A Sharp Upper Bound on Algebraic Connectivity Using Domination Number

M. Aouchiche[a], P. Hansen[a,b] and D. Stevanović[c,d]

[a] GERAD and HEC Montreal, Montreal, Qc, CANADA
[b] LIX, École Polytechnique, Palaiseau, FRANCE
mustapha.aouchiche@gerad.ca and pierre.hansen@gerad.ca
[c] PMF, University of Niš, Serbia
[d] FAMNIT, University of Primorska, SLOVENIA
dragance106@yahoo.com

Abstract

Let G be a connected graph of order n. The algebraic connectivity of G is the second smallest eigenvalue of the Laplacian matrix of G. A dominating set in G is a vertex subset S such that each vertex of G that is not in S is adjacent to a vertex in S. The least cardinality of a dominating set is the domination number. In this paper, we prove a sharp upper bound on the algebraic connectivity of a connected graph in terms of the domination number and characterize the associated extremal graphs.

MSC: 05C35; 05C50

Key words: Algebraic Connectivity, Laplacian, Domination number, Extremal graph

1 Introduction

Let $G = (V, E)$ be a simple connected graph of order (number of vertices) n. The degree of a vertex v_i is denoted $d_G(v_i)$ or d_i when no confusion is possible. The *minimum degree* is denoted δ. The *Laplacian matrix* of G is defined by $L = L(G) = D - A$, where D is the diagonal matrix which entries are the degrees of the vertices of G, *i.e.*, $D = diag(d_1, d_2, \cdots d_n)$, and A is the *adjacency matrix* of G defined by $a_{ij} = 1$ if $v_iv_j \in E$, otherwise $a_{ij} = 0$. The Laplacian spectrum of G is the spectrum of $L(G)$ and is denoted $\Lambda = (\lambda_1, \lambda_2, \cdots \lambda_n)$ such that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. It is well known that $\lambda_1 = 0$ and its multiplicity is equal to the number of connected components of G (see for example [9,10]).

Preprint submitted to Elsevier *11 December 2009*

The second smallest Laplacian eigenvalue of G, $a = a(G) = \lambda_2$, is called *algebraic connectivity* of G. Note that $a \geq 0$ with equality if and only if G is not connected [10]. A dominating set in G is a vertex subset S such that each vertex of G that is not in S is adjacent to a vertex in S. The least cardinality of a dominating set is the *domination number* and is denoted by $\beta = \beta(G)$.

Let H be a graph on k vertices $v_1, v_2, \cdots v_k$. *The even corona graph* of H, denoted $EC(H)$, is the graph obtained from H by adding k vertices $v'_1, v'_2, \cdots v'_k$ and the edges $v_i v'_i$ for $i = 1, \cdots k$. Note that the number of vertices in $EC(H)$ is even and equals $2k$. *The odd corona graph* of H, denoted $OC(H)$, is the graph obtained from H by adding $k-1$ vertices $v'_1, v'_2, \cdots v'_{k-1}$ and the edges $v_i v'_i$ for $i = 1, \cdots k-1$. Note that the number of vertices in $OC(H)$ is odd and equals $2k-1$. *The pseudo corona graph* of H, denoted $PC(H)$, is the graph obtained from the odd corona graph of H by adding the edge $v_k v'_{k-1}$. Note that the number of vertices in $PC(H)$ is odd and equals $2k-1$. Let $PC(H) - e^*$ be the graph obtained from $PC(H)$ by deleting the edge $v_{k-1}v_k$, assuming it exists, *i.e.*, the edge of $PC(H)$ whose vertices are adjacent to the same vertex v'_{k-1} that does not belong to the original graph H.

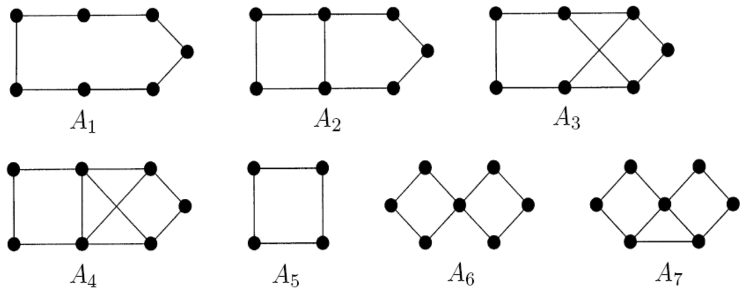

Fig. 1. The graphs of family $\mathcal{A}$

Table 1
Values of a, β and n for the graphs in $\mathcal{A}$.

G	A_1	A_2	A_3	A_4	A_5	A_6	A_7
$a(G)$	0.753020	0.753020	1	1	2	0.585786	0.829914
$\beta(G)$	3	3	3	3	2	3	3
n	7	7	7	7	4	7	7

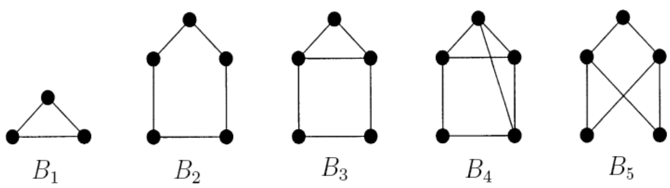

Fig. 2. The graphs of family $\mathcal{B}$

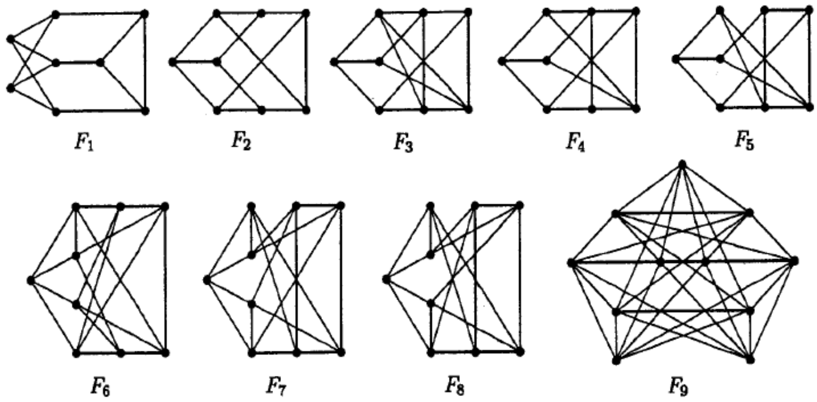

Fig. 3. The graphs of family $\mathcal{F}$

Table 2
Values of a, β and n for the graphs in $\mathcal{F}$.

G	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
$a(G)$	1.438447	2	2.267949	2	2	3	3	3	4.763932
$\beta(G)$	3	3	3	3	3	3	3	3	3
n	8	8	8	8	8	9	9	9	11

In [12] the following classes of graphs were defined.

• $\mathcal{G}_1 = \{C_4\} \cup \{G : G \cong EC(H)$ where H is connected$\}$, where C_4 denotes the cycle on 4 vertices. Note that the order of each graph in $\mathcal{G}_1$ is even and if $G \in \mathcal{G}_1 - \{C_4\}$, then G is a spanning graph of $EC(K_{n/2})$, where n is the order of G.

• $\mathcal{G}_2 = \mathcal{A} \cup \mathcal{B} - \{C_4\}$, where $\mathcal{A}$ and $\mathcal{B}$ are the sets of graphs shown in Fig. 1 and Fig. 2 respectively.

• For any graph H, Let $\mathcal{S}(H)$ be the set of connected graphs, each of which can be obtained from $EC(H)$ by adding a vertex v and edges joining v to one or more vertices from H. Then define $\mathcal{G}_3 = \cup_H \mathcal{S}(H)$. Note that the order of each graph in $\mathcal{G}_3$ is odd and if $G \in \mathcal{G}_3$, then G is a spanning graph of $OC(K_{(n+1)/2})$, where n is the order of G.

• $\mathcal{G}_4 = \{\theta(G) : G \in \mathcal{G}_3\}$, where $\theta(G)$ is the graph obtained from C_4 and G by adding a single edge between a vertex from C_4 and a vertex from G.

• Consider the path $P_3 = uvw$ and any graph H. Let $\mathcal{P}(H)$ be the set of connected graphs obtained from $EC(H)$ by joining each of u and w to one or more vertices of H. Then define $\mathcal{G}_5 = \cup_H \mathcal{P}(H)$. Note that the order of each graph in $\mathcal{G}_5$ is odd and if $G \in \mathcal{G}_5$, then G is a spanning graph of $PC(K_{\frac{n+1}{2}}) - e^*$, where n is the order of G.

• Let H be a graph and $X \in \mathcal{B}$. Let $\mathcal{R}(H, X)$ be the set of connected graphs which may be formed from $EC(H)$ by joining each vertex of $U \subset V(X)$ to one or more vertices of H such that no set with fewer than $\beta(X)$ vertices of X dominates $V(X) - U$. Then define $\mathcal{G}_6 = \cup_{H,X} \mathcal{R}(H, X)$.

Finding bounds on the algebraic connectivity has been widely studied (see [1] for references) since it was introduced by M. Fiedler [10]. In this paper, we are interested in upper bounds on algebraic connectivity in terms of domination number. Such a bound is given in the following theorem.

Theorem 1 [13] : *If G is a connected graph on $n \geq 2$ vertices with algebraic connectivity a and domination number β, then*

$$a \leq n - \beta + \frac{n - \beta^2}{n - \beta}.$$

For $\beta \leq \sqrt{n}$ this bound was improved in the next theorem.

Theorem 2 [14] : *If G is a connected graph on $n \geq 2$ vertices with algebraic connectivity a and domination number β, then*

$$a \leq \begin{cases} n & \text{if } \beta = 1, \\ n - \beta & \text{if } \beta \geq 2. \end{cases}$$

If $\beta = 1$ equality holds if and only if $G \equiv K_n$. If $\beta = 2$ equality holds if and only if G is the complement of a perfect matching. If $\beta \geq 3$, the inequality is always strict.

In this paper, the bounds given in theorems 1 and 2 are improved in the case $\beta \geq 3$.

Since $a \leq \delta$ for any graph $G \not\equiv K_n$, a natural question arises. How tight are the upper bounds, in terms of domination number, on the minimum degree when considered as bounds on the algebraic connectivity? One of these bounds, due to C. Payan [15], is

$$\delta \leq n - 2\beta + 1.$$

In order to know how tight this bound is if δ is replaced by a, we used AutoGraphiX (a conjecture making system in graph theory [2–6]) to look for extremal graphs for (the graphs that maximize) $a + 2\beta$ under the constraint $\beta \geq 3$ (the case $\beta \leq 2$ is entirely solved by Theorem 2). The "presumably" extremal graphs provided by AutoGraphiX have a regular structure and are well defined by their order. For n even, the extremal graphs are $EC(K_{\frac{n}{2}})$ (see Fig. 4 for an example with $n = 10$). If n is odd, there are three families: $OC(K_{\frac{n+1}{2}})$, $PC(K_{\frac{n+1}{2}})$ and $PC(K_{\frac{n+1}{2}}) - e^*$ (see Fig. 4 for examples with $n = 9$ and 10).

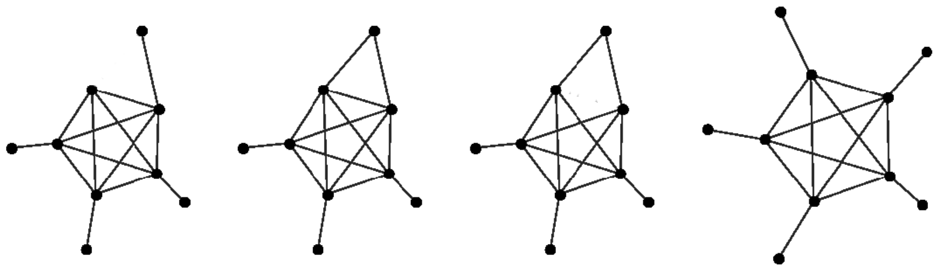

Fig. 4. Extremal graphs for $a + 2\beta$ with $n = 9$ (3, left) and $n = 10$ (right).

2 Preliminary results

In this section, we recall some known results that discuss bounds on the minimum degree of a graph G, in terms of domination number β. Some results about extremal graphs for given domination number are also given. All these results will be used in the next section.

The following two theorems characterize the graphs of order n for which the domination number $\beta = \lfloor n/2 \rfloor$.

Theorem 3 [11,16] : *For a graph G with even order n and no isolated vertices, the domination number $\beta = n/2$ if and only if the components of G are C_4 or the corona graph $EC(H)$ for any connected graph H.*

This theorem can be generalized as follows.

Theorem 4 [12] : *A connected graph G satisfies $\beta = \lfloor n/2 \rfloor$ if and only if $G \in \mathcal{G} = \cup_{i=1}^{6} \mathcal{G}_i$.*

The following theorems provide an upper bound on the domination number β in terms of the number of vertices n and the minimum degree δ.

Theorem 5 [15,18] : *If G is a connected graph on n vertices with minimum degree δ and domination number β, then*

$$\beta \leq \frac{n + 1 - \delta}{2},$$

with exception of the case that G is the complement of a one–regular graph.

The bound in the above theorem is improved, with few exceptions, in the case of $\delta \geq 2$ as follows.

Theorem 6 [18] : *If G is a connected graph on n vertices with minimum* *degree $\delta \geq 2$, maximum degree Δ and domination number β, then*

$$\beta \leq \frac{n-\delta}{2},$$

with exception of the cases that G is either a member of the families $\mathcal{A}$ (Fig. 1), $\mathcal{B}$ (Fig. 2) or $\mathcal{F}$ (Fig. 3), or G is the complete graph or a graph G with $n-3 \leq \delta \leq \Delta = n-2$.

3 New results

In this section, new results are proved. The main theorems provide upper bound on the algebraic connectivity a in terms of the number of vertices n and the domination number β. Families of extremal graphs are given according to the parity of the number of vertices n.

First, we prove some results related to the spectra of some graphs defined in Section 1.

Theorem 7 : *Let $G = (V, E)$ be a graph on k vertices with Laplacian spectrum λ_i, $i = 1, \cdots k$.*
(i) The Laplacian spectrum of $EC(G)$ is

$$\mu_i = \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4}}{2} + 1 \qquad i = 1, \cdots k. \tag{1}$$

(ii) If λ is a Laplacian eigenvalue of G such that the associated eigenvectors $X = (x_1, x_2, \cdots x_k)^T$ satisfy $x_k = 0$, then

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 + 4}}{2} + 1 \tag{2}$$

are Laplacian eigenvalues of $OC(G)$.

(iii) If λ is a Laplacian eigenvalue of G such that the associated eigenvectors $X = (x_1, x_2, \cdots x_k)^T$ satisfy $x_{k-1} = x_k = 0$, then

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 + 4}}{2} + 1 \tag{3}$$

are Laplacian eigenvalues of $PC(G)$.

Proof :
(i) For all $i = 1, \cdots k$, let $d_i = d(v_i)$ denote the degree of the vertex v_i in G. So $d_{EC(G)}(v_i) = d_i + 1$ and $d_{EC(G)}(v'_i) = 1$ for all $i = 1 \cdots k$. Let $\mu \neq 1$ be an eigenvalue of $EC(G)$ and $L_{EC(G)}$ the Laplacian matrix of $EC(G)$. So, if $X = (x_1, \cdots x_k, x'_1, \cdots x'_k)^T$ is a μ–eigenvector of $EC(G)$, we have $L_{EC(G)}X = \mu X$ or, equivalently

$$\begin{array}{rcl} (d_i+1)x_i - \sum_{v_iv_j \in E} x_j - x'_i & = & \mu x_i \\ -x_i + x'_i & = & \mu x'_i \end{array} \qquad i = 1, \cdots k. \tag{4}$$

From the second equation of (4), $x'_i = x_i/(1-\mu)$. Then by substitution in the first equation of (4), we have for every $i = 1, \cdots k$

$$\begin{array}{rcl} (d_i+1)x_i - \sum\limits_{v_iv_j \in E} x_j - \dfrac{x_i}{1-\mu} & = & \mu x_i \\ d_i x_i - \sum\limits_{v_iv_j \in E} x_j & = & \left(\mu - 1 - \dfrac{1}{\mu - 1}\right) x_i \end{array} \tag{5}$$

Note that equations (5) are the eigenvalue equations for the Laplacian of G. So

$$\lambda = \left(\mu - 1 - \frac{1}{\mu - 1}\right) \tag{6}$$

is an eigenvalue of G.
By solving equation (6), where μ is the unknown and λ is a parameter, the eigenvalues of $EC(G)$ are of the form

$$\mu_i = \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4}}{2} + 1 \qquad i = 1, \cdots k.$$

Note that, since 0 is a Laplacian eigenvalue of G, 0 and 2 are Laplacian eigenvalues of $EC(G)$.

(ii) Note that $d_{OC(G)}(v_i) = d(v_i) + 1$ and $d_{OC(G)}(v'_i) = 1$ for $i = 1, \cdots k - 1$ and $d_{OC(G)}(v_k) = d(v_k)$. Let $\mu \neq 1$ be a Laplacian eigenvalue of $OC(G)$ and $X = (x_1, \cdots x_k, x'_1, \cdots x'_{k-1})^T$ a μ–eigenvector. Then

$$\begin{array}{rcl} (d_i+1)x_i - \sum_{v_iv_j \in E} x_j - x'_i & = & \mu x_i \\ -x_i + x'_i & = & \mu x'_i \\ d_k x_k - \sum_{v_kv_j \in E} x_j & = & \mu x_k. \end{array} \qquad i = 1, \cdots k-1. \tag{7}$$

Proceeding as in (i), we get

$$\begin{array}{rcll} d_i x_i - \sum_{v_i v_j \in E} x_j & = & \left(\mu - 1 - \frac{1}{\mu-1}\right) x_i & i = 1, \cdots k-1, \\ d_k x_k - \sum_{v_k v_j \in E} x_j & = & \mu x_k. & \end{array} \tag{8}$$

Note that the equations in (8) are the eigenvalue equations for the Laplacian of G if and only if $\mu = 0$ or $x_k = 0$. It is obvious that $\mu = 0$ is a Laplacian eigenvalue of $OC(G)$, so consider the case $x_k = 0$, in which the equations in (8) become

$$\begin{array}{rcll} d_i x_i - \sum_{v_i v_j \in E} x_j & = & \left(\mu - 1 - \frac{1}{\mu-1}\right) x_i & i = 1, \cdots k-1, \\ \sum_{v_k v_j \in E} x_j & = & 0. & \end{array} \tag{9}$$

The equations in (9) characterize the eigenvalues λ of G whose eigenvectors k–th entry is 0. Then by solving $\mu - 1 - \frac{1}{\mu-1} = \lambda$, we get that $\mu = \frac{\lambda \pm \sqrt{\lambda^2+4}}{2} + 1$ are Laplacian eigenvalues of $OC(G)$. The multiplicity of each is equal to the dimension of the subspace, of the eigenspace associated to λ, whose eigenvectors k–th entry is 0.

(iii) Note that $d_{PC(G)}(v_i) = d(v_i) + 1$ for $i = 1, \cdots k$, $d_{PC(G)}(v_i') = 1$ for $i = 1, \cdots k-2$ and $d_{PC(G)}(v_{k-1}) = 2$. Let $\mu \notin \{1, 2\}$ be a Laplacian eigenvalue of $PC(G)$ and $X = (x_1, \cdots x_k, x_1', \cdots x_{k-1}')^T$ a μ–eigenvector. Then

$$\begin{array}{rcll} (d_i + 1)x_i - \sum\limits_{v_i v_j \in E} x_j - x_i' & = & \mu x_i & i = 1, \cdots k-2, \\ (d_{k-1} + 1)x_{k-1} - \sum\limits_{v_{k-1} v_j \in E} x_j - x_{k-1}' & = & \mu x_{k-1} & \\ (d_k + 1)x_k - \sum\limits_{v_k v_j \in E} x_j - x_{k-1}' & = & \mu x_k & \\ -x_i + x_i' & = & \mu x_i' & i = 1, \cdots k-2, \\ -x_{k-1} - x_k + 2x_{k-1}' & = & \mu x_{k-1}'. & \end{array}$$

From the last equation, we have

$$x_{k-1}' = -\frac{1}{\mu-2} x_{k-1} - \frac{1}{\mu-2} x_k$$

and then, proceeding as in (i) and (ii), we get

$$d_i x_i - \sum_{v_i v_j \in E} x_j = \left(\mu - 1 - \frac{1}{\mu-1}\right) x_i \qquad i = 1, \cdots k-2,$$

$$\begin{aligned}
d_{k-1}x_{k-1} - \sum_{v_{k-1}v_j \in E} x_j &= \left(\mu - 1 - \frac{1}{\mu-2}\right) x_{k-1} - \frac{1}{\mu-2} x_k, \\
d_k x_k - \sum_{v_k v_j \in E} x_j &= \left(\mu - 1 - \frac{1}{\mu-2}\right) x_k - \frac{1}{\mu-2} x_{k-1}.
\end{aligned}$$

If $x_{k-1} = x_k = 0$, these equations are the eigenvalue equations for the Laplacian of G. In this case, the equations are

$$\begin{aligned}
d_i x_i - \sum_{v_i v_j \in E} x_j &= \left(\mu - 1 - \frac{1}{\mu-1}\right) x_i \qquad i = 1, \cdots k-2, \\
\sum_{v_{k-1}v_j \in E} x_j &= 0. \\
\sum_{v_k v_j \in E} x_j &= 0.
\end{aligned}$$

These equations characterize the eigenvalues λ of G whose eigenvectors $(k-1)$–th and k–th entries are 0. Then by solving $\mu - 1 - \frac{1}{\mu-1} = \lambda$, we get $\mu = \frac{\lambda \pm \sqrt{\lambda^2+4}}{2} + 1$ are Laplacian eigenvalues of $PC(G)$. The multiplicity of each is equal to the dimension of the subspace, of the eigenspace associated to λ, whose eigenvectors $(k-1)$–th and k–th entries are 0. $\square$

Corollary 8 : *Let $k \geq 4$ be an integer.*
(i) The Laplacian spectrum of $EC(K_k)$ is

$$\begin{pmatrix} 0 & \frac{k+2-\sqrt{k^2+4}}{2} & 2 & \frac{k+2+\sqrt{k^2+4}}{2} \\ 1 & k-1 & 1 & k-1 \end{pmatrix}.$$

(ii) The Laplacian spectrum of $OC(K_{k+1})$ is

$$\begin{pmatrix} 0 & \frac{k+3-\sqrt{(k+1)^2+4}}{2} & \frac{k+3-\sqrt{(k-1)^2+4}}{2} & \frac{k+3+\sqrt{(k-1)^2+4}}{2} & \frac{k+3+\sqrt{(k+1)^2+4}}{2} \\ 1 & k-1 & 1 & 1 & k-1 \end{pmatrix}.$$

(iii) $\frac{k+3-\sqrt{(k+1)^2+4}}{2}$, $\frac{k+3+\sqrt{(k+1)^2+4}}{2}$ and $k+2$ are eigenvalues of $PC(K_{k+1})$ with multiplicities $k-2$, $k-2$ and 1 respectively.

(iv) $\frac{k+3-\sqrt{(k+1)^2+4}}{2}$, $\frac{k+3+\sqrt{(k+1)^2+4}}{2}$ and k are eigenvalues of $PC(K_{k+1}) - e^*$ with multiplicities $k-2$, $k-2$ and 1 respectively.

Proof :
(i) The result follows from Theorem 7-(i) by replacing G by K_k, the spectrum of which is $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_k = k$.

(ii) Here we use Theorem 7-(ii) by replacing G by K_{k+1} whose Laplacian spectrum is $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_{k+1} = k+1$. First, we have to show that there are Laplacian eigenvectors associated to $\lambda = k+1$ such that the $(k+1)$-th entry of each is 0. Under these conditions the Laplacian eigenvalues equations are

$$\begin{aligned} kx_i - \sum_{v_iv_j \in E} x_j &= (k+1)x_i \qquad i = 1, \cdots k, \\ \sum_{v_{k+1}v_j \in E} x_j &= 0. \end{aligned}$$

which are equivalent to the single equation

$$\sum_{j=1}^{k} x_j = 0.$$

This equation has exactly $k-1$ independent solutions, *i.e.*, there are exactly $k-1$ Laplacian eigenvectors associated to $\lambda = k+1$ such that the $(k+1)$-th entry of each is 0. Then by Theorem 7-(ii), $\frac{k+3-\sqrt{(k+1)^2+4}}{2}$ and $\frac{k+3+\sqrt{(k+1)^2+4}}{2}$ are eigenvalues of $OC(K_{k+1})$ and the multiplicity of each is $k-1$.
Obviously, 0 is a Laplacian eigenvalue of $OC(K_{k+1})$. To compute the remaining two eigenvalues, we use

$$\begin{aligned} \sum_{i=1}^{2k+1} \mu_i &= Tr(L_{OC(K_{k+1})}) = k^2 + 3k, \\ \sum_{i=1}^{2k+1} \mu_i^2 &= Tr(L^2_{OC(K_{k+1})}) = k^3 + 3k^2 + 5k. \end{aligned}$$

Some easy computations give that the two eigenvalues are $\frac{k+3-\sqrt{(k-1)^2+4}}{2}$ and $\frac{k+3+\sqrt{(k-1)^2+4}}{2}$.

(iii) We proceed exactly as in (ii) above. The only difference is that (since $x_k = x_{k+1} = 0$) the eigenvalue equations are equivalent to

$$\sum_{j=1}^{k-1} x_j = 0$$

for which there are exactly $k-2$ independent eigenvectors whose k-th and $(k+1)$-th entries are 0. Then by Theorem 7-(iii), $\frac{k+3-\sqrt{(k+1)^2+4}}{2}$ and $\frac{k+3+\sqrt{(k+1)^2+4}}{2}$ are eigenvalues of $PC(K_{k+1})$ and the multiplicity of each is $k-2$.
It is easy to see that the k-th and $(k+1)$-th lines (columns) of $L(PC(K_{k+1})) - (k+2)I$ are identical. So $k+2$ is a Laplacian eigenvalue of $PC(K_{k+1})$. Let

b, c and d be the remaining non-zero eigenvalues (with possible repetitions) of $PC(K_{k+1})$. To show that the multiplicity of $k+2$ is 1, it suffices to prove that $k+2 \notin \{b, c, d\}$. Indeed, suppose the contrary and let (without loss of generality) $d = k+2$. Then using the relations

$$Tr(L^p_{PC(K_{k+1})}) = \sum_{i=1}^{2k+1} \lambda_i^p \qquad \text{for } p = 1, 2, 3,$$

we get the following equations

$$\begin{aligned} b + c &= 4 \\ b^2 + c^2 &= 12 \\ b^3 + c^3 &= 6k + 34 \end{aligned}$$

which are unsolvable if $k > 1$.

(iv) This case is proved exactly like (iii). □

Proposition 9 : *If $\lambda \notin \{k, k+2\}$, then λ is a Laplacian eigenvalue of $PC(K_{k+1})$ if and only if λ is a Laplacian eigenvalue of $PC(K_{k+1}) - e^*$. In addition, the graphs $OC(K_{k+1})$, $PC(K_{k+1})$ and $PC(K_{k+1}) - e^*$ have the same algebraic connectivity $a = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$.*

Proof : The eigenvalue equations of $PC(K_{k+1})$ and $PC(K_{k+1}) - e^*$ differ only in the equations corresponding to the vertices v_k and v_{k+1}. These two equations for a $PC(K_{k+1})$ Laplacian eigenvalue $\lambda \neq k+2$ are

$$(k+1)x_k - \sum_{j=1, j\neq k}^{k+1} x_j - x'_k = \lambda x_k \qquad (10)$$

$$(k+1)x_{k+1} - \sum_{j=1}^{k} x_j - x'_k = \lambda x_{k+1} \qquad (11)$$

Taking the difference between (10) and (11), we have

$$(k+2)(x_k - x_{k+1}) = \lambda(x_k - x_{k+1}). \qquad (12)$$

Since $\lambda \neq k+2$ and a Laplacian eigenvalue of $PC(K_{k+1})$, necessarily $x_k = x_{k+1}$ and therefore (10) and (11) become

$$kx_k - \sum_{j=1}^{k-1} x_j - x'_k = \lambda x_k$$

$$kx_{k+1} - \sum_{j=1}^{k-1} x_j - x'_k \quad = \quad \lambda x_{k+1}$$

which are exactly the eigenvalue equations of $PC(K_{k+1}) - e^*$ corresponding to the vertices v_k and v_{k+1}. Then, and since the remaining eigenvalue equations are the same, λ is a Laplacian eigenvalue of $PC(K_{k+1}) - e^*$.
Similarly, we can prove that if $\lambda \neq k$ is a Laplacian eigenvalue of $PC(K_{k+1}) - e^*$ so it is for $PC(K_{k+1})$.
Obviously $a(PC(K_{k+1})) = a(PC(K_{k+1}) - e^*)$ holds. On the other hand, it follows from Corollary 8–(ii) that $a(OC(K_{k+1})) = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$. Thus to be done, it suffices to prove that $a(PC(K_{k+1})) = a(OC(K_{k+1}))$. Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2k+1}$ and $0 = \lambda'_1 \leq \lambda'_2 \leq \cdots \leq \lambda'_{2k+1}$ be the spectra of $OC(K_{k+1})$ and $PC(K_{k+1})$ respectively.
Using the Courant–Weyl inequalities (see, e.g., [9, Theorem 2.1]) and the fact that $PC(K_{k+1}) = OC(K_{k+1}) + v_{k+1}v'_k$, it follows that

$$0 = \lambda_1 = \lambda'_1 \leq \lambda_2 \leq \lambda'_2 \leq \cdots \leq \lambda_{2k+1} \leq \lambda'_{2k+1}.$$

On the other hand, according to Corollary 8–(ii), we have $\lambda_2 = \lambda_3 = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$.
It follows that $a(PC(K_{k+1})) = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$. $\square$

Lemma 10 : *If G is a connected graph on n vertices with algebraic connectivity a, then $a(EC(G)) \leq \frac{n+2-\sqrt{n^2+4}}{2}$ with equality if and only if $G \equiv K_n$.*

Proof : If $G \equiv K_n$ the equality follows from Corollary 8–(i). Now, let $G \not\equiv K_n$, then $a(G) \leq n-2$. By Theorem 7–(i), $\frac{a(G)+2-\sqrt{(a(G))^2+4}}{2}$ is an eigenvalue of $EC(G)$. On the other hand, the function $f(t) = t - \sqrt{t^2+4}$ is increasing. Thus $a < \frac{n+2-\sqrt{n^2+4}}{2}$. $\square$.

Theorem 11 : *Let G be a connected graph with even order $n = 2k \geq 6$, algebraic connectivity a and domination number $\beta \geq 3$. Then*

$$a \leq 2k - 2\beta + \frac{k+2-\sqrt{k^2+4}}{2}$$

with equality if and only if G is $EC(K_k)$.

Proof :
• If the minimum degree $\delta \geq 2$ and $G \notin \{F_1, F_2, \cdots F_5\}$, then by Theorem 6 and the fact that $a \leq \delta$ $(G \not\equiv K_n)$, we have

$$a + 2\beta \leq \delta + 2\beta \leq n = 2k < 2k + \frac{k+2-\sqrt{k^2+4}}{2}.$$

Thus the bound is not reached in this case.

• If $G \in \{F_1, F_2, \cdots F_5\}$, the bound is true from Table 2.
• If the minimum degree $\delta = 1$, then by Theorem 5,

$$a + 2\beta \leq \delta + 2\beta \leq n + 1.$$

If $\delta + 2\beta \leq n$, then $a + 2\beta < \frac{k+2-\sqrt{k^2+4}}{2}$. If $\delta + 2\beta = n + 1$, then there exists a graph H such that $G \equiv EC(H)$. Thus by Lemma 10 $a \leq a(EP(K_k))$ with equality if and only if $H \equiv K_k$. Therefore the result follows. □

Lemma 12 : *Let $G \in \cup_{i=3}^{6} \mathcal{G}_i$ with order $n = 2k + 1$.*
(i) If $G \in \mathcal{G}_3$, then $a(G) \leq a(OC(K_{k+1})) = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$.

(ii) If $G \in \mathcal{G}_4$, then $a(G) \leq a(EC(K_{k-1})) = \frac{k+1-\sqrt{(k-1)^2+4}}{2}$.

(iii) If $G \in \mathcal{G}_5$, then $a(G) \leq a(PC(K_{k+1} - e^*)) = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$.

(iv) If $G \in \mathcal{G}_6$, then $a(G) \leq \frac{k+3-\sqrt{(k+1)^2+4}}{2}$.

Proof :
(i) It is easy to see that if $G \in \mathcal{G}_3$ with $2k + 1$ vertices, then G is a spanning subgraph of $OC(K_{k+1})$. Thus the inequality follows.

(ii) Let H be the graph in $\mathcal{G}_4$ corresponding to $OC(K_{k-1})$ in $\mathcal{G}_3$. Then H is the union of the graph H_1 composed of C_4 and $2k - 3$ isolated vertices, and the graph H_2 composed of $EC(K_{k-1})$ and 3 isolated vertices. The Laplacian spectrum of H_1 is $\lambda_1(H_1) = \cdots \lambda_{2k-2}(H_1) = 0$, $\lambda_{2k-1}(H_1) = \lambda_{2k}(H_1) = 2$ and $\lambda_{2k+1}(H_1) = 4$; and the Lapalcian spectrum of H_2 (using Corollary 8) is $\lambda_1(H_2) = \cdots = \lambda_4(H_2) = 0$ and $\lambda_5(H_2) = \cdots = \lambda_{k+2}(H_2) = \frac{k+1-\sqrt{(k-1)^2+4}}{2}$, $\lambda_{k+3} = 2$ and $\lambda_{k+4}(H_2) = \cdots = \lambda_{2k+1}(H_2) = \frac{k+1+\sqrt{(k-1)^2+4}}{2}$. Now, using the Courant–Weyl inequalities (see, e.g., [9, Theorem 2.1]) we have

$$\lambda_2(H) \leq \lambda_{2k-2}(H_1) + \lambda_5(H_2) = \frac{k + 1 - \sqrt{(k-1)^2 + 4}}{2}.$$

(iii) The inequality follows from the fact that any graph in $\mathcal{G}_5$ with $2k + 1$ vertices, is a spanning subgraph of $PC(K_{k+1} - e^*)$.

(iv) First consider a graph G of $\mathcal{G}_6$ obtained using B_1 (see Fig. 2). There are two cases.
If $|U| = 1$, let $\{v_1, v_2, v_3\}$ be the vertex–set of B_1 and assume (without a loss of generality) that $U = \{v_1\}$. Then G is a spanning graph of H, where $H \in \mathcal{G}_6$ obtained from $EC(K_{k-1})$ by adding all possible edges between $\{v_1\}$ and the vertices of the K_{k-1}. Thereafter, we proceed as in (ii) above by considering H_1 as the path P_3 together with $2k - 2$ isolated vertices, and H_2 as $EC(K_k)$

together with an isolated vertex (v_2 or v_3). It is easy to see that $\lambda_1(H_1) = \cdots \lambda_{2k-1}(H_1) = 0$, $\lambda_{2k}(H_1) = 1$ and $\lambda_{2k+1}(H_1) = 3$; and $\lambda_1(H_2) = \lambda_2(H_2) = 0$ and $\lambda_3(H_2) = \cdots = \lambda_{k+1}(H_2) = \frac{k+2-\sqrt{k^2+4}}{2}$, $\lambda_{k+2} = 2$ and $\lambda_{k+3}(H_2) = \cdots = \lambda_{2k+1}(H_2) = \frac{k+2+\sqrt{k^2+4}}{2}$. Now, using the Courant–Weyl inequalities, we have

$$\lambda_2(H) \le \lambda_{2k-1}(H_1) + \lambda_3(H_2) = \frac{k+2-\sqrt{k^2+4}}{2} < \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

If $|U| = 2$, then G is a spanning graph of $PC(K_{k+1})$ and therefore

$$\lambda_2(G) \le \lambda_2(PC(K_{k+1})) = \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

Now, assume that G is obtained using one of the graphs B_2, B_3, B_4 or B_5. Let $\{v_1, \cdots v_5\}$ denote the set of vertices of B_i, $i = 2, \cdots 5$. At this level we will consider different cases according the cardinality of U. First note that any two vertices of B_i, $= 2, \cdots 5$, have a common neighbor. Therefore $|U| \le 2$.
If $|U| = 1$, assume, without a loss of generality, that v_1 is connected to vertices from G. Let v_2 be a neighbor of v_1 in B_i. Then G is the union of H_1 composed of $B_i - v_1v_2$ together with $2k-4$ isolated vertices, and H_2 composed of $EC(K_{k-1})$ together with 3 isolated vertices v_3, v_4 and v_5. Using the Courant–Weyl inequalities applied to the Laplacian spectra of H_1 and H_2, we have

$$a(G) \le a(H) \le \lambda_{2k-4}(H_1) + \lambda_5(H_2) = \frac{k+1-\sqrt{(k-1)^2+4}}{2} < \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

If $|U| = 2$, say $U = \{v_1, v_2\}$ with $v_1v_3, v_2v_4 \in E(B_1)$ (we relabel the vertices of B_i if needed). Then we proceed as above by choosing H_1 composed of $B_i - \{v_1v_3, v_2v_4\}$ and $2k-4$ isolated vertices, and H_2 composed of $EC(K_k)$ together with an isolated vertex $\{v_5\}$. Thus

$$a(G) \le a(H) \le \lambda_{2k-4}(H_1) + \lambda_5(H_2) = \frac{k+2-\sqrt{k^2+4}}{2} < \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

This completes the proof of the lemma. □

Theorem 13 : *Let G be a connected graph with odd order $n = 2k+1 \ge 9$, algebraic connectivity a, minimum degree δ and domination number $\beta \ge 3$. If $\delta \in \{1, 3, 5\}$ or δ is even and $G \notin \{F_6, F_7, F_8\}$, then*

$$a \le 2k - 2\beta + \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

The bound is the best possible as shown by $OC(K_{k+1})$, $PC(K_{k+1})$ and $PC(K_{k+1}) - e^*$.

Proof :
If δ is even and $G \notin \{F_6, F_7, F_8, F_9\}$, then using Theorem 6, $2\beta \le (n-1-\delta)$. Thus

$$a + 2\beta \le \delta + 2\beta \le n - 1 < 2k + \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

If $G \equiv F_9$, the bound is true from Table 2.
If $\delta = 1$, the result follows from Theorem 4, Corollary 8 and Lemma 12.
If $\delta = 3$, it is known that $\beta \leq 3n/8$ (see [17] and [12, p. 48]). Thus

$$a + 2\beta \leq 3 + \frac{3n}{4} < n - 1 \qquad \text{for all } n \geq 17.$$

For $n \leq 15$, we use the maximum possible value for β, denoted by β^*, in a graph on n vertices with minimum degree $\delta = 3$, provided in [8]. Table 3 is obtained. So in fact, we habe to check only for $n = 9$ and $n = 11$.
Using McKay's program *nauty* (available at *"http://cs.anu.edu.au/~bdm/nauty/"*),

Table 3
Values of $\delta + 2\beta^*$ for $n = 9, 11, \cdots 15$ and $\delta = 3$.

n	9	11	13	15
$\delta + 2\beta^*$	9	11	11	13

we generated all graphs on $n = 9$ vertices with $\delta = 3$ and maximum degree at most $\Delta = 6$ (if $\Delta = 7$, necessarily $\beta = 2$, and therefore the inequality is strict). There are exactly 41113 such graphs, among which there are exactly 484 with $\beta = 3$. Over all these 484 graphs, the algebraic connectivity is at most $a = 2.4604154$ which is reached for only two graphs (presented in Fig 5), while the corresponding value of the bound is approximately 2.8074176.

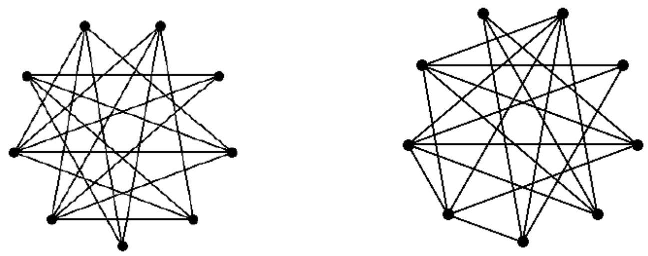

Fig. 5. The two graphs that maximize a for $n = 9$, $\delta = 3$ and $\beta = 3$.

For $n = 11$, we have to chek for graphs with $\delta = 3$ and $\beta = 4$. Using *nauty*, we generated all graphs with minimum degree $\delta = 3$, maximum degree $\Delta \leq 7$ (since $\Delta \geq 8$ implies $\beta \leq 3$) and size $17 \leq m \leq 31$ (due to the inequality $\beta \leq n + 1 - \sqrt{2m + 1}$ [12, p. 55]). There are exactly 205 662 831 such graphs, among which only 8 have domination number $\beta = 4$ and they are given in Fig. 6. We did not explore all the 205 662 831 graphs. We first computed the domination for the graphs on up to 21 edges. Since any graph G on $m = 22$ edges with minimum degree 3 contains at least an edge uv such that $d(u) \geq 4$ and $d(v) \geq 4$, so can be obtained from a graph G' on $m - 1 = 21$ edges with minimum degree 3, by adding an edge. Since adding an edges does not increase the domination number, $\beta(G) \leq \beta(G')$. Among all the 1 225 809 graphs on 11 vertices and 21 edges with $\delta = 3$ and $\Delta \leq 7$, there was no graph with $\beta = 4$. Thus, it is so for $m = 22$ and recursively for $m = 23, \cdots 31$.

Table 4
Values of $a(G_i)$ for $i = 1, 2, \cdots 8$.

i	1	2	3	4	5	6	7	8
$a(G_i)$	0.7382	1.1864	1.3446	1.5013	1.3075	1.3937	0.7382	1.6672

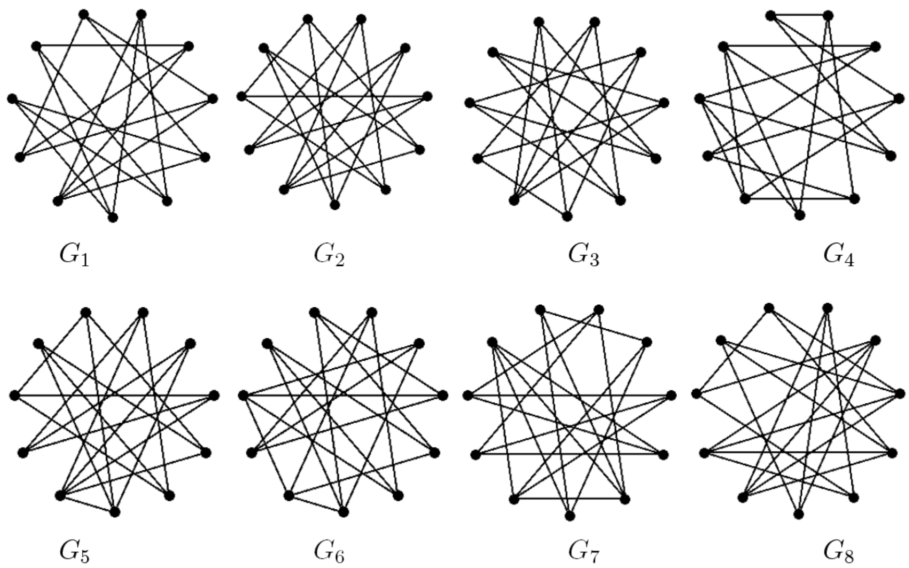

Fig. 6. All the graphs on $n = 11$ vertices with $\delta = 3$ and $\beta = 4$

The algebraic connectivities of the 8 graphs (given in Fig. 6) on $n = 11$ vertices with minimum degree $\delta = 3$ and domination $\beta = 4$ are given in Table 4. Thus, the bound is not reached for any of these graphs.

If $\delta = 5$, it is proved in [19] that $\beta \leq 5n/14$. Thus

$$a + 2\beta \leq 5 + \frac{5n}{7} \leq n - 1 \qquad \text{for all } n \geq 21.$$

Therefore, the bound is true, with strict inequality, for all $n \geq 21$. For $n \in \{7, 9, \cdots 19\}$, we use the values of β^* from [8] for $n = 9, \cdots 15$, and an upper bound on β^* for $n \in \{17, 19\}$ computed using the following formula from [7], $\beta^* \leq \min\{p, g_p = 0\}$, where g_p is defined by

$$g_0 = n \qquad \text{and} \qquad g_{p+1} = \left\lfloor g_p \left(1 - \frac{\delta + 1}{n - p}\right)\right\rfloor \tag{13}$$

The corresponding values are given in Table 5. Since for $n = 9$, $\beta^* = 2$ and the

Table 5
Values of $\delta + 2\beta^*$ for $n = 11, 13, \cdots 19$ and $\delta = 5$.

n	11	13	15	17	19
$\delta + 2\beta^*$	11	11	13	15	17

desired inequality is strict for $n \in \{13, 15, 17, 19\}$, we have to check only for graphs

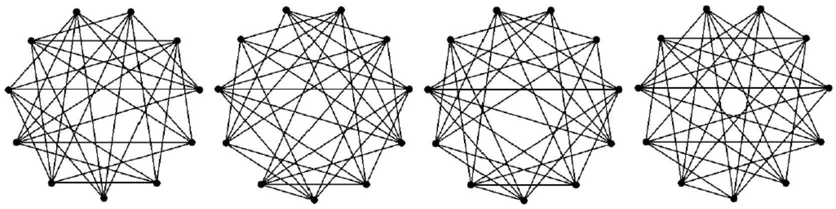

Fig. 7. The four graphs with $n = 11$, $\delta = 5$ and $\beta = 3$ that maximize a

on $n = 11$ vertices with $\delta = 5$ and $\beta = 3$. Note in addition to these conditions, if $\Delta \geq 8$ then $\beta \leq 2$. So using *nauty*, we enumerated all graphs on $n = 11$ vertices with $\delta = 5$ and $\Delta \leq 7$. There are exactly 3 982 767 graphs satisfying these conditions, 2 098 of which have $\beta = 3$. Among these 2 098 graphs, the algerbaic connectivity is maximum for the four presented in Fig. 7 for which $a = 0.26795$, while the value of the bound corresponding to $n = 11$ is 0.83772.
This completes the proof. □

Note that the condition $\beta \geq 3$ in Theorem 13 is necessary. Indeed, when exploring graphs on $n = 9$ vertices with minimum degree $\delta = 5$, we found exactly 16 graphs (Fig. 8) with $\beta = 2$ and $a = \delta = 5$. Thus graphs for which the bound in Theorem 13 is not true.

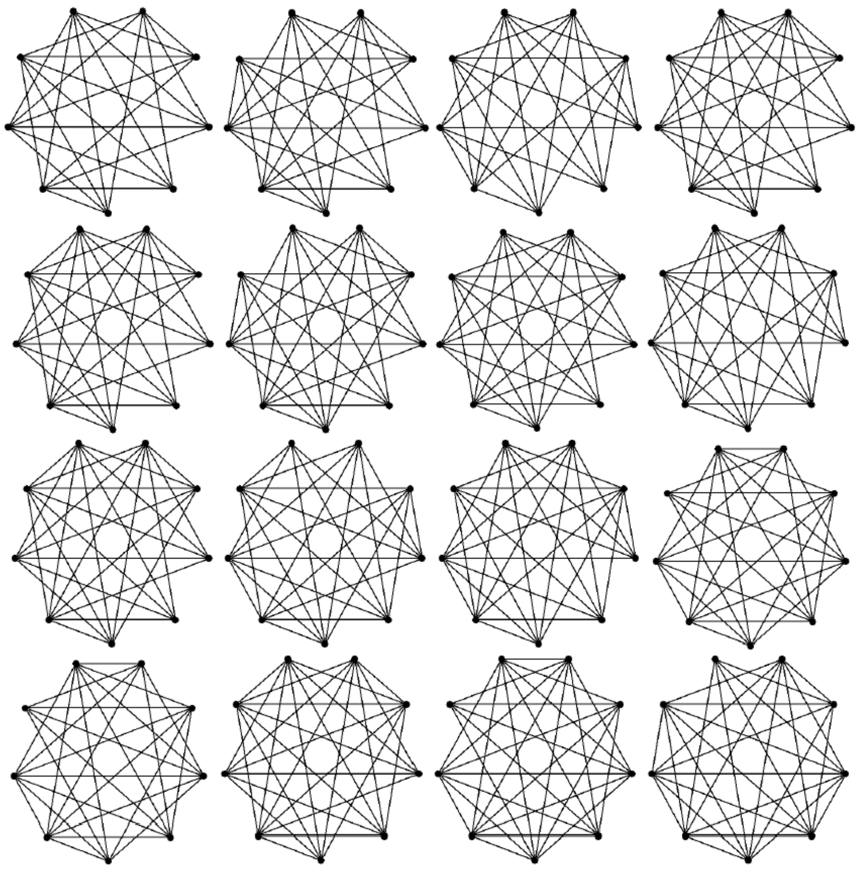

Fig. 8. The 16 graphs on $n = 9$ vertices with $a = \delta = 5$ and $\beta = 2$.

We are convinced that Theorem 13 is true for all values of the minimum degree, however we do not yet have the proof, so we close with the following conjecture.

Conjecture 14 :
Let G be a connected graph with odd order $n = 2k+1 \geq 9$, algebraic connectivity a, minimum degree δ and domination number $\beta \geq 3$. If $G \notin \{A_3, A_4, F_6, F_7, F_8\}$, then

$$a \leq 2k - 2\beta + \frac{k + 3 - \sqrt{(k+1)^2 + 4}}{2}$$

with equality if and only if G is $OC(K_{k+1})$, $PC(K_{k+1})$ or $PC(K_{k+1}) - e^*$.

References

[1] N. M. M. de Abreu, *Old and New Results on Algebraic Connectivity of Graphs.* Linear Algebra Appl. 423 (2007) 53–73.

[2] M. Aouchiche, J.-M. Bonnefoy, A. Fidahoussen, G. Caporossi, P. Hansen, L. Hiesse, J. Lacheré and A. Monhait, *Variable Neighborhood Search for Extremal Graphs. 14. The AutoGraphiX 2 System.* In L. Liberti and N. Maculan (editors), Global Optimization: From Theory to Implementation, Springer (2006) 281–310.

[3] M. Aouchiche, G. Caporossi and P. Hansen, *Variable Neighborhood Search for Extremal Graphs. 20. Automated Comparison of Graph Invariants.* MATCH Commun. Math. Comput. Chem. 58 (2007) 365–384.

[4] M. Aouchiche, G. Caporossi, P. Hansen and M. Laffay, *AutoGraphiX: a survey.* Elec. Notes in Disc. Math. 22 (2005) 515–520.

[5] G. Caporossi and P. Hansen, *Variable Neighborhood Search for Extremal Graphs. I. The AutoGraphiX System.* Disc. Math. 212 (2000) 29–44.

[6] G. Caporossi and P. Hansen, *Variable Neighborhood Search for Extremal Graphs. 5. Three Ways to Automate Finding Conjectures.* Disc. Math. 276 (2004), no. 1-3, 81–94.

[7] W. E. Clark, B. Shekhtman, S. Suen and D. C. Fisher, *Upper Bounds for the Domination Number of a Graph.* Congr. Numer. 132 (1998) 99–123.

[8] W. E. Clark and A. Dunning, *Tight Upper Bounds for the Domination Numbers of Graphs with Given Order and Minimum Degree.* Electron. J. Combin. 4 (1997) Research Paper 26, 25 pp. (electronic).

[9] D. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs — Theory and Application.* Academic Press, New York, 1980.

[10] M. Fiedler, *Algebraic Connectivity of Graphs.* Czechoslovak Math. J. 23 (1973) 298–305.

[11] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, *On Graphs Having Domination Half their Order.* Period. Math. Hungar., 16 (1985) 287–293.

[12] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs.* Monographs and Textbooks in Pure and Applied Mathematics, 208. Marcel Dekker, Inc., New York, 1998.

[13] M. Lu, H. Liu and F. Tian, *Bounds on Laplacian Spectrum of Graphs Based on Domination Number.* Linear Algebra Appl. 402 (2005) 390–396.

[14] V. Nikiforov, *Bounds on Graph Eigenvalues. I.* Linear Algebra Appl. 420 (2007) 667–671.

[15] C. Payan, *Sur le Nombre d'Absorption d'un Graphe Simple.* Cahiers Centre Études Recherche Opér. 17 (1975) 307–317.

[16] C. Payan and N. H. Xuong, *Domination–Balanced Graphs.* J. Graph Theory, 6 (1982) 23–32.

[17] B. Reed, *Paths, Stars and the Number Three.* Combin. Probab. Comput. 5 (1996) 277–295.

[18] L. Volkmann, *Upper Bounds on the Domination Number of a Graph in Terms of Order and Minimum Degree.* Ars Combin. 81 (2006) 3–22.

[19] H. M. Xing, L. Sun and X. G. Chen, *Domination in Graphs of Minimum Degree Five.* Graphs Combin. 22 (2006) 127–143.