Mass-Luminosity Relation for White-Dwarf Stars

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Improving the model of Milne in which the white-dwarf star is considered as consisting of an envelope of perfect gas and an internal part of degenerate electron gas, we investigated the relation between its mass and luminosity. In conclusion, we found thai, though the masses are the same, the luminosity of the white dwarfs which contain light elements in their interior is greater than that of those which contain heavier elements.

Aperfeiçoando o modèlo de Milne em que a estrêla anã branca é considerada como sendo constituida de um envelope de gás perfeito e de uma parte interna de gás de elétrons degenerado, investigamos a relação entre a sua massa e luminosidade. Como conclusão, achamos que, mesmo quando as massas sejam iguais, a luminosidade das anãs brancas que contenham elementos leves no seu interior é maior que a daquelas que contenham elementos pesados.

1. Introduction

Differing from the usual stars, we have no simple relation between their masses and luminosities in the case of the white-dwarf stars. Namely, the luminosity of a white dwarf frequently differs from that of another with the almost same mass and apparent chemical composition by a factor of one hundred. The purpose of this paper is to explain qualitatively the reason for this irregularity. In order to solve this problem, we have used here a simple model of white dwarfs such as Milne considered formerly[1]. That is, we divided the white-dwarf star into two parts, the degenerate core where Salpeter's mode1[2] holds and the envelope where the perfect gas model is applicable. We adopted the continuities of temperatures, pressures, densities and opacities, as the conditions to decide the position of the interface of these two regions.

As is well-known, if we use the simple Kramers opacity in the envelope, the radial dependences of temperature, pressure and density in this region are easily given in terms of the mass, radius and luminosity of the star and the chemical composition of the envelope. On the other hand, in the degenerate core, the relation between pressure and density and the dependence of the opacity on density and temperature (or on an auxiliary parameter, chemical potential) are known. Using these relations and the above continuity conditions, we can eliminate the position of the interface and, as result, obtain a simple proportionality relation between mass and luminosity. The proportionality constant is fortunately independent of the radius and is a function of chemical composition only.

In conclusion, we will show qualitatively that the ratio of mass to luminosity is small when the central part of the white-dwarf consists of light elements and it is large when heavier elements exist there.

2. Calculations

If we use the Kramers law, $\kappa_r = \kappa_0 \rho T^{-7/2}$, for the opacity, the radial dependences of temperature, density and pressure in the envelope are given in terms of Schwarzschild variables[3] as follows[4]:

$$t = \frac{4}{17}\left(\frac{1}{x}-1\right), \tag{1}$$

$$\rho = \left(\frac{4}{17}\right)^{15/4} \frac{1}{C^{1/2}}\left(\frac{1}{x}-1\right)^{13/4}, \tag{2}$$

$$p = \left(\frac{4}{17}\right)^{19/4} \frac{1}{C^{1/2}}\left(\frac{1}{x}-1\right)^{17/4}. \tag{3}$$

For simplicity, in place of the radius R, the mass M and the luminosity L, we use hereafter the dimensionless quantities r, m and 1 defined by the following equations:

$$R = r\left(\frac{3\pi\hbar^3}{2^7 m_e^2 cGH^2}\right)^{1/2} = r \times 1.37 \times 10^8 \, cm \,, \tag{4}$$

$$M = m\left(\frac{3\pi\hbar^3 c^3}{2^{15} G^3 H^4}\right)^{1/2} = m \times 6.27 \times 10^{31} g \,, \tag{5}$$

$$L = l\left(\frac{3^3 \pi^{11} a^2 c^{19} m_e \hbar^{15} \mu^{15}}{2^{63} \kappa_0^2 k^{15} GH^6}\right)^{1/2}$$

$$= l \times \frac{\mu^{15/2}}{(X + Y + Z\langle Z_A^2/A\rangle)(1 + X)} \times 1.02 \times 10^{35} \, ergs/sec, \tag{6}$$

where $\mu = 4/(8X + 3Y + 2Z)$ is the mean molecular weight of the envelope, m_e is the electron mass and other notations are the same as Referente 4. In these terms, the Schwarzschild constant C is written as

$$C = lr^{1/2}/m^{11/2}. \tag{7}$$

In the degenerate core, according to Salpeter[2], the density and pressure are given in terms of an auxiliary parameter s as follows:

$$\rho = (r^3/m)s^3, \tag{8}$$

$$p = (r^4/m^2) f(s), \tag{9}$$

where

$$f(s) = s(2s^2 - 3)\sqrt{s^2 + 1} + 3\, sh^{-1} s \; - \frac{24}{10}\left(\frac{4}{9\pi}\right)^{1/3} \alpha Z^{2/3} s^4$$

$$- \frac{432}{175}\left(\frac{4}{9\pi}\right)^{2/3} \alpha^2 Z^{4/3} \frac{s^5}{\sqrt{1 + s^2}} - \frac{6}{\pi}\alpha\chi(s) \; - (0.0311)\cdot\frac{8}{3}\alpha^2 s^3,$$

$$\chi(s) = \frac{1}{32}(b^4 + b^{-4}) + \frac{1}{4}(b^2 + b^{-2}) - \frac{9}{16} \; - \frac{3}{4}(b^2 - b^{-2}) \ln b$$

$$+ \frac{3}{2}(\ln b)^2 - \frac{1}{3}\,\frac{s}{\sqrt{1 + s^2}}\left[\frac{1}{8}(b^4 - b^{-4}) \; - \frac{1}{4}(b^2 - b^{-2})\right.$$

$$\left. - \frac{3}{2}(b^2 - 2 + b^{-2}) \ln b \right], \; b = s + \sqrt{1 + s^2}. \tag{10}$$

The electron conduction opacity κ_c is given, according to Meste1[5], by

$$T\kappa_c = \frac{ac\pi\hbar^3 e^4}{32 m_e H k^5}\cdot\frac{Z'^2}{A'}\cdot\frac{12F_2}{15F_2F_4 - 16F_3^2}$$

$$\times\left[\frac{1}{3}\ln(12\pi) + \frac{1}{3}\ln Z' \; + \frac{1}{2}\ln F_{3/2} - \frac{}{6}\ln F_{1/2}\right], \tag{11}$$

where Z' and A' are, respectively, the atomic and mass numbers of the element contained in the degenerate core, T is the absolute temperature in the usual units, $F_n = \int_0^\infty dy\, y^n/(e^{y-\psi} + 1)$ and ψ is the chemical potential in units of kT. For the determination of ψ, the usual relation,

$$\rho t^{-3/2} = (3^2 mr^3 \mu^3/2^9)^{1/2} F, \quad F = F_{1/2} \tag{12}$$

is used.

Since t and ρ of Eqs. *(1)* and *(2)* are equal to those of Eq. *(12)* at the interface on account of the continuity conditions, by substituting the former into the latter, we have the following relation:

$$\frac{1}{r}\left(\frac{1}{x} - 1\right) = \left(\frac{3F}{16}\right)^{4/7}\left(\frac{\mu}{2}\right)^{6/7}\left(\frac{17}{4m}\right)^{9/7} l^{2/7}. \tag{13}$$

Substituting this into the continuity equations for ρ and p, we have

$$s^3 = \left(\frac{3F}{16}\right)^{13/7}\left(\frac{\mu}{2}\right)^{39/14}\left(\frac{17l}{4m}\right)^{3/7} \tag{14}$$

$$f(s) = \left(\frac{3F}{16}\right)^{17/7}\left(\frac{\mu}{2}\right)^{51/14}\left(\frac{17l}{4m}\right)^{5/7} \tag{15}$$

Note that r also has been already eliminated in these relations. Eliminating further m/l from Eqs. *(14)* and (15), we have

$$f(s)/s^5 = 6.105\, F^{-2/3} \mu^{-1}. \tag{16}$$

On the other hand, from the continuity condition for opacity, we have

$$s = 445.2\, \{(X + Y + Z\langle Z_A^2/A\rangle)(1 + X) F^{5/3}/T\kappa_c\}^{1/2} \tag{17}$$

where $T\kappa_c$ is the expression of Eq. *(11)*.

Solving numerically[7] Eqs. *(16)* and (17), we can determine the values of s and ψ at the interface. Inserting these values into the equation which is obtained by solving Eq. *(14)* inversely, we have the following mass-luminosity relation:

$$M^*/L^* = (m/m_\odot)/(l/l_\odot) = 3.94 \times 10^{-8} F^{13/3} s^{-7} \mu^{-1} \times$$
$$\times (X + Y + Z\langle Z_A^2/A\rangle)(1 + X) \;\cdot \tag{18}$$

3. Conclusions

Assuming that $Z = 0$ and $A' = 2Z'$, we have plotted the mass-luminosity relation of Eq. (18) as a function of X and Z' in Fig. 1. Because of the greatly simplified model adopted here, we should interpret this graph qualitatively rather than quantitatively. As a general tendency, we may say that, though the masses are the same, the luminosity of the white dwarfs which contain only light elements in their interior is larger than that of those which contain heavier elements. Also, we can see that the white dwarfs whicb contain more helium in their surface are darker than those with less helium,

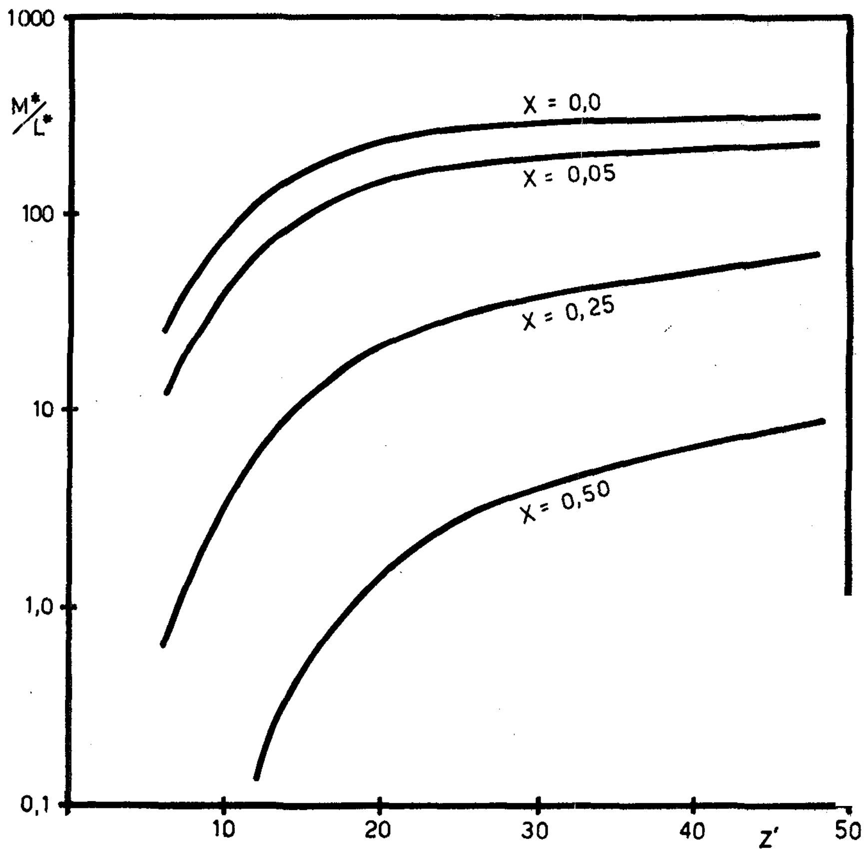

Figure 1 - M^*/L^* means the ratio of mass to luminosity of white dwarfs in solar units, Z' is the atomic number of the element contained in their interior and X, the mass concentration of hydrogen in the surface.

even though other conditions are the same. Note that the variations of M^*/L^* with X and Z' are rather drastic as shown in Fig. 1. This could explain the large variations in luminosity of the white-dwarfs which are observed.

References and Notes

1. See, for example, Menzel, Bhatnagar and Sen: Stellar Interiors (1963, John Wiley & Sons Inc.) § 12.4
2. E. E. Salpeter: Astrophys. Journ. 134, 669, (1961)
3. M. Schwarzschild: Astrophys. Journ. 104, 203, (1946)
4. See, for example, M. H. Wrubel: Handb. d. Physik, Vol. LI (1958, Springer-Verlag) p. 58
5. L. Mestel: Proc. Cambridge Phil. Soc. 46, 331, (1950)
6. The numerical calculation was carried out on a computer (SEMA, São Paulo).