosh(\delta_2(\rho_1 - d_1(x)))/\cosh(\delta_2 \rho_1)], \tag{2.5}$$

$$\sigma S_\Omega \delta_2/(\sigma+\delta_2) \leqslant \Lambda^2 \int_\Omega u(x)\, dx \leqslant \sigma S_\Omega \delta_1/(\sigma+\delta_1), \tag{2.6}$$

$$1 < v(x) \leqslant 1 + \delta_3, \tag{2.7}$$

for all $x \in \bar{\Omega}$, where ρ_1 and S_Ω are as defined in assumption (H.1) *and Eq.* (1.10) *respectively, $d(x)$ is the distance from x to $\partial\Omega$, $d_1(x) = \min\{d(x), \rho_1\}$ and the positive constants δ_1, δ_2 and δ_3 satisfy*

$$\delta_2 = \Lambda/\sqrt{m}, \quad \delta_3 = \varepsilon\sigma\delta_1/(\sigma+\delta_1)\nu \quad \text{and} \quad |\delta_1 - \Lambda| = O(\Lambda^{-1}) \quad \text{as } \Lambda \to \infty, \tag{2.8}$$

uniformly in $\varepsilon > 0$, $\sigma > 0$ and $\nu > 0$.

Proof. If the dimension m is equal to 1, then (2.3) and (2.4) are solved in closed-form and (2.5)–(2.7) are readily obtained. If $m \geqslant 2$, let $u_m = \min\{u(x) : x \in \partial\Omega\} > 0$ and for each $x_0 \in \partial\Omega$, let S_2 be the outer hypersphere, of radius ρ_2, that is tangent to $\partial\Omega$ at x_0 (assumption (H.1)). If r is the distance to the center of S_2, let the function $w = u_m \exp[-\delta_1(r-\rho_2)]$, where $\delta_1 = (m-1)/2\rho_2 + \sqrt{(m-1)^2/4\rho_2^2 + \Lambda^2}$. Then δ_1 satisfies (2.8) and w is such that

$$\Delta w \geqslant \Lambda^2 w \quad \text{in} \quad \Omega_1 = \{x \in \mathbb{R}^m : r > \rho_2\}, \qquad w = u_m \quad \text{at } \partial\Omega_1.$$

In addition $\Omega \subset \Omega_1$ and $w \leqslant u_m \leqslant u$ at $\partial\Omega$. As a consequence, maximum principles [12] readily imply that $u \geqslant w$ in Ω, and the first inequality (2.5) follows provided that

$$u_m \geqslant \sigma/(\sigma+\delta_1). \tag{2.9}$$

In order to obtain this inequality, let x_0 be a point where the minimum u_m is attained. Then $u = w = u_m$ at x_0 and, since $u \geqslant w$ in Ω, we have $\partial w/\partial n \geqslant \partial u/\partial n$ at x_0, i.e., $\delta_1 u_m \geqslant \sigma(1-u_m)$ and (2.9) follows. Thus the first inequality (2.5) has been obtained.

The second inequality (2.5) is obtained in a similar way. Let $u_M = \max\{u(x): x \in \bar{\Omega}\}$; notice that such maximum is attained at $\partial\Omega$ because $\Delta u > 0$ in Ω. For each $x_0 \in \partial\Omega$, let S_1 be the inner hypersphere of radius ρ_1 that is tangent to $\partial\Omega$ at x_0 (assumption (H.1)), and let the function w be defined as $w = u_M \cosh(\delta_2 r)/\cosh(\delta_2 \rho_1)$, where r is the distance to the center of S_1 and the constant δ_2 is as defined in Eq. (2.8). Then

$$\Delta w \leqslant \Lambda^2 w \qquad \text{in} \quad \Omega_1 = \{x \in \mathbb{R}^m : r < \rho_1\} \subset \Omega, \qquad w = u_M \geqslant u \qquad \text{at} \quad \partial\Omega_1,$$

and maximum principles imply that $u \leqslant w$ in Ω_1. But if x_0 is a point where the maximum u_M is attained, then $w(x_0) = u(x_0)$ and $\partial w/\partial n \leqslant \partial u/\partial n$ at x_0, i.e., $\delta_2 u_M \leqslant \sigma(1-u_M)$, or $u_M \leqslant \sigma/(\sigma+\delta_2)$. Since, in addition, $u \leqslant w$ for all $x_0 \in \partial\Omega$, the second inequality (2.7) follows when $d(x) \leqslant \rho_1$. In order to prove that this inequality also holds when $x \in \Omega_2 = \{x \in \Omega : d(x) > \rho_1\}$, notice that if $\Omega_2 \neq \emptyset$, then the maximum of u in $\bar{\Omega}_2$ is attained at $\partial\Omega_2$ (because $\Delta u > 0$ in Ω_2) and $u \leqslant (\sigma/(\sigma+\delta_2)) \cosh(\delta_2 \rho_1)$ at $\partial\Omega_2$. Thus the second inequality (2.5) has been obtained.

In order to prove that (2.6) holds integrate Eq. (2.3) in Ω, integrate by parts and take into account the boundary condition to obtain $\Lambda^2 \int_\Omega u\, dx = \sigma \int_{\partial\Omega} (1-u)$, and apply (2.5).

Finally, the first inequality (2.7) is readily obtained via maximum principles when taking into account that $\varepsilon\Lambda^2 u > 0$ in $\bar{\Omega}$. In order to obtain the second inequality (2.7), notice that the function $v_1 = v - \varepsilon(1-u)$ satisfies

$$\Delta v_1 = 0 \qquad \text{in } \Omega, \qquad \partial v_1/\partial n = v(1-v_1) + \varepsilon(\sigma - v)(1-u),$$

and, since $u \geqslant \sigma/(\sigma+\delta_1)$ at $\partial\Omega$ (see (2.9)) and $\sigma > v$, maximum principles readily imply that $v_1 \leqslant 1 + \varepsilon(\sigma - v)\,\delta_1/(\sigma+\delta_1)v$ in $\bar{\Omega}$, or $v \leqslant 1 + \varepsilon\sigma\delta_1/(\sigma+\delta_1)v$ in $\bar{\Omega}$. Thus, the proof is complete.

Let us now prove some estimates on the solution of (1.1)–(1.3). In particular we show that, after some time, u becomes quite small except in a boundary layer near $\partial\Omega$ (Lemma 2.2) and $|v - V|$ also becomes quite small (Lemma 2.3), where V is the spatial average of v.

Lemma 2.2. *Under the assumptions* (H.1)–(H.4) *and* (H.6) (*at the end of Section* 1) *there is a constant* T, *depending only on*

$$\|u_0\|_{C(\bar{\Omega})}, \quad \|v_0\|_{C(\bar{\Omega})}, \quad \phi,\ \sigma,\ L,\ \beta, \text{ and } v, \tag{2.10}$$

and satisfying

$$T = O(\phi^{-2}\log(2+\phi/\sigma+\sigma/\phi)+\phi^{-1}+(\nu L)^{-1}\log(2+\beta\phi^2/\nu)) \quad (2.11)$$

in the limit (1.16), *such that every solution of* (1.1)–(1.3) *satisfies*

$$u_1 < u(\cdot, t) < u_2, \qquad 1/2 < v(\cdot, t) < 1 + v_1 \qquad \text{in } \Omega \quad \text{if } t \geq T, \quad (2.12)$$

where u_1, u_2 and v_1 are the unique solutions of

$$\Delta u_1 = 2k_1\phi^2 u_1 \qquad \text{in } \Omega, \qquad \partial u_1/\partial n = \sigma(1-u_1) \qquad \text{at } \partial\Omega, \quad (2.13)$$

$$\Delta u_2 = k_2\phi^2 u_2/2 \qquad \text{in } \Omega, \qquad \partial u_2/\partial n = \sigma(1-u_2) \qquad \text{at } \partial\Omega, \quad (2.14)$$

$$\Delta v_1 + \beta k_1\phi^2 u_2 = 0 \qquad \text{in } \Omega, \qquad \partial v_1/\partial n = \nu(1-v_1) \qquad \text{at } \partial\Omega, \quad (2.15)$$

with the constants $k_1 > 0$ and $k_2 > 0$ as defined in assumption (**H**.4).

Proof. Let $\alpha_1 > 0$ be the smallest eigenvalue of

$$\Delta\varphi_1 + \alpha_1\varphi_1 = 0 \qquad \text{in } \Omega, \qquad \partial\varphi_1/\partial n + \nu\varphi_1 = 0 \qquad \text{at } \partial\Omega, \quad (2.16)$$

and let φ_1 be the associated eigenfunction such that

$$\varphi_1 > 0 \qquad \text{in } \bar{\Omega}, \qquad \max\{\varphi_1(x) : x \in \bar{\Omega}\} = 1. \quad (2.17)$$

φ_1 and α_1 are readily seen to satisfy

$$\min\{\varphi_1(x) : x \in \bar{\Omega}\} \to 1 \qquad \text{and} \qquad \alpha_1 = S_\Omega \nu/V_\Omega + o(\nu^2) \qquad \text{as} \quad \nu \to 0, \quad (2.18)$$

where S_Ω and V_Ω are as defined in (1.10). The proof proceeds in five steps.

Step 1. *u and v are such that*

$$u < A_1 \qquad \text{and} \qquad v < A_2 \qquad \text{if} \quad x \in \bar{\Omega} \quad \text{and} \quad t \geq 0, \quad (2.19)$$

where the constants A_1 and A_2 depend only on the quantities (2.10) *and satisfy*

$$1 < A_1 = O(1) \quad \text{and} \quad 1 < A_2 = O(1+\beta\phi^2/\nu), \quad \text{in the limit } (1.16). \quad (2.20)$$

Let $A_1 = 1 + \max\{u_0(x) : x \in \bar{\Omega}\}$, that satisfies (2.20) according to assumption (**H**.6). The function w, defined as $w = A_1 - u$, is readily seen to satisfy $w > 0$ in $\bar{\Omega}$ if $t = 0$, $\partial w/\partial t - \Delta w > 0$ in Ω if $t > 0$ and $\partial w/\partial n + \sigma w > 0$ in $\partial\Omega$ if $t > 0$ and, consequently, maximum principles [12] imply that $w > 0$ in Ω if $t > 0$. Then, the first inequality (2.19) follows.

In order to obtain the second inequality (2.19), let the function v_2 be defined as the unique solution of the linear problem

$$\Delta v_2 + \beta\phi^2 g_1(A_1) = 0 \qquad \text{in } \Omega, \qquad \partial v_2/\partial n + \nu(v_2 - 1) = 0 \qquad \text{at } \partial\Omega,$$

where the function g_1 is as defined in assumption (H.3). Maximum principles readily imply that

$$v_2 \leqslant 1 + \beta\phi^2 g_1(A_1)\, \varphi_1/[\alpha_1 \min\{\varphi_1(x) : x \in \bar{\Omega}\}] \qquad \text{in } \bar{\Omega},$$

where α_1 and φ_1 are as defined above, or, according to (2.17)–(2.18) and the first estimate (2.20),

$$\max\{v_2(x) : x \in \bar{\Omega}\} = O(1 + \beta\phi^2/\nu) \qquad \text{in the limit (1.16).} \qquad (2.21)$$

On the other hand, the function w, defined as $w = \max\{v_0(x) : x \in \bar{\Omega}\} + v_2 - v$ satisfies (see assumption (H.3))

$$w > 0 \quad \text{in } \Omega, \quad \text{if} \quad t = 0, \quad L^{-1}\, \partial w/\partial t > \Delta w \quad \text{in } \Omega, \quad \text{if} \quad t > 0, \qquad (2.22)$$

$$\partial w/\partial n + \nu w \geqslant 0 \qquad \text{at } \partial\Omega, \qquad \text{if} \quad t > 0, \qquad (2.23)$$

and consequently maximum principles imply that $w \geqslant 0$ in $\bar{\Omega}$ if $t \geqslant 0$. Then assumption (H.6) and Eq. (2.21) yield the second inequality (2.19), with A_2 satisfying (2.20), and the step is complete.

Step 2. There is a constant T_1, depending only on the quantities (2.10), *such that $T_1 = O(1/\nu L)$ in the limit* (1.16), *and*

$$v \geqslant 1/2 \qquad \text{in } \Omega \qquad \text{if} \quad t \geqslant T_1. \qquad (2.24)$$

Let the constant $A_3 \geqslant 1/2$ be such that $A_3\varphi_1 > 1 - v_0$ in $\bar{\Omega}$. Notice that A_3 may be chosen to be bounded, according to assumption (H.6) and Eq. (2.18). The function w defined as $w = v - 1 + A_3\varphi_1 \exp(-\alpha_1 L t)$ is readily seen to satisfy (2.22) and (2.23) and, as above, maximum principles imply that $w \geqslant 0$ in Ω if $t \geqslant 0$. Then (2.24) holds with $T_1 = (\alpha_1 L)^{-1} \log(2A_3)$ and the result follows.

Step 3. There is a constnat T_2, depending only on the quantities (2.10), *such that $0 < T_2 - T_1 = O(\phi^{-2})$ in the limit* (1.16), *and*

$$u \leqslant 2 \qquad \text{in } \Omega \qquad \text{if} \quad t \geqslant T_2.$$

Let the constant $A_4 > 0$ be defined as $A_4 = g_2(A_1)$. According to the assumption (H.2) and the results in steps 1 and 2, $f(u, v) > A_4 u$ in $\bar{\Omega}$ for all $t \geqslant T_1$, and the function $w = A_1 \exp[-A_4\phi^2(t-T_1)] + 1 - u$ satisfies

$$w > 0 \quad \text{in } \bar{\Omega} \quad \text{if } t = 0, \quad \partial w/\partial t > \Delta w - A_4\phi^2 w \quad \text{in } \Omega \quad \text{if} \quad t \geqslant T_1, \tag{2.25}$$

$$\partial w/\partial n + \sigma w \geqslant 0 \qquad \text{at } \partial\Omega, \qquad \text{if} \quad t > T_1. \tag{2.26}$$

As a consequence, maximum principles imply that $w \geqslant 0$ if $x \in \bar{\Omega}$ and $t \geqslant T_1$, and the result follows with $T_2 = T_1 + (A_4\phi^2)^{-1} \log A_1$.

Step 4. *There is a constant* T_3, *depending only on the quantities* (2.10), *such that* $0 < T_3 - T_2 = O(\phi^{-2} \log(2 + \phi/\sigma + \sigma/\phi) + \phi^{-1})$ *in the limit* (2.4) *and*

$$u_1 \leqslant u \leqslant u_2 \qquad \text{in } \bar{\Omega} \qquad \text{if} \quad t \geqslant T_3, \tag{2.27}$$

where u_1 *and* u_2 *are as given by* (2.13)–(2.14).

For the sake of brevity we shall only obtain the second inequality (2.27); the first inequality is obtained in a completely similar way. Let the function u_3 be the unique solution of the linear problem

$$\Delta u_3 = k_2\phi^2 u_3 \qquad \text{in } \Omega, \qquad \partial u_3/\partial n = \sigma(1-u_3) \qquad \text{at } \partial\Omega,$$

and let the constant δ be defined as

$$\delta = \min\{u_3(x) : x \in \bar{\Omega}\}/2.$$

According to Lemma 2.1, δ is such that

$$\delta \geqslant [\sigma/2(\sigma + \delta_1)] \exp(-\delta_1 D/2), \tag{2.28}$$

where D is the diameter of the domain Ω and $|\delta_1 - \sqrt{k_2}\phi| = O(\phi^{-1})$. Then the function $w = (1+\delta)\, u_1 - u_3 - \delta$ is such that $\Delta w < k_1\phi^2 w$ in Ω, $\partial w/\partial n + \sigma w = 0$ at $\partial\Omega$, and maximum principles imply that $w \geqslant 0$ in $\bar{\Omega}$, i.e., $u_2 - u_3 \geqslant \delta(1 - u_2)$ in $\bar{\Omega}$, or according to Lemma 2.1,

$$u_2 - u_3 \geqslant \delta\phi/(\phi + \sqrt{2m/k_2}\,\sigma). \tag{2.29}$$

Now, if the constant $A_5 > 0$ is such that $A_5 > u(\cdot, T_2) - u_3$ in $\bar{\Omega}$ (A_5 may be chosen to be such $A_5 \leqslant 2$, according to the result in step 3), then the function $w = u_3 - u + A_5 \exp[-\phi^2 k_2(t - T_2)]$ satisfies (2.25)–(2.26) with A_4 and T_1 replaced by k_1 and T_2 respectively, and again maximum principles imply that $w \geqslant 0$ in Ω if $t \geqslant T_2$. Then the second inequality (2.27) holds provided that

$$T_3 - T_2 = (\phi^2 k_2)^{-1}\, |\log[A_5(\phi + \sqrt{2m/k_2}\,\sigma)/\delta\phi]|$$

and, when taking into account (2.28), the result follows.

Step 5. There is a constant T, depending only on the quantities (2.10), *such that* $0 \leqslant T - T_3 = O(1/\nu L) \log(2 + \beta\phi^2/\nu)$ *in the limit* (1.16), *and*

$$v \leqslant 1 + v_1 \qquad \text{in } \bar{\Omega} \qquad \text{if} \quad t \geqslant T. \tag{2.30}$$

According to the results in steps 1–4, if $t \geqslant T_3$ then $u \leqslant u_2$ ($\leqslant 1$) and $1/2 \leqslant v \leqslant A_2$, and according to assumption (H.4), $f(u, v) \leqslant k_1 u$ in $\bar{\Omega}$. Let the function w be defined as

$$w = v_1 - v + A_6 \varphi_1 \exp[-\alpha_1 L(t - T_3)],$$

where $A_6 \geqslant 1$ satisfies $A_6 \varphi_1 \geqslant A_2$ in $\bar{\Omega}$; notice that, according to (2.18), A_6 may be chosen such that

$$A_6 = O(A_2) = O(1 + \beta\phi^2/\nu). \tag{2.31}$$

Also the function w satisfies (2.22)–(2.23) with t replaced by $t - T_3$, and maximum principles imply that $w \geqslant 0$ in $\bar{\Omega}$ for all $t \geqslant T_3$. As a consequence, (2.30) holds with $T = T_3 + (\alpha_1 L)^{-1} \log(1 + A_6)$. Finally, $T - T_3 = O(1/\nu L)$ $\log(2 + \beta\phi^2/\nu)$ in the limit (1.16), as obtained when taking into account (2.18) and (2.31). Thus the step and the proof of the Lemma are complete.

LEMMA 2.3. *Under the assumptions of Lemma 2.2 there are two constants,* $\mu > 0$ *and* $T' \geqslant T$ *such that* (i) μ *and* $T' - T$ *depend only on* (*the domain* Ω *and*)

$$\phi,\ \sigma,\ L,\ \beta \text{ and } \nu, \tag{2.32}$$

(ii) $\mu = O(\nu + \beta\phi\sigma/(\phi + \sigma))$ *and* $T' - T = O(L^{-1})$ $\log[(1 + (\beta\sigma\phi/(\sigma + \phi)\nu))/\mu]$ *in the limit* (1.16); *and* (iii) *if* $t \geqslant T'$ *then*

$$|v - V| \leqslant \mu \qquad \text{for all} \quad x \in \bar{\Omega}, \tag{2.33}$$

where $V(t)$ *is the spatial average of* v, *i.e.*,

$$V(t) = V_\Omega^{-1} \int_\Omega v(x, t)\, dx, \qquad \text{with} \quad V_\Omega = \int_\Omega dx. \tag{2.34}$$

Proof. Let us define the new time variable

$$\tau = Lt - T. \tag{2.35}$$

Then the spatial average of v satisfies

$$dV/d\tau = \left[\nu \int_{\partial\Omega} (1 - v)\, ds + \beta\phi^2 \int_\Omega f(u, v)\, dx\right] \Big/ V_\Omega, \tag{2.36}$$

as obtained upon integration of (2.2) in Ω, integration by parts, substitution of the boundary condition and multiplication by V_Ω^{-1}. If (2.36) is substracted from (2.2) then we obtain

$$\partial(v-V)/\partial\tau = \Delta(v-V) + \beta\phi^2 f(u,v)$$
$$- V_\Omega^{-1}\left[\nu\int_{\partial\Omega}(1-v)\,ds + \beta\phi^2\int_\Omega f(u,v)\,dx\right], \quad (2.37)$$

$$\partial(v-V)/\partial n = \nu(1-v) \qquad \text{at } \partial\Omega. \quad (2.38)$$

On the other hand, according to assumption (H.4) (at the end of Section 1) and Lemmas 2.1 and 2.2, we have

$$f(u,v) \leqslant k_1 u_2 \quad \text{in } \bar{\Omega} \quad \text{if} \quad \tau \geqslant 0, \quad 0 < u_2 < 1 \quad \text{in } \bar{\Omega},$$
$$\|1-u_2\|_{C(\partial\Omega)} = O(\phi/(\phi+\sigma)), \quad (2.39)$$

$$\int_\Omega u_2(x)\,dx = O(\sigma/\phi(\sigma+\phi)), \qquad \|v_1\|_{C(\bar{\Omega})} = O(1+\beta\phi\sigma/(\phi+\sigma)\nu) \quad (2.40)$$

in the limit (1.16), where u_2 and v_1 are as given by (2.14) and (2.15).

The proof proceeds in three steps.

Step 1. *The following inequality holds*

$$\int_\Omega (v-V)^2\,dx \leqslant B_2\exp(-2\gamma_1\tau) + 2\gamma_1 B_1$$
$$\times\int_0^\tau \|v(\cdot,\xi) - V(\xi)\|_{C(\bar{\Omega})}\exp[2\gamma_1(\xi-\tau)]\,d\xi \quad (2.41)$$

if $\tau \geqslant 0$, *where* $\gamma_1 > 0$ *depends only on the domain* Ω, $B_1 > 0$ *and* $B_2 > 0$ *depend only on (the domain* Ω *and) the quantities* (2.32), *and* $B_1 = O(\nu + \beta\sigma\phi/(\sigma+\phi))$ *and* $B_2 = O(1+\beta\sigma\phi/(\sigma+\phi)\nu)^2$ *in the limit* (1.16).

If (2.37) is multiplied by $v-V$, the resulting equation is integrated in Ω, integration by parts is applied and (2.38) is substituted, then the following equation results

$$\frac{1}{2}\frac{d}{d\tau}\int_\Omega (v-V)^2\,dx = -\int_\Omega |\nabla v|^2\,dx + \beta\phi^2\int_\Omega (v-V)\,f(u,v)\,dx$$
$$+ \nu\int_{\partial\Omega}(v-V)(1-v)\,ds, \quad (2.42)$$

where we have taken into account that, according to (2.34)

$$\int_{\Omega} (v - V)\, dx = 0 \qquad \text{for all} \quad \tau \geqslant 0. \tag{2.43}$$

Now we take into account the following Poincaré–Friedrichs-type property. There is a constant $\gamma > 0$, depending only on the domain Ω, such that for every $\varphi \in W_2^1(\Omega)$ such that $\int_\Omega \varphi(x)\, dx = 0$, the following inequality holds (see, e.g., [13, p. 45, Eq. (2.12)]) $\gamma \int_\Omega \varphi(x)^2\, dx \leqslant \int_\Omega |\nabla\varphi|^2\, dx$. If, in addition, (2.12), (2.39), and (2.40) are taken into account, then the following inequality results from (2.42)

$$\frac{1}{2}\frac{d}{d\tau}\int_{\Omega} (v - V)^2$$

$$\leqslant -\gamma \int_{\Omega} (v - V)^2\, dx + B_1 \, \|v(\cdot,\ \tau) - V(\tau)\|_{C(\bar{\Omega})} \qquad \text{if} \quad \tau \geqslant 0 \tag{2.44}$$

where $B_1 = \beta\phi^2 k_1 \int_\Omega u_2(x)\, dx + v \int_{\partial\Omega} |v_1(s) - 1|\, ds = O(v + \beta\sigma\phi/(\sigma + \phi))$ in the limit (1.16) (see (2.40)), as stated. And we only need to apply Gronwall's lemma to obtain (2.41) with $B_2 = \int_\Omega v_1(x)^2\, dx \geqslant \int_\Omega [v(x, 0) - V(0)]^2\, dx$ (see (2.12) and (2.32)). According to (2.40), $B_2 = O(1 + \beta\sigma\phi/(\sigma + \phi)v)^2$ in the limit (2.16), and the step is complete.

Step 2. The following inequality holds

$$\|v - V\|_{C(\bar{\Omega}) \times [\tau,\ \tau+1]} \leqslant \gamma_2 [B_3 + \|v - V\|_{L_2(\Omega \times]\tau-1,\ \tau+1[}] \tag{2.45}$$

for all $\tau > 1$, *where* $\gamma_2 > 0$ *depends only on the domain* Ω *and* $B_3 > 0$ *depends on* (Ω *and*) *the quantities* (2.32), *and satisfies* $B_3 = O(v + \beta\sigma\phi/(\sigma + \phi))$ *in the limit* (1.16).

In order to obtain (2.45) we decompose $v - V$ as

$$v - V = -vV\varphi + w_1 + w_2, \tag{2.46}$$

where φ and w_1 are uniquely defined by the linear problems

$$\begin{aligned} &\Delta\varphi = \varphi \qquad \text{in } \Omega, \qquad \partial\varphi/\partial n + v\varphi = 1 \qquad \text{at } \partial\Omega, \\ &\partial w_1/\partial\tau = \Delta w_1 - w_1 + \beta\phi^2 f(u, v) \qquad \text{in } \Omega, \qquad \text{if} \quad \tau > 0, \\ &\partial w_1/\partial n + v w_1 = 0 \quad \text{at } \partial\Omega, \quad \text{if} \quad \tau \geqslant 0, \quad w_1 = 0 \quad \text{in } \bar{\Omega}, \qquad \text{if} \quad \tau = 0, \end{aligned} \tag{2.47}$$

and w_2 satisfies, for all $\tau \geqslant 0$,

$$\partial w_2/\partial\tau - \Delta w_2 = -\nu V\varphi + w_1 + (\nu\varphi - 1)\, V_\Omega^{-1} \times \left[\beta\phi^2 \int_\Omega f(u, v)\, dx + \nu \int_{\partial\Omega} (1-v)\, ds \right] \quad \text{in } \Omega, \tag{2.48}$$

$$\partial w_2/\partial n + \nu w_2 = \nu \qquad \text{at } \partial\Omega.$$

Now, φ and w_1 are such that

$$\varphi > 0 \qquad \text{in } \Omega, \qquad \|\varphi\|_{C(\bar\Omega)} = O(1) \qquad \text{as} \quad \nu \to 0, \tag{2.49}$$

$$0 \leqslant w_1 \leqslant (2k_2/k_1)\, \beta(1 + B_4\varphi) \qquad \text{in } \bar\Omega \qquad \text{if} \quad \tau \geqslant 0, \tag{2.50}$$

where k_1 and k_2 and u_2 are as defined in assumption (**H**.4) (at the end of Section 1) and Lemma 2.2, and B_4 is given $B_4 = \sigma \max\{1 - u_2(x) : x \in \partial\Omega\}/\min\{\varphi(x) : x \in \partial\Omega\}$ and satisfies (see (2.39))

$$B_4 = O(\phi\sigma/(\phi+\sigma)), \qquad \text{in the limit (1.16).} \tag{2.51}$$

In order to obtain (2.49) we only need to apply maximum principles to (2.47), and take into account that $\nu \to 0$ is a regular limit of (2.47). Similarly, (2.50) follows when taking into account that if either $w = w_1$ or $w = (2k_1/k_2)\beta(1 - u_2 + B_4\varphi) - w_1$ then $\partial w/\partial\tau - \Delta w + w \geqslant 0$ in $\bar\Omega$, $\partial w/\partial n + \nu w \geqslant 0$ at $\partial\Omega$ if $\tau \geqslant 0$, and $w \geqslant 0$ in $\bar\Omega$ if $\tau = 0$, and applying maximum principles.

Finally, (2.34), (2.39)–(2.40), and (2.49)–(2.51) imply that the sup norm of the right side of (2.48) is bounded above by a constant $B_5 > 0$ that depends only on the quantities (2.32) and satisfies

$$B_5 = O(\nu + \beta\sigma\phi/(\phi+\sigma)), \qquad \text{in the limit (1.16).} \tag{2.52}$$

Then, local parabolic estimates [14, p. 355] readily imply that for each $p \geqslant 2$ there is a constant δ_p, depending only on the domain Ω, such that

$$\|w_2\|_{W_p^{2,1}(\Omega \times]\tau, \tau+1[)} \leqslant \delta_p(B_5 + \nu + \|w_2\|_{L_2(\Omega \times]\tau-1, \tau+1[)})$$

for all $\tau \geqslant 2$. If $p \geqslant 2$ is taken such that $p > (m+2)/2$ (m = dimension of Ω), then imbedding theorems [14, p. 80] imply that there is a constant δ'_p, depending only on Ω, such that

$$\|w_2\|_{C(\bar\Omega) \times [\tau, \tau+1]} \leqslant \delta'_p \, \|w_2\|_{W_p^{2,1}(\Omega \times]\tau, \tau+1[)}.$$

These two inequalities and (2.46), (2.49)–(2.52) readily imply the stated result, and the step is complete.

Step 3. The result in the statement of this lemma holds.

For each positive integer k, let $P_k \geqslant 0$ and $Q_k \geqslant 0$ be defined as

$$P_k = \|v - V\|_{C(\bar{\Omega} \times [k, k+1])}, \qquad Q_k = \|v - V\|_{L_2(\Omega \times]k, k+1[)}. \tag{2.53}$$

Then, according to (2.12), (2.34) and (2.40), we have

$$P_0 + P_1 \leqslant B_6, \tag{2.54}$$

where B_6 depends only on the quantities (2.32) and satisfies

$$B_6 = O(1 + \beta\phi\sigma/(\phi + \sigma)\nu), \qquad \text{in the limit (1.16)}, \tag{2.55}$$

and, according to the results in steps 1 and 2,

$$Q_k^2 \leqslant \int_k^{k+1} \left[B_2 + 2\gamma_1 B_1 \int_0^{k-1} \|v(\cdot, \xi) - V(\xi)\|_{C(\bar{\Omega})} \exp(2\gamma_1 \xi)\, d\xi \right] \times \exp(-2\gamma_1 \tau)\, d\tau \tag{2.56}$$

$$\leqslant (2\gamma_1)^{-1} \exp(2\gamma_1) \left[B_2 + B_1 \sum_{q=0}^{k} P_q \exp(2\gamma_1 q) \right] \exp(-2\gamma_1 k), \tag{2.56}$$

$$P_k \leqslant \gamma_2(B_3 + Q_{k-1} + Q_k), \tag{2.57}$$

for all $k \geqslant 1$. Also, if (2.57) is substituted into (2.56) and (2.54) is taken into account, then we obtain

$$Q_k^2 \leqslant B_7 Q_k + \left(2B_7 \sum_{s=1}^{k-1} Q_s \exp(2\gamma_1 s) + B_8 \right) \exp(-2\gamma_1 k) \qquad \text{for all} \quad k \geqslant 2, \tag{2.58}$$

where the constants B_7 and B_8 depend only on the quantities (2.32) and satisfy

$$B_7 = O(\nu + \beta\sigma\phi/(\sigma + \phi)), \quad B_8 = O(1 + \beta\sigma\phi/(\sigma + \phi)\nu)^2, \quad \text{in the limit (1.16)}. \tag{2.59}$$

The inequalities (2.54) and (2.58) imply that

$$Q_k \leqslant 5B_7 + (1 + B_6 + B_8^{1/2}) \exp[-\gamma_1(k-1)] \qquad \text{for all} \quad k \geqslant 1,$$

as readily seen by means of an induction argument. Then we only need to take into account (2.35), (2.53), (2.55), (2.57) and (2.59) to complete this step and the proof of the lemma.

2.2. *Derivation of the Asymptotic Model*

Let us define the function U as the unique solution of the following semi-linear parabolic problem

$$\partial U/\partial t = \Delta U - \phi^2 f(U, V) \qquad \text{in } \Omega \qquad \text{if} \quad t > 0, \tag{2.60}$$

$$\partial U/\partial n = \sigma(1-U) \qquad \text{at } \partial\Omega, \qquad \text{if} \quad t > 0; \qquad u = u_0 \qquad \text{in } \bar{\Omega}, \qquad \text{if } t = 0, \tag{2.61}$$

where V is the spatially averaged temperature, defined in (2.34), that satisfies (see (2.35)–(2.36))

$$(V_\Omega/L)(dV/dt) = \nu \int_{\partial\Omega} (1-v)\, ds + \beta\phi^2 \int_\Omega f(u, v)\, dx. \tag{2.62}$$

In order to derive the asymptotic model (1.8)–(1.9) we shall first prove that there is a constant T'' such that if $t \geqslant T''$ then the following properties hold: (i) $v > 1$ in $\bar{\Omega}$, $V > 1$ and U is very small except in a boundary layer near the boundary of Ω (Lemma 2.4); (ii) $|u - U|$ is appropriately small in the boundary layer (Lemma 2.5); and (iii) $|\tilde{\nabla} U|$ is appropriately small in the boundary layer, where $\tilde{\nabla}$ is the spatial gradient along the hypersurfaces parallel to $\partial\Omega$ (Lemma 2.8). Then the asymptotic model will be obtained in Theorem 1.1 as follows. As a consequence of property (iii), U depends only on the distance to $\partial\Omega$ in first approximation and thus U satisfies a 1-D parabolic equation in first approximation. In addition, since $|v - V|$ is appropriately small (Lemma 2.3), u and v can be replaced by U and V in Eq. (2.62) in first approximation, and the model (1.12)–(1.14) follows. Let us begin with property (i).

Lemma 2.4. *Under the assumptions* (H.1)–(H.4) *and* (H.6) (*at the end of Section* 1) *there is a constant* $T_1'' \geqslant 2T'$ *such that* $T_1'' - 2T'$ *depends only on the quantities* (2.32) *and satisfies* $T_1'' - 2T' = O(\nu L)^{-1} \log[2 + \nu(\phi + \sigma)/\beta\phi\sigma]$ *in the limit* (1.16), *and*

$$V > 1, \quad v > 1 \quad \text{and} \quad u_1 \leqslant U \leqslant u_2 \quad \text{in } \Omega, \qquad \text{for all} \quad t \geqslant T_1'', \tag{2.63}$$

where u_1, u_2 *and* T' *are as defined in Lemmas* 2.2 *and* 2.3.

Proof. According to the results in Lemmas 2.1–2.3, if $t \geqslant T'$ then

$$\beta\phi^2 \int_\Omega f(u, v)\, dx \geqslant \beta k_2 S_\Omega \sigma\phi/2[\phi + \sigma\sqrt{m/2k_1}],$$

where $S_\Omega = \int_{\partial\Omega} 1 ds$. As a consequence V is such that (see also (2.33)–(2.34) and (2.62))

$$(V_\Omega/L)(dV/dt) \geqslant \nu S_\Omega(1 - V - \mu) + \beta S_\Omega k_2 \sigma\phi/2[\phi + \sigma\sqrt{m/2k_1}],$$

where μ is as defined in Lemma 2.3. Then we only need to apply Gronwall's lemma and take into account that $\nu \to 0$ in the limit (1.16) and Eq. (2.33) to obtain the first two inequalities (2.63) for $t \geqslant T_1''$, with T_1'' as stated. Finally, the last two inequalities (2.63) readily follow by the argument in the proof of Lemma 2.2, step 4. Thus the proof is complete.

Now, we show that $|u - U|$ is conveniently small.

LEMMA 2.5. *Under the assumptions* (H.1)–(H.6) (*at the end of Section* 1), *let μ and T_1'' be as defined in Lemmas* 2.3 *and* 2.4. *Then there is a constant $T_2'' \geqslant T_1''$ such that $T_2'' - T_1''$ depends only on the quantities* (2.32) *and satisfies $T_2'' - T_1'' = O(\phi^{-1}) \log(1 + 1/\mu)$ in the limit* (1.16), *and*

$$|U - u| \leqslant 4k_5\mu u_2/k_3 \qquad \text{in } \bar{\Omega}, \qquad \text{for all} \quad t \geqslant T_2'', \tag{2.64}$$

where k_3, k_5 and u_2, are as defined in assumption (H.5) *and Lemma* 2.2.

Proof. Since, according to Lemma 2.1, $u_2 \leqslant \sigma/(\sigma + \phi\sqrt{k_2/2m})$, when using assumption (H.5) (at the end of Section 1) and the results in Lemmas 2.2–2.4, and applying the mean function theorem, we have

$$f(u, v) - f(U, v) = h(x, t)(u - U), \qquad \text{with} \quad h(x, t) \geqslant k_3,$$

$$|f(U v) - f(U, V)| \leqslant k_5\mu u_2$$

in $\bar{\Omega}$, for all $t \geqslant T_1''$. As a consequence, the functions

$$w_\pm = 2k_5\mu u_2/k_3 + [\sigma/(\sigma + \phi\sqrt{k_2/2m})] \exp[-k_3\phi^2(t - T_1'')/2] \pm (u - U)$$

are readily seen to satisfy

$$\partial w_\pm/\partial t - \Delta w_\pm + \phi^2 h(x, t)\, w_\pm \geqslant 0 \qquad \text{in } \Omega,$$

$$\partial w_\pm/\partial n + \sigma w_\pm > 0 \qquad \text{at } \partial\Omega, \qquad \text{if} \quad t \geqslant T_1'',$$

$$w_\pm = 0 \qquad \text{in } \bar{\Omega} \qquad \text{if} \quad t = T_1'',$$

and maximum principles imply that $w_\pm \geqslant 0$ in $\bar{\Omega}$ for all $t \geqslant T_1''$. Since, in addition, $u_2 \geqslant [\sigma/(\sigma + \delta_1)] \exp(-\delta_1 D/2)$, where D is the diameter of the domain Ω and $|\delta_1 - \sqrt{k_2/2}\,\phi| = O(\phi^{-1})$ as $\phi \to \infty$ (Lemma 2.1), the result follows. Thus, the proof is complete.

The following result gives a bound on the spatial derivatives of the solution of (2.60)–(2.61).

Lemma 2.6. *Under the assumptions of Lemma* 2.4, *there is a constant* $\mu_1 > 0$, *depending only on the quantities* (2.32), *such that* $\mu_1 = O(\sigma\phi/(\phi+\sigma))$ *in the limit* (1.16) *and*

$$|\nabla U(x,t)| \leqslant \mu_1 \exp[-\phi d_1(x)\sqrt{k_2/2m}] \quad \textit{if} \quad x \in \bar{\Omega} \quad \textit{and} \quad t \geqslant T_1'' + 2/\phi^2, \tag{2.65}$$

where $d_1(x) = \min\{\rho_1, d(x)\}$, $d(x)$ *is the distance from* x *to* $\partial\Omega$ *and* ρ_1, k_1 *and* T_1'' *are as defined in assumptions* (**H**.1) *and* (**H**.4) (*at the end of Section* 1) *and in Lemma* 2.4.

Proof. According to assumption (H.4) and the results in Lemmas 2.1 and 2.4, we have

$$\begin{gathered} 0 \leqslant f(U,V) \leqslant k_1 U, \\ 0 \leqslant U \leqslant [2\sigma/(\sigma+\phi\sqrt{k_2/2m})] \exp[-\phi d_1(x)\sqrt{k_2/2m}], \end{gathered} \tag{2.66}$$

$$0 \leqslant 1 - U \leqslant [1+\sigma d(x)]\, \delta_1/(\sigma+\delta_1), \tag{2.67}$$

in $\bar{\Omega}$, if $t \geqslant T_1''$, where δ_1 depends only on the quantities (2.32) and satisfies $|\delta_1 - \sqrt{2k_1}\phi| = O(1/\phi)$ in the limit (1.16).

In order to bound $|\nabla U(x_0, t_0)|$, with $x_0 \in \bar{\Omega}$ and $t_0 \geqslant T_1'' + 2/\phi^2$, we shall distinguish three cases, depending on the relative values of $d(x_0)$, $1/\phi$ and $1/\sigma$.

Case 1. $d(x_0) \leqslant \min\{1/\sigma, 1/\phi\}$. In this case we introduce the new variables w, ξ and τ defined as

$$w = 1 - U, \qquad x = x_0 + \varepsilon\xi \qquad \text{and} \qquad t = t_0 + \varepsilon^2\tau, \tag{2.68}$$

where $\varepsilon = \min\{1/\sigma, 1/\phi\}$, to rewrite (2.60)–(2.61) as

$$\partial w/\partial\tau = \Delta_\xi w + (\varepsilon\phi)^2 f(U,V) \quad \text{in } \Omega_\varepsilon, \quad \partial w/\partial n + \varepsilon\sigma w = 0 \quad \text{at } \partial\Omega_\varepsilon. \tag{2.69}$$

Now if B and B' are the balls with center at the origin and radii 1 and 2 respectively, local L_p estimates up to the boundary [14, p. 355] and imbedding theorems [14, p. 80] imply that there is a fixed constant K such that

$$\|\nabla_\xi w\|_{C(\bar{B}_\varepsilon \times [0,1])} \leqslant K[(\varepsilon\phi)^2 \|f(U,V)\|_{L_p(B'_\varepsilon \times]-1,1[)} + \|w\|_{L_1(B'_\varepsilon \times]-1,1[)}],$$

where $B_\varepsilon = B \cap \Omega_\varepsilon$, $B'_\varepsilon = B' \cap \Omega_\varepsilon$ and $p = (m+3)/2$. Notice that although $\partial\Omega_\varepsilon \cap B'$ and the coefficient $\varepsilon\sigma$ in the boundary condition depend on ε, the constant K may be chosen to be independt of ε because ε is bounded above (see (1.16)) and, as $\varepsilon \to 0$, $\partial\Omega_\varepsilon \cap B'$ converges to a part of a hyperplane and $\varepsilon\sigma$ remains bounded above. Now, when using (2.66) and (2.67) and taking into account that $\varepsilon\phi \leqslant 1$ we obtain

$$\|\nabla_\xi w\|_{C(B_\varepsilon \times [0,1])} \leqslant 2K[(2\gamma_m)^{1/p} k_1 \varepsilon\phi\sigma/(\sigma+\phi\sqrt{k_2/2m}) + 2\gamma_m(1+2\varepsilon\sigma)\,\delta_1/(\sigma+\delta_1)],$$

where γ_m is the measure of the unit ball of $\mathbb{R}^m$, or when coming back to the original variables,

$$|\nabla U(x_0, t_0)| \leqslant 2K[(2\gamma_m)^{1/p} k_1 \phi\sigma/(\sigma+\phi\sqrt{k_2/2m}) + 2\gamma_m(\max\{\phi,\sigma\}+2\sigma)\,\delta_1/(\sigma+\delta_1)]. \tag{2.70}$$

Case 2. $\sigma > \phi$ *and* $1/\sigma < d(x_0) \leqslant 1/\phi$. In this case we take $\varepsilon = d(x_0)/2$ and use the variables (2.68). Thus (2.69) holds again. But now $B' \cap \partial\Omega_\varepsilon$ is void and local interior estimates and imbedding theorems yield

$$\|\nabla_\xi w\|_{C(B\times[0,1])} \leqslant K[(\varepsilon\phi)^2\,\|f(U,V)\|_{L_p(B'\times]-1,1[)} + \|w\|_{L_1(B'\times]-1,1[)}],$$

where K is a fixed constant and $p=(m+3)/2$ as above. When using (2.66) and (2.67) and taking into account that $\varepsilon\phi \leqslant 1/2$, we obtain

$$\|\nabla_\xi w\|_{C(B\times[0,1])} \leqslant K[(2\gamma_m)^{1/p} k_1 \varepsilon\phi\sigma/(\sigma+\phi\sqrt{k_2/2m}) + 2\gamma_m(1+2\varepsilon\sigma)\,\delta_1/(\sigma+\delta_1)],$$

or, when coming back to the original variables,

$$|\nabla U(x_0, t_0)| \leqslant K[(2\gamma_m)^{1/p} k_1 \phi\sigma/(\sigma+\phi\sqrt{k_2/2m}) + 2\gamma_m\sigma 3\delta_1/(\sigma+\delta_1)]. \tag{2.71}$$

Case 3. $d(x_0) > 1/\phi$. Now we take $\varepsilon = 1/\phi$ and use the variables ξ and τ defined in (2.68) to rewrite (2.60) as

$$\partial U/\partial\tau = \Delta U - f(U,V) \qquad \text{in } \Omega_\varepsilon.$$

Since $B' \cap \Omega_\varepsilon$ is void, local interior estimates and imbedding theorems yield

$$\|\nabla_\xi U\|_{C(B\times[0,1])} \leqslant K[\|f(U,V)\|_{L_p(B'\times]-1,1[)} + \|U\|_{L_1(B'\times]-1,1[)}],$$

where again K is a fixed constant and $p=(m+3)/2$, or when using (2.67)

$$\|\nabla_\xi U\|_{C(\bar{B}\times[0,1])} \leqslant 2K(2\gamma_m)^{1/p}\, k_1[\sigma/(\sigma+\phi\sqrt{k_2/2m})]\,\exp[-\phi(d_1(x_0)-\varepsilon)\sqrt{k_2/2m}].$$

Then we only need to come back to the original variables to obtain

$$|\nabla U(x_0,t_0)| \leqslant 2K(2\gamma_m)^{1/p}\, k_1[\phi\sigma/(\sigma+\phi\sqrt{k_2/2m})] \times \exp[-\phi(d_1(x_0)-1/\phi)\sqrt{k_2/2m}], \qquad (2.72)$$

where we have taken into account that $d_1(x_0-\varepsilon)\geqslant d_1(x_0)-\varepsilon=d_1(x_0)-1/\phi$.

Finally, since one of the cases 1–3 above necessarily holds, Eqs. (2.70)–(2.72) yield the stated result, and the proof is complete.

In order to bound the gradient of U along the hypersurfaces parallel to $\partial\Omega$ (i.e., orthogonal to the normals to $\partial\Omega$ at each point) we first collect some facts from differential geometry. Let Ω_1 be defined as

$$\Omega_1=\{x\in\Omega : d(x)<\rho_1/2\} \qquad (2.73)$$

where ρ_1 is defined as in assumption (**H**.1) (at the end of Section 1) and, as above, $d(x)$ is the distance from x to $\partial\Omega$. According to assumption (H.1), the hypersurfaces parallel to $\partial\Omega$ are of class C^4, and simply cover $\bar{\Omega}_1$. Notice that if $x=x_0(\eta^2,\ldots,\eta^m)$ is a C^4-regular parametric representation of a part of one of these hypersurfaces, H, and $n=n(\eta^2,\ldots,\eta^m)$ is the outward unit normal to H, then

$$x=\eta^1 n(\eta^2,\ldots,\eta^m)+x_0(\eta^2,\ldots,\eta^m) \qquad (2.74)$$

defines a local C^3-coordinate system of $\mathbb{R}^m$ such that the hypersurfaces $\eta^1=$ constant are precisely those parallel to H (and to $\partial\Omega$). Also, the covariant components of the metric tensor associated with these co-ordinate system are such that

$$g_{11}=n\cdot n\equiv 1 \qquad \text{and} \qquad g_{1k}=n\cdot(\eta^1 n_{\eta^k}+x_{0\eta^k})\equiv 0 \qquad \text{if} \quad k\neq 1, \quad (2.75)$$

where the dot stands for the inner product of $\mathbb{R}^m$. Then the contravariant components of the metric tensor satisfy

$$g^{11}\equiv 1, \qquad g^{1k}\equiv 0 \qquad \text{if} \quad k\neq 1. \qquad (2.76)$$

With these facts in mind we can prove the following result.

LEMMA 2.7. *Let $\tilde{\iota}_0$ be a unit vector that is tangent to a hypersurface, H, parallel to $\partial\Omega$, at $p \in \bar{\Omega}_1$. Then there are a neighborhood N of p in $\mathbb{R}^m$, a C^3-vector field $\tilde{\iota}$: $N \to \mathbb{R}^3$, two vectors a_1 and a_2 and two scalars, b_1 and b_2, such that the following properties hold:*

(i) *a_1, a_2, b_1 and b_2 depend continuously on p and $\tilde{\iota}_0$.*

(ii) *$\tilde{\iota} = \tilde{\iota}_0$ at p, $\tilde{\iota} \cdot \tilde{\iota} = 1$ in N and, for each $q \in N \cap \bar{\Omega}_1$, $\tilde{\iota}(q)$ is tangent to the hypersurface parallel to $\partial\Omega$ passing through q .*

(iii) *If $I \subset \mathbb{R}$ is an open interval and $U: (N \cap \bar{\Omega}_1) \times I \to \mathbb{R}$ is a $C^{3,1}$-function satisfying*

$$\partial U/\partial t = \Delta U + \varphi \qquad \text{in} \quad (N \cap \bar{\Omega}_1) \times I \tag{2.77}$$

then the $C^{2,1}$-function $w = \nabla U \cdot \tilde{\iota}$ satisfies

$$\partial w/\partial t = \Delta w + a_1 \cdot \nabla w + a_2 \cdot \nabla U + b_1 w + \nabla\varphi \cdot \tilde{\iota} \text{ at } p, \quad \text{for all } t \in I, \tag{2.78}$$

$$\partial w/\partial n = \nabla(\partial U/\partial n) \cdot \tilde{\iota} + b_2 w \text{ at } p, \quad \text{for all} \quad t \in I, \tag{2.79}$$

where n is the outward unit normal to H at p.

Proof. Let H be the hypersurface parallel to $\partial\Omega$ passing through p and let $x = x_0(\eta^2, ..., \eta^m)$ be a C^3-parametric representation of H in a neighborhood of p, where $(\eta^2, ..., \eta^m)$ are *Fermi geodesic coordinates* defined as follows. The first coordinate η^2 is an arclength along the geodesic of H, C, that is tangent to t_0 at p. If $m > 2$, then the remaining coordinates are arclengths along $m-2$ geodesics that are tangent at each point of C to $e_3, ..., e_m$, where $\{e_2, ..., e_m\}$ is an orthonormal frame moved along C by parallelism on H, such that $e_2 = \tilde{\iota}_0$ at p. In addition, $(\eta^2, \dots, \eta^m)$ are chosen with origin at p and such that the line $\eta^3 = \cdots = \eta^m = 0$ is the geodesic C. That coordinate system is well defined in a neighborhood of p and such that

$$x_{0\eta^i} \cdot x_{0\eta^j} = \delta_{ij} \qquad \text{if} \quad \eta^3 = \cdots = \eta^m = 0, \qquad x_{0\eta^2} = \tilde{\iota}_0 \qquad \text{at } p, \tag{2.80}$$

where δ_{ij} is the Kronecker symbol (see, e.g., [15, p. 335]). Then (2.74) defines a C^3-coordinate system of $\mathbb{R}^m$ in a neighborhood N of p, whose metric tensor satisfies (see (2.75)–(2.76) and (2.80))

$$g_{ij} = g^{ij} = \delta_{ij} \quad \text{for all} \quad i, j = 1, ..., m, \quad \text{if} \quad \eta^1 = \eta^3 = \cdots = \eta^m = 0. \tag{2.81}$$

Also, (2.77) may be written as

$$\frac{\partial U}{\partial t} = \sum_{i,j=1}^{m} \left[g^{ij} \frac{\partial^2 U}{\partial\eta^i \partial\eta^j} + G^{-1/2} \frac{\partial}{\partial\eta^i} (G^{1/2} g^{ij}) \frac{\partial U}{\partial\eta^j} \right] + \varphi,$$

where G is the determinant of the $m\times m$ matrix (g_{ij}). If this equation is derivated with respect to η^2 and it is taken into account that, according to (2.81), $\partial g^{ij}/\partial\eta^2=\partial G/\partial\eta^2=0$ at p, then the following equation results

$$\partial w_1/\partial t=\Delta w_1+A\cdot\nabla U+\nabla\varphi\cdot\tilde{\imath}_0,\qquad \partial w_1/\partial n=\nabla(\partial U/\partial n)\cdot\tilde{\imath}_0\qquad\text{at } p,$$

where $w_1=\partial U/\partial\eta^2\equiv\nabla U\cdot(\eta^1 n_{\eta^2}+x_{0\eta^2})$ and A is the vector field whose contravariant components are

$$A^j=\sum_{i=1}^{m}\frac{\partial}{\partial\eta^2}\left[G^{-1/2}\frac{\partial}{\partial\eta^i}(G^{1/2}g^{ij})\right],\tag{2.82}$$

and $w=\nabla U\cdot(\eta^1 n_{\eta^2}+x_{0\eta^2})/\sqrt{g_{22}}\equiv w_1/\sqrt{g_{22}}$ is readily seen to satisfy (2.78) and (2.79) at p with

$$a_1=2(g_{22}(p))^{-1/2}\,\nabla(\sqrt{g_{22}(p)}),\qquad a_2=(g_{22}(p))^{-1/2}A,$$
$$b_1=(g_{22}(p))^{-1/2}\,\Delta(\sqrt{g_{22}(p)}),\qquad b_2=-(g_{22}(p))^{-1/2}\,\partial\sqrt{g_{22}(p)}/\partial n\tag{2.83}$$
$$\text{and}\quad \tilde{\imath}=(\eta^1 n_{\eta^2}+x_{0\eta^2})/\sqrt{g_{22}}.$$

Notice that $\tilde{\imath}$ is a unit vector tangent to the hypersurfaces $\eta^1=\text{constant}$ (that are parallel to H) along the parametric lines associated with the coordinate η^2. Finally, a_1, a_2, b_1 and b_2 depend continuously on p and $\tilde{\imath}_0$ (see (2.82)–(2.83) and take into account that g_{ij}, g^{ij} and their first and second order derivatives depend continuously on p and $\tilde{\imath}_0$). Thus the proof is complete.

A bound to the gradient of U along the hypersurfaces parallel to $\partial\Omega$ is given in the following result

LEMMA 2.8. *Under the assumptions* (H.1)–(H.6) (*at the end of Section* 1), *let* ρ_1, k_1, k_3, T_1'' *and* μ_1 *be as defined in assumptions* (H.1), (H.4), *and* (H.5) *and in Lemmas* 2.4 *and* 2.6. *Then there are two constants*, $T''\geqslant T_1''+1/\phi^2$ *and* $\mu_2>0$, *depending only on the quantities* (2.32) *such that* $T''-T_1''=O(\phi^{-2})\log\phi$ *and* $\mu_2=O(\sigma/(\phi+\sigma)\phi)$ *in the limit* (1.16), *and*

$$|\tilde{\nabla}U|\leqslant\mu_2\exp[-\sqrt{k}\,\phi d(x)]\quad\text{if}\quad x\in\bar{\Omega}_1,\quad\text{and}\quad t\geqslant T'',\tag{2.84}$$

where U *is a solution of* (2.60)–(2.61), $\tilde{\nabla}U$ *is the gradient of* U *along the hyper-surfaces parallel to* $\partial\Omega$,

$$k=\min\{k_2/4m,\,k_3/3\}>0,\tag{2.85}$$

Ω_1 *is as defined in* (2.73) *and, as above*, $d(x)$ *is the distance from* x *to* $\partial\Omega$.

Proof. Let us consider the function

$$w_1 = [\mu_2/2 + \mu_1 \exp(-k\phi^2(t - 1/\phi^2 - T_1''))] \exp(-\sqrt{k}\phi d(x)), \qquad (2.86)$$

where $\mu_1 > 0$ is as defined in Lemma 2.6 and $\mu_2 > 0$ is to be defined below (see Eq. (2.88)). When taking into account that $\bar{\Omega}_1$ may be covered by a finite number of coordinate systems such as that in (2.74), with $\eta^1 = -d(x)$, it is readily seen that

$$\begin{aligned} &\Delta w_1 \leqslant (k\phi^2 + k_6\phi)\, w_1, \qquad |\nabla w_1| = \sqrt{k}\phi w_1 \qquad \text{in } \Omega_1, \\ &\partial w_1/\partial n = \sqrt{k}\phi w_1 \qquad \text{at } \partial\Omega, \qquad \text{for all } t, \end{aligned} \qquad (2.87)$$

where $k_6 = \sqrt{k} \max\{|G_{\eta^1}/G| \text{ in } \bar{\Omega}_1\}$. Let the continuous functions a_1, a_2: $\bar{\Omega}_1 \to \mathbb{R}^m$ and b_1, b_2: $\bar{\Omega}_1 \to \mathbb{R}$ be as defined in Lemma 2.7, and let $k_7 > 0$ be a common upper bound of $|a_1|$, $|a_2|$, and $|b_1|$ in Ω_1, and of $|b_2|$ on $\partial\Omega$. Notice that k_7 depends only on $\partial\Omega$. If

$$\phi > \max\{k_7/\sqrt{k},\ 6(1 + k_6 + k_7)\sqrt{k}/k_3,\ \sqrt{6k_7/k_3}\}$$

(recall that $\phi \to \infty$ in the limit (1.16)) and μ_2 is defined as

$$\begin{aligned} \mu_2 = (4\mu_1/\phi^2) \max\{&k_7\phi^2/[k_3\phi^2/3 - (k_6 + k_7)\sqrt{k}\,\phi - k_7], \\ &\phi^2 \exp(-\sqrt{k}\phi\rho_1/4)\} \end{aligned} \qquad (2.88)$$

then $\mu_2 > 0$ and w_1 satisfies

$$\partial w_1/\partial t > \Delta w_1 + a_1 \cdot \nabla w_1 + a_2 \cdot \nabla U + b_1 w_1 - k_3\phi^2 w_1 \text{ at } p, \qquad \text{for all} \quad p \in \Omega_1, \qquad (2.89)$$

$$w_1 > |\nabla U| \qquad \text{if} \quad d(x) = \rho_1/2, \qquad (2.90)$$

$$\partial w_1/\partial n - b_2 w_1 > 0 \text{ at } p, \qquad \text{for all} \quad p \in \partial\Omega, \qquad (2.91)$$

provided that $t \geqslant T_1'' + 1/\phi^2$, as readily seen when taking into account (2.65) and (2.85)–(2.87).

Now, if we show that

$$|\tilde{\nabla} U| \leqslant w_1 \qquad \text{in } \bar{\Omega}_1 \qquad \text{for all} \quad t \geqslant T_1'' + 1/\phi^2 \qquad (2.92)$$

then (2.84) follows with μ_2 as given by (2.88) and $T'' = T_1'' + 1/\phi^2 + (k\phi^2)^{-1} \log(1 + \mu_1/\mu_2)$. In order to show that (2.92) holds first notice that, since $|\tilde{\nabla} U| \leqslant |\nabla U|$, according to (2.86) and the result in Lemma 2.6,

$w_1 > |\tilde{\nabla} U|$ in $\bar{\Omega}_1$ if $t = 1/\phi^2 + T_1''$. Assume for contradiction that there is a first value of t, T_0, and a point $p \in \bar{\Omega}_1$ such that

$$|\tilde{\nabla} U(p, T_0)| = w_1, \qquad |\tilde{\nabla} U| \leqslant w_1 \qquad \text{in } \bar{\Omega}_1 \qquad \text{if} \quad t \leqslant T_0. \tag{2.93}$$

Let H be the hypersurface parallel to $\partial\Omega$ at p. According to the definition of $\tilde{\nabla}$, there is a unit vector, $\tilde{\imath}_0$, that is tangent to H at p and such that

$$|\tilde{\nabla} U(p, T_0)| = \tilde{\nabla} U(p, T_0) \cdot \tilde{\imath}_0 = \nabla U(p, T_0) \cdot \tilde{\imath}_0,$$

and if $\tilde{\imath}$ and N are as given in Lemma 2.7, then

$$\nabla U \cdot \tilde{\imath} = \tilde{\nabla} U \cdot \tilde{\imath} \leqslant |\tilde{\nabla} U| \qquad \text{in } \bar{\Omega}_1 \cap N, \qquad \text{for all } t. \tag{2.94}$$

As a consequence, $w = \nabla U \cdot \tilde{\imath}$ satisfies (see (2.93))

$$w = w_1 \quad \text{at} \quad (x, t) = (p, T_0), \quad w \leqslant w_1 \quad \text{in } \bar{\Omega}_1 \cap N \quad \text{if} \quad t \leqslant T_0. \tag{2.95}$$

Also, the result (iii) in Lemma 2.7 implies that

$$\partial w/\partial t \leqslant \Delta w + a_1 \cdot \nabla w + a_2 \cdot \nabla U + b_1 w - \phi^2 k_3 w$$

$$\text{at} \quad (x, t) = (p, T_0) \qquad \text{if} \quad p \in \Omega_1, \tag{2.96}$$

$$\partial w/\partial n = -\sigma w + b_2 w \leqslant b_2 w \qquad \text{at} \quad (x, t) = (p, T_0) \qquad \text{if} \quad p \in \partial\Omega, \tag{2.97}$$

where we have taken into account (2.60)–(2.61) and that, according to Lemmas 2.1 and 2.4 and to assumption (H.5) (at the end of Section 1), $f_U(U, V) > k_3$ and $w = w_1 > 0$ at $(x, t) = (p, T_0)$. In order to get contradiction, we shall distinguish three cases:

(i) If $p \in \Omega_1$ then $w_2 = w_1 - w$ satisfies $\nabla w_2(p, T_0) = 0$, $\Delta w_2(p, T_0) \geqslant 0$ and $\partial w_2(p, T_0)/\partial t \leqslant 0$ (see (2.95)) and this is in contradiction with the inequality that is obtained upon substraction of (2.89) and (2.96).

(ii) If $p \in \partial\Omega$ then $w_2 = w_1 - w$ satisfies $\partial w_2(p, T_0)/\partial n \geqslant 0 = b_2 w_2(p, T_0)$ (see (2.95)) and this is again in contradiction with the inequality that is obtained upon substraction of (2.91) and (2.97).

(iii) Finally, if $d(p) = \rho_1/2$ then $w_1 = \nabla U \cdot \tilde{\imath} \leqslant |\tilde{\nabla} U| \leqslant |\nabla U|$ at $(x, t) = (p, T_0)$ (see (2.94) and (2.95)), and this is in contradiction with (2.90).

But, according to the definition of Ω_1, in (2.73), one of the three cases, (i), (ii), or (iii) above, necessarily holds. Then a contradiction has been obtained and the proof is complete.

Now we have the ingredients to derive the model (1.17)–(1.19). The remainder ψ, in (1.19), is such that $|\psi(t)|$ is appropriately small, and $\tilde{U}$ and V are appropriately close to u and v respectively, as stated in Theorem 1.1.

Proof of Theorem 1.1. If $0 \leqslant d(x) \leqslant \rho_1/2$, let $\eta = -d(x)$ be a coordinate along the outward unit normal to $\partial\Omega$, and for each η_0 let $H(\eta_0)$ be the hypersurface parallel to $\partial\Omega$, defined by $\eta = \eta_0$. Notice that the Laplacian operator may be written as

$$\Delta U = G^{-1} \frac{\partial}{\partial \eta} \left(G \frac{\partial U}{\partial \eta} \right) + \tilde{\Delta} U. \tag{2.98}$$

Here, for each $x \in \Omega$ such that $d(x) \leqslant \rho_1/2$, $\tilde{\Delta}$ is the Laplace–Beltrami operator on the hypersurface $H(-d(x))$, and $G = |(1 - k_1 \eta) \cdots (1 - k_{m-1} \eta)|$, where $k_1, ..., k_{m-1}$ are the principal curvatures of $\partial\Omega$ at the point of $\partial\Omega$ that shares with x the normal to $\partial\Omega$. Notice also that

$$G, G^{-1}, \quad |G_\eta| \quad \text{and} \quad |G_{\eta\eta}| \text{ are bounded if } -\rho_1/2 \leqslant \eta \leqslant 0. \tag{2.99}$$

Now, let $U = U(x, t)$ be as defined by (2.60)–(2.61), and let $U_1 = U_1(\eta, t)$ be as given by

$$U_1 = S(\eta)^{-1} \int_{H(\eta)} U(s, t)\, ds, \qquad \text{where} \quad S(\eta) = \int_{H(\eta)} ds \tag{2.100}$$

is the measure of the hypersurface $H(\eta)$. If, for each $\eta \in]-\rho_1/2, 0[$ we integrate (2.60) on $H(\eta)$ and divide by $S(\eta)$, then after some manipulations we obtain

$$\partial U_1/\partial t = \partial^2 U_1/\partial \eta^2 - \phi^2 f(U_1, V) + \varphi_1(\eta, t) \qquad \text{in} \quad -\rho_1/2 < \eta < 0, \tag{2.101}$$

where we have taken into account that $\int_{H(\eta)} \tilde{\Delta} U ds = 0$ and

$$\varphi_1(\eta, t) = \phi^2 \left[f(U_1, V) - \frac{1}{S(\eta)} \int_{H(\eta)} f(U, V)\, ds \right] + \frac{2S'(\eta)}{S(\eta)} \frac{\partial U_1}{\partial \eta}$$
$$+ \frac{S''(\eta)}{S(\eta)} U_1 - \frac{1}{2S(\eta)} \frac{\partial}{\partial \eta} \int_{H(\eta)} \frac{U}{G} \frac{\partial G}{\partial \eta}\, ds. \tag{2.102}$$

Similarly, if we integrate the boundary condition (2.60) in $\partial\Omega = H(0)$ and divide by $S_\Omega = S(0)$, then we obtain

$$\partial U_1/\partial \eta = \sigma(1 - U_1) + \varphi_2(t) \qquad \text{at} \quad \eta = 0, \tag{2.103}$$

where

$$\varphi_2(t) = \frac{1}{2S(0)} \int_{H(0)} \frac{U}{G} (\partial G/\partial \eta)\, ds - [S'(0)/S(0)]\, U_1(0, t). \tag{2.104}$$

Now, when taking into account assumptions (**H**.1) and (**H**.5) and the results in Lemmas 2.1, 2.4–2.6, and 2.8, we obtain

$$|U_1| \leqslant 2[\sigma/(\phi\sqrt{k_2/2m}+\sigma)] \exp(-\sqrt{k_2/2m}\phi d(x)), \tag{2.105}$$

$$\begin{aligned} &\phi\,|U_1(-d(x), t) - U(x, t)| \\ &\quad \leqslant \mu_3 \exp(-\sqrt{k}\phi d(x)) \quad \text{if } x \in \Omega \quad \text{and} \quad d(x) \leqslant \rho_1/2, \end{aligned} \tag{2.106}$$

$$|\varphi_1(\eta, t)| \leqslant \phi\mu_3 \exp(-\sqrt{k}\phi\eta) \quad \text{if } -\rho_1/2 \leqslant \eta \leqslant 0, \quad |\varphi_2(t)| \leqslant \mu_3, \tag{2.107}$$

for all $t \geqslant T''$, where T'', k and k_1 are as defined in Lemma 2.8 and assumption (**H**.4) and $\mu_3 > 0$ depends only on the domain Ω and on the quantities (1.20) and satisfies $\mu_3 = O(\sigma/(\sigma+\phi))$ in the limit (1.16).

Now, let $\bar{U}$ be the unique solution of

$$\begin{aligned} \partial\bar{U}/\partial t &= \partial^2\bar{U}/\partial\eta^2 - \phi^2 f(\bar{U}, V) \quad \text{in} \quad -\infty < \eta < 0, \\ \partial\bar{U}/\partial\eta &= \sigma(1-\bar{U}) \quad \text{at} \quad \eta = 0, \end{aligned} \tag{2.108}$$

if $t \geqslant T''$, with initial conditions

$$\begin{aligned} &\bar{U}(\eta, T'') = U_1(\eta, T'') \quad \text{if} \quad -\rho_1/2 \leqslant \eta \leqslant 0, \\ &\bar{U}(\eta, T'') = U_1(-\rho_1/2, T'') \exp[\sqrt{k_1/2m}\,\phi(\eta+\rho_1/2)] \\ &\qquad \text{if} \quad -\infty < \eta < -\rho_1/2. \end{aligned} \tag{2.109}$$

Since $f(\bar{U}, V) \leqslant k_1\bar{U}$ (assumption (H.4)) and $\bar{U}(\eta, T'') \leqslant 2[\sigma/(\phi\sqrt{k_2/2m}+\sigma)]$ $\exp(-\sqrt{k_2/2m}\,\phi d(x))$ (see (2.105)), maximum principles readily imply that

$$\begin{aligned} &0 < \bar{U}(\eta, t) < 2[\sigma/(\phi\sqrt{k_2/2m}+\sigma)] \exp(\sqrt{k_2/2m}\,\phi\eta) \\ &\qquad \text{if} \quad -\infty < \eta \leqslant 0 \quad \text{and} \quad t \geqslant T'', \end{aligned} \tag{2.110}$$

and assumption (H.5) and the mean function theorem readily imply that

$$\begin{aligned} &f(U_1, V) - f(\bar{U}, V) = h(x, t)(U_1 - \bar{U}), \quad \text{with} \quad h(x, t) \geqslant k_3 \\ &\qquad \text{if} \quad -\rho_1/2 \leqslant \eta \leqslant 0. \end{aligned} \tag{2.111}$$

Then if

$$\begin{aligned} &\lambda = \sqrt{k}/2 \quad \text{and} \quad \delta = \max\{8\mu_3/k\phi,\ 4\mu_3/\sqrt{k}\phi, \\ &[4\sigma/(\sigma+\phi\sqrt{k_2/2m})] \exp(-\sqrt{k}\phi\rho_1/4)\}, \end{aligned}$$

the functions $w_\pm = \delta \exp(\lambda\phi\eta) \pm (U_1 - \tilde{U})$ are readily seen to satisfy

$$\partial w_\pm/\partial t - \partial^2 w_\pm/\partial\eta^2 + \phi^2 h(x,t)\, w_\pm > 0 \quad \text{in} \quad -\rho_1/2 \leqslant \eta \leqslant 0, \quad \text{if } t \geqslant T'',$$

$$w_\pm > 0 \quad \text{at } \eta = -\rho_1/2, \quad \partial w_\pm/\partial\eta + \sigma w_\pm > 0 \quad \text{at } \eta = 0, \quad \text{if} \quad t \geqslant T'',$$

$$w_\pm > 0 \quad \text{in} \quad -\rho_1/2 \leqslant \eta \leqslant 0, \quad \text{if} \quad t = T'',$$

and maximum principles readily imply that $w_+ \geqslant 0$, i.e., that

$$|U_1 - \tilde{U}| \leqslant \delta \exp(\lambda\phi\eta) \quad \text{in} \quad -\rho_1/2 \leqslant \eta \leqslant 0, \quad \text{if} \quad t \geqslant T''. \tag{2.112}$$

If, in addition, we take into account (2.64) and (2.106), then (1.23) follows. In order to obtain (1.24) we only need to take into account that V satisfies (2.62), and that u, $v - V$ and $u - U$ satisfy (2.12), (2.33) and (1.23). Thus the proof is complete.

Remark 2.10. In the following, we shall ignore the initial transient $0 \leqslant t \leqslant \tilde{T}$, where $\tilde{T}$ is as given in Theorem 1.1. *Then the estimates* (1.23)–(1.24) *and* (2.110) *will be assumed to hold for all* $\tau \geqslant 0$.

3. ANALYSIS OF THE ASYMPTOTIC MODEL

The asymptotic model (1.17)–(1.19) will be considered now. We shall first analyze, in Section 3.1, the *distinguished limit* when all terms are of the same order and then we shall consider, in Section 3.2, other sub-limits leading to still simpler sub-models. Finally, in Section 3.3 we shall analyze the particular case when the non-linearity is as given by one of the expressions (1.4)–(1.6), and the activation energy γ is large.

3.1. *The Distinguished Limit*

Let us consider the following sub-limit of (1.16)

$$\sigma/\phi \to s, \qquad vL/\phi^2 \to V_\Omega l/S_\Omega, \qquad \beta\phi/v \to \lambda, \tag{3.1}$$

where

$$s > 0, \quad l > 0 \quad \text{and} \quad \lambda > 0 \text{ are bounded, and } \phi \to \infty, \quad L \to \infty, \quad v \to 0. \tag{3.2}$$

Then the model (1.17)–(1.19) may be written as

$$\partial \bar{U}/\partial \tau = \partial^2 \bar{U}/\partial \xi^2 - f(\bar{U}, V) \qquad \text{in} \quad -\infty < \xi < 0, \tag{3.3}$$

$$\bar{U} = 0 \qquad \text{at} \quad \xi = -\infty, \qquad \partial \bar{U}/\partial \xi = s(1 - \bar{U}) \qquad \text{at} \quad \xi = 0, \tag{3.4}$$

$$l^{-1}\, dV/d\tau = 1 - V + \lambda \int_{-\infty}^{0} f(\bar{U}, V)\, d\xi + \psi_1(\tau), \tag{3.5}$$

where

$$\xi = \phi \eta, \qquad \tau = \phi^2 t, \qquad \psi_1 = \psi/\nu \tag{3.6}$$

and, according to Remark 2.10,

$$0 < \bar{U} < [2s/(\sqrt{k_2/2m} + s)] \exp(\sqrt{k_2/2m}\, \xi) \quad \text{if} \quad -\infty < \xi < 0 \quad \text{and} \quad \tau \geqslant 0, \tag{3.7}$$

$$|\psi_1(\tau)| \leqslant \varepsilon_1 = O(\nu) \qquad \text{uniformly in} \quad 0 \leqslant \tau < \infty. \tag{3.8}$$

Two remarks concerning this model are in order:

(a) According to Theorem 1.1, the attractors as $\tau \to \infty$ of (3.3)–(3.5) are close to the attractors of the original model (1.1)–(1.2), in the sense of the estimates (1.23)–(1.24).

(b) If we ignore the remainder ψ_1 then (3.5) may be rewritten as

$$l^{-1}\, dV/d\tau = 1 - V + \lambda \int_{-\infty}^{0} f(\bar{U}, V)\, d\xi. \tag{3.5$'$}$$

Notice that condition (3.7) defines an *invariant set* of both (3.3)–(3.5) and (3.3)–(3.4), (3.5′) (that is, if the first two inequalities in (3.7) hold at $\tau = \tau_0$, then they also hold for all $\tau \geqslant \tau_0$), as readily seen when applying a maximum principle. Then we may consider only those solutions of both (3.3)–(3.5) and (3.3)–(3.4), (3.5) that satisfy (3.7) and (for comparison of the solutions of both problems) define the distance associated with the norm

$$\|(\bar{U}(\cdot, \tau),\ V(\tau))\| = \sup\{\bar{U}(\xi, \tau) \exp(-\sqrt{k_2/2m}\, \xi) : -\infty < \xi < 0\} + |V(\tau)|. \tag{3.9}$$

With that distance, the solution of both problems remain close to each other in finite time intervals, as readily seen by the argument leading to Eq. (2.112), in in the proof of Theorem 1.1. As a consequence, (with the distance associated with (3.9)) *the exponential attractors, as $\tau \to \infty$, of both*

(3.3)–(3.5) *and* (3.3)–(3.4), (3.5′) *are close to each other*; of course, non-exponential attractors need not be close. This is the sense in which the asymptotic behavior as $\tau \to \infty$ of (3.3)–(3.5) (or that of (1.1)–(1.3), according to Remark (a) above) may be approximated by that of (3.3)–(3.4), (3.5′).

Even with a fairly simple nonlinearity, such as that in (1.4), with $p = 1$, the model (3.3)–(3.4), (3.5′) exhibits at least multiple steady states and Hopf bifurcations, see [2].

3.2. *Some Particular Sub-limits of* (3.1)–(3.2)

Let us now consider the model (3.3)–(3.5) in the particular sub-limits $s \to 0$, $s \to \infty$ and $l \to 0$. As

$$s \to 0 \tag{3.10}$$

$\bar{U}$ is small (see (3.7)) and, according to assumption (H.5), at the end of Section 1, the nonlinearity f may be written as

$$f(\bar{U}, V) = f_u(0, V)\, \bar{U} + O(|\bar{U}|^2). \tag{3.11}$$

If λ is fixed then the right-hand side of (3.5) equals $1 - V$ in first approximation and the dynamics of the resulting model is trivial. If, instead, λ is large, such that

$$s\lambda \to \lambda_1 \neq 0,\ \infty, \qquad \text{with} \quad 0 \neq l = \text{fixed}, \tag{3.12}$$

then the model (3.3)–(3.5) may be rewritten as

$$\partial \bar{U}_1/\partial\tau = \partial^2 \bar{U}_1/\partial\xi^2 - \bar{U}_1 f_1(V) + \psi_2(\xi, \tau) \exp(\sqrt{k_2/2m}\,\xi) \quad \text{in} \ -\infty < \xi < 0, \tag{3.13}$$

$$\bar{U}_1 = 0 \quad \text{at} \quad \xi = -\infty, \qquad \partial\bar{U}_1/\partial\xi = 1 + \psi_3(\tau) \quad \text{at} \quad \xi = 0, \tag{3.14}$$

$$l^{-1}\, dV/d\tau = 1 - V + \lambda_1 f_1(V) \int_{-\infty}^{0} \bar{U}_1\, d\xi + \psi_1(\tau) + \psi_4(\tau), \tag{3.15}$$

where

$$\bar{U}_1 = \bar{U}/s, \qquad f_1(V) = f_u(0, V) \tag{3.16}$$

and, according to (3.7) and (3.11), the remainders ψ_2, ψ_3 and ψ_4 are such that

$$|\psi_2(\xi, \tau)| + |\psi_3(\tau)| + |\psi_4(\tau)| = O(s) \quad \text{uniformly in} \ -\infty < \xi < 0, \quad \tau \geqslant 0 \tag{3.17}$$

in the limit (3.10), (3.12). Then, if the remainders $\psi_1, \ldots, \psi_5$ are ignored in (3.13)–(3.15), we obtain an asymptotic model that is seen to approximate the large time behavior in a sense similar to that described in Remarks (a) and (b), at the end of Section 3.1.

In the limit

$$s \to \infty, \qquad \text{with} \quad l \neq 0 \qquad \text{and} \qquad \lambda \neq 0 \qquad \text{fixed,} \tag{3.18}$$

we have

$$|\partial \bar{U}(0, \tau)/\partial \xi| = \text{uniformly bounded in } \tau \geqslant 0, \tag{3.19}$$

as readily seen by an argument similar to that in the proof of Lemma 2.6, Case 1. Then the boundary conditions (3.4) may be written as

$$\bar{U} = 0 \qquad \text{at} \quad \xi = -\infty, \qquad \bar{U} = 1 + \psi_5(\tau) \qquad \text{at} \quad \xi = 0, \tag{3.4'}$$

where the remainder ψ_5 is given by

$$\psi_5(\tau) = s^{-1}\, \partial \bar{U}(0, \tau)/\partial \xi \tag{3.20}$$

and, according to (3.19), satisfies

$$|\psi_5(\tau)| = O(s^{-1}) \qquad \text{uniformly in} \quad \tau \geqslant 0, \tag{3.21}$$

in the limit (3.18). Again, if the remainders ψ_1 and ψ_5 are ignored in the model (3.3), (3.4′), (3.5), then we obtain an asymptotic model that approximates the large time behavior of (1.1)–(1.2), in a sense similar to that described in Remarks (a) and (b), at the end of Section 3.1.

In the limit.

$$l \to 0, \qquad \text{with } s \text{ and } \lambda \text{ fixed,} \tag{3.22}$$

let us replace Eq. (3.3) by

$$0 = \partial^2 \bar{U}/\partial \xi^2 - f(\bar{U}, V) \qquad \text{in} \quad -\infty < \xi < 0. \tag{3.3'}$$

For each function $V = V(\tau)$ the problem (3.3′) (3.4) is readily seen to uniquely define

$$\bar{U} = H(\xi, V) \tag{3.23}$$

and, according to assumptions (H.4) and (H.5) at the end of Section 1, the following estimates are seen to hold

$$0 < H \exp(-\sqrt{k_2}\,\xi) = O(1), \qquad |\partial H/\partial V| \exp(-\sqrt{k}\,\xi) = O(1) \tag{3.24}$$

uniformly in $-\infty < \xi < 0$, $\tau > 0$, $V > 1/2$, in the limit (3.22), where $O(1)$ stands for a bounded quantity. If $(\bar{U}, V)$ is a solution of (3.3)–(3.5) and $\bar{U}_1 = H(\xi, V(\tau))$ is the associated solution of (3.3′), (3.4), then $|dV/d\tau| = O(l)$ uniformly in $\tau > 0$, as readily seen from (3.5) when taking into account assumption (H.4) and Eq. (3.7), and (3.24) implies that

$$|\partial \bar{U}_1/\partial \tau| \exp(-\sqrt{k_2}\, \xi) = O(l) \qquad \text{uniformly in} \quad -\infty < \xi < 0, \ \tau > 0, \tag{3.25}$$

in the limit (3.22). Then we only need to apply maximum principles, as in the argument leading to Eq. (2.108), in the proof of Theorem 1.1, to obtain

$$|\bar{U} - \bar{U}_1| = O(1) \exp(-k\tau + \sqrt{k}\, \xi) + O(l) \exp(\sqrt{k}\, \xi)$$

uniformly in $-\infty < \xi < 0$, $\tau > 0$, where $k = \min\{k_2, k_2\}/2$ (see assumptions (H.4) and (H.5)), and consequently

$$|\bar{U} - \bar{U}_1| \exp(-\sqrt{k}\, \xi) = O(l)$$
$$\text{uniformly in} \quad -\infty < \xi < 0, \quad \tau > \tau_0 = O(|\log l|). \tag{3.26}$$

Then we only need to replace $\bar{U}$ by $\bar{U}_1$ in (3.5) to obtain

$$dV/d\tilde{\tau} = 1 - V + \lambda \int_{-\infty}^{0} f(H(\xi, V), V)\, d\xi + \psi_1(\tau) + \psi_6(\tau), \tag{3.27}$$

where H is as given in (3.23); the new time variable $\tilde{\tau}$ and the remainder ψ_6 are given by

$$\tilde{\tau} = lt, \quad \psi_6(\tau) = \lambda \int_{-\infty}^{0} [f(\bar{U}, V) - f(\bar{U}_1, V)]\, d\xi$$

and, according to (3.26) the remainder is small, i.e.,

$$|\psi_6(\tau)| = O(l) \qquad \text{uniformly in} \quad -\infty < \xi < 0, \ \ \tau > \tau_0, \tag{3.28}$$

in the limit (3.22). If the remainders ψ_1 and ψ_6 are ignored in (3.27) then an autonomous ODE is obtained that may exhibit multiple steady states and yields trivial dynamics (namely, $V(\tau)$ converges to a steady state as $\tau \to \infty$). As above, this implies that the dynamics of (1.1)–(1.2) is essentially trivial in first approximation.

Similarly, as $l \to \infty$ one could try to prove that the time derivative in the left hand side of (3.5) may be just omitted in first approximation. Then, after solving the resulting equation and replacing its solutions into (3.3), a non-local semilinear equation would be obtained. Unfortunately, this would require the non-linearity f to be such that $f_v < 0$, while f_v is usually positive (see (1.4)–(1.6)). In this case, when the time derivative is omitted in (3.5), the resulting equation may possess multiple solutions and, as a consequence, the complete problem may possess *relaxation oscillations* whose analysis is beyond the scope of this paper (see, e.g., Hastings [16] and Grasman [17] for a formal analysis of these oscillations in related problems).

3.3. *Large Activation Energies*

As mentioned in Section 1, the activation energy γ may be fairly large (1.4)–(1.6). Let us now consider the limit

$$\gamma \to \infty \tag{3.29}$$

in (1.4) (with $p = 1$ for assumptions (H.4) and (**H**.5) to hold); the analysis of the non-linearities (1.5) and (1.6) is completely similar.

In the limit (3.29) we shall consider two *distinguished limits*. In a first *extinction limit* we rescale ϕ^2, β, and v as

$$\phi_1^2 = \phi^2 \exp(-\gamma), \qquad \beta_1 = \beta/\gamma \qquad \text{and} \qquad v_1 = v/\gamma, \tag{3.30}$$

to rewrite (1.1)–(1.2), (1.4) as

$$\partial u/\partial t = \Delta u - \phi_1^2 u \exp(-1/v_1) \qquad \text{in } \Omega,$$

$$\partial u/\partial n = \sigma(1-u) \qquad \text{at } \partial\Omega, \tag{3.31}$$

$$L^{-1}\partial v_1/\partial t = \Delta v_1 + \beta_1 \phi_1^2 u \exp(-1/v_1) \qquad \text{in } \Omega,$$

$$\partial v_1/\partial n = \nu(1/\gamma - v_1) \qquad \text{at } \partial\Omega. \tag{3.32}$$

In the limit

$$\gamma \to \infty, \quad \phi_1^2 \to \infty, \quad \beta_1\phi_1\sigma/(\phi_1 + \sigma) \to 0, \quad \nu \to 0, \quad \sigma^{-1} = O(1) \tag{3.33}$$

the results in Section 2 (and, in particular, Theorem 1.1) apply to yield the asymptotic model

$$\partial \bar{U}/\partial t = \partial^2 \bar{U}/\partial \eta^2 - \phi_1^2 \bar{U} \exp(-1/V_1) \qquad \text{in} \quad -\infty < \eta < 0, \tag{3.34}$$

$$\bar{U} = 0 \qquad \text{at} \quad \eta = -\infty, \qquad \partial \bar{U}/\partial \eta = \sigma(1 - \bar{U}) \qquad \text{at} \quad \eta = 0, \tag{3.35}$$

$$(V_\Omega/S_\Omega L)\, dV_1/dt = -\nu V_1 + \beta_1 \phi_1^2 \exp(-1/V_1) \int_{-\infty}^{0} \bar{U}\, d\xi + \nu/\gamma + \psi(t), \tag{3.36}$$

where V_1 is the spatial average of v_1 and $|\psi(t)+v/\gamma|$, $|u-\bar{U}|$ and $|v_1-V_1|$ are appropriately small, according to Theorem 1.1. Notice that (3.34)–(3.36) is essentially (except for the constant v on the right-hand side of (1.19)) a particular case of the model (1.17)–(1.19), and thus the analysis in Sections 3.1–3.2 above applies to this new asymptotic model.

In a second *ignition limit*, β and $v-1$ are rescaled as

$$\beta_2 = \gamma\beta \qquad \text{and} \qquad v_2 = \gamma(v-1), \tag{3.37}$$

to rewrite (1.1)–(1.2), (1.4) as

$$\partial u/\partial t = \Delta u - \phi^2 u \exp[v_2/(1+v_2/\gamma)] \qquad \text{in } \Omega,$$

$$\partial u/\partial n = \sigma(1-u) \qquad \text{at } \partial\Omega, \tag{3.38}$$

$$L^{-1}\partial v_2/\partial t = \Delta v_2 + \beta_2\phi^2 u \exp[v_2/(1+v_2/\gamma)] \qquad \text{in } \Omega,$$

$$\partial v_2/\partial n = -v v_2 \qquad \text{at } \partial\Omega. \tag{3.39}$$

Notice that now the non-linearity is not bounded (as $v_2 \to \infty$) in the limit (3.29), as required by assumption (H.3) and thus the results in Section 2 do not apply to (3.38)–(3.39). But the boundedness assumption was used in Section 2 only to prove (in Lemma 2.2) that v is bounded. Then, *if we only consider those solutions of* (3.38)–(3.39) *such that* $|v_2(t)|$ *is bounded in* $0<t<\infty$, then Theorem 1.1 still applies (after slight changes to account for the fact that the non-linearity depends on the small parameter $1/\gamma$) in the limit

$$\gamma \to \infty, \quad \phi^2 \to \infty, \quad \beta_2\phi\sigma/(\phi+\sigma) \to 0, \quad v \to 0, \quad \sigma^{-1} = O(1), \tag{3.40}$$

to obtain the asymptotic model

$$\partial \bar{U}/\partial t = \partial^2 \bar{U}/\partial \eta^2 - \phi^2 \bar{U} \exp(V_2) \qquad \text{in} \quad -\infty < \eta < 0, \tag{3.41}$$

$$\bar{U} = 0 \qquad \text{at} \quad \eta = -\infty, \qquad \partial \bar{U}/\partial \eta = \sigma(1-\bar{U}) \qquad \text{at} \quad \eta = 0, \tag{3.42}$$

$$(V_\Omega/S_\Omega L)\, dV_2/dt = -vV_2 + \beta_2\phi^2 \exp(V_2) \textstyle\int_{-\infty}^{0} \bar{U}\, d\eta + \psi(t), \tag{3.43}$$

where V_2 is the spatial average of v_2 and $|\psi|$, $|u-\bar{U}|$ and $|v_2-V_2|$ are appropriately small. Again, the analysis in Section 3.1 and 3.2 above still applies to (3.41)–(3.43).

In addition, (3.38)–(3.39) possess solutions that are not bounded, but become very large in finite time. This phenomenon is known as *ignition* in the Combustion literature and its analysis in connection with (3.38)–(3.39) is (again) beyond the scope of this paper.

4. CONCLUDING REMARKS

We have considered the model (1.1)–(1.2) in the limit (1.16). The spatial domain, the non-linearity and the initial data have been assumed to satisfy assumptions (H.1)–(H.6). Some of these assumptions could be relaxed as explained at the end of Section 1, and have been imposed for the sake of both brevity and clarity. The assumption $f_u > k_3 > 0$ (in (**H**.5)) instead, is necessary for some of the ideas in the paper to apply, but perhaps it is not necessary for the main result to hold.

In Section 2 we have first obtained some estimates on the solutions of (1.1)–(1.2) implying that, after an initial transient (i) the reactant concentration u becomes quite small except in a boundary layer, near the boundary of the domain, $\partial\Omega$, (ii) the temperature v becomes approximately spatially constant, and (iii) the gradient of u along the hypersurfaces parallel to $\partial\Omega$ becomes small. Then the asymptotic model (1.17)–(1.19) was obtained. The 1-D parabolic semilinear equation (1.17) yields the reactant concentration in the above-mentioned boundary layer, and the ODE (1.19) gives the spatial average of the temperature.

The asymptotic model was analyzed (in Section 3.1) in the *distinguished limit* when all terms are comparable (except for the remainder ψ, that is smaller) and in some representative sub-limits (in Section 3.2). Namely, (i) when $s \to 0$ (i.e., when chemical reaction is much faster than material exchange through the boundary) and $s \to \infty$ the mixed boundary condition at $\eta = 0$ (in (1.18)) can be replaced by Neumann and Dirichlet boundary conditions respectively, and (ii) when $l \to 0$ (i.e., when diffusion is much faster that thermal exchange through the boundary) the reactant concentration becomes quasi-steady and the asymptotic model is reduced to an ODE; the opposite limit, $l \to \infty$, is much more subtle, as explained at the end of Section 3.2. Finally, in Section 3.3 we considered the case when the chemical reaction obeys a first-order Arrhenius kinetic law and the activation energy is large.

Let us point out that the asymptotic model was derived in a quite realistic limit, and that it is much simpler than the original reaction–diffusion system. Thus we expect this model to be useful in the analysis of the dynamics of catalytic pellets, which are of great interest in chemical reactor theory.

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