

PARAMETERIZATIONS OF GENERALIZED NURBS VOLUMES OF REVOLUTION

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In this paper, we propose several algorithms for computing parameterizations of NURBS domains (surfaces and volumes) which are motivated by the recent research in isogeometric analysis. We describe methods for finding parameterizations of planar NURBS domains bounded by either a given closed NURBS curve, or four NURBS curves fulfilling compatibility conditions. Further, parameterizations of NURBS volumes of revolution based on two different parameterizations of a disc are presented. The main result of the paper is the formulation of an algorithm for computing parameterizations of the so-called generalized NURBS volumes of revolution.

Keywords: CAD/CAM, parametrization, B-spline and NURBS objects

1. Introduction

In 1950s and 1960s, the developments of the so-called *finite element method (FEM)* started with the progress in aerospace industry. In late 1960s, the first commercial software packages based on the finite element analysis appeared (e.g. NASTRAN). Nowadays, the finite element method (see e.g. [5]) is also used in many other engineering applications.

On the other hand, computer aided design (CAD) and computer aided geometric design (CAGD) have their origins mainly in 1970s and 1980s. The motivation came again from the industry – in this case from automotive industry. The modern history of this discipline started with the definition of the so-called Bézier curves and fruitful research continues till the present day.

Since CAD and CAGD originated later than FEM, this is probably the reason why the connections between CAD/CAGD and FEM were not so strong. Nevertheless, Hughes et al. in [7] recently proposed a method called *isogeometric analysis* which bridges this gap and which strongly connects CAD and FEM. The thorough analysis has followed, see e.g. [3, 4, 1]. This method is based on NURBS objects (see [10]), which are standard objects used in current CAD/CAM systems. The overall strategy is similar to FEM in standard cases – the exact geometry in CAD system is used to construct a mesh of the so-called *NURBS elements* and this mesh can be subsequently refined by h-, p- or k-refinements. Further, using weak form of a boundary value problem a system of linear equations for control points of a new NURBS object representing an approximate solution of the original problem is set. One of the main advantages of the method is that we can avoid generation of a triangular (quadrangular) mesh from CAD representation to be able to perform the finite element

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analysis and create this mesh again and again after each change of the geometry, because this task is very time-consuming. Another advantage is that the geometry is represented exactly, not only approximately, which is suitable e.g. for thin shell analysis.

Solving boundary value problems with isogeometric analysis is closely related to the geometrical problem of a description of a computational domain with the help of NURBS surfaces/volumes. In CAD/CAM, only the boundary of an object is usually stored and to perform any computation we need to be able to derive automatically a NURBS surface describing a computational domain, if the boundary NURBS curves are given, and similarly for a NURBS volume, if the boundary NURBS surface(s) describing an object is(are) given. This problem has attracted attention of researchers in the last years, see e.g. [6, 8, 11, 1, 9]. It was revealed that the choice of inner control points of a NURBS description of such a domain can significantly influence results and convergence rates of the follow-up computations in isogeometric analysis.

In this paper, we will present several algorithms which propose a solution of this problem in some cases. Firstly, we will show how to derive a NURBS surface describing a domain bounded by either one closed NURBS curve, or four connected NURBS curves forming a closed domain. Further, we will focus on NURBS volumes and, with the help of results for planar NURBS domains, we formulate algorithms for finding a NURBS volume parameterization of solids bounded by NURBS surfaces of revolution and by the so-called generalized NURBS surfaces of revolution which is the main contribution of the paper.

The paper is organized as follows. Section 2 gives a brief overview of the theory of B-spline and NURBS objects. In Section 3, the algorithm for finding a NURBS description of a domain bounded by one closed NURBS curve or four NURBS curves fulfilling compatibility conditions is shown. Section 4 presents the algorithm for a computation of a NURBS description of a domain bounded by a NURBS surface of revolution. In Section 5, the algorithm is modified for the so-called generalized NURBS volumes of revolution. Finally, in Section 6 we conclude the paper.

2. B-spline and NURBS curves/surfaces/volumes

Current CAD/CAM systems mostly use the so-called B-spline curves and surfaces for representation of free-form curves and surfaces. In fact, cubic B-spline curves and bicubic B-spline surfaces are usually used, even though the systems usually offer the possibility to work also with NURBS curves and surfaces. In this section, we briefly review fundamentals of B-spline and NURBS objects (for more details see e.g. [10]).

Definition 1 A NURBS curve of degree p is determined by a control polygon of control points \mathbf{P}_i , their associated weights w_i , $i = 0, \dots, n$, and a knot vector $\mathbf{T} = (t_0, \dots, t_{n+p+1})$. Then its parameterization is of the form

$$\mathbf{c}(u) = \frac{\sum_{i=0}^n N_{i,p}(u) w_i \mathbf{P}_i}{\sum_{i=0}^n N_{i,p}(u) w_i}, \quad (1)$$

where $N_{i,p}(u)$ are B-spline basis functions of degree p corresponding to the knot vector \mathbf{T} .

For the given knot vector $\mathbf{T} = (t_0, \dots, t_{n+p+1})$, the B-spline basis functions are defined by a recursive formula

$$N_{i,0}(t) = \begin{cases} 1, & t_i \leq t < t_{i+1}, \\ 0, & \text{otherwise,} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t), \quad k = 1, \dots, p.$$

The knot vector \mathbf{T} is a non-decreasing sequence of real numbers which determines the distribution of a parameter of the corresponding curve/surface/volume. In general, two different forms of knot vectors are commonly distinguished:

- *periodic* knot vector $\mathbf{T} = (t_0, \dots, t_{n+p+1})$ where we have to add the desired degree p of B-spline basis functions (see Fig. 1 (top row)),
- *non-periodic* knot vector $\mathbf{T} = (\underbrace{a, \dots, a}_{p+1}, t_{p+1}, \dots, t_n, \underbrace{b, \dots, b}_{p+1})$ where p is a degree of the associated B-spline basis functions (see Fig. 1 (bottom row)).

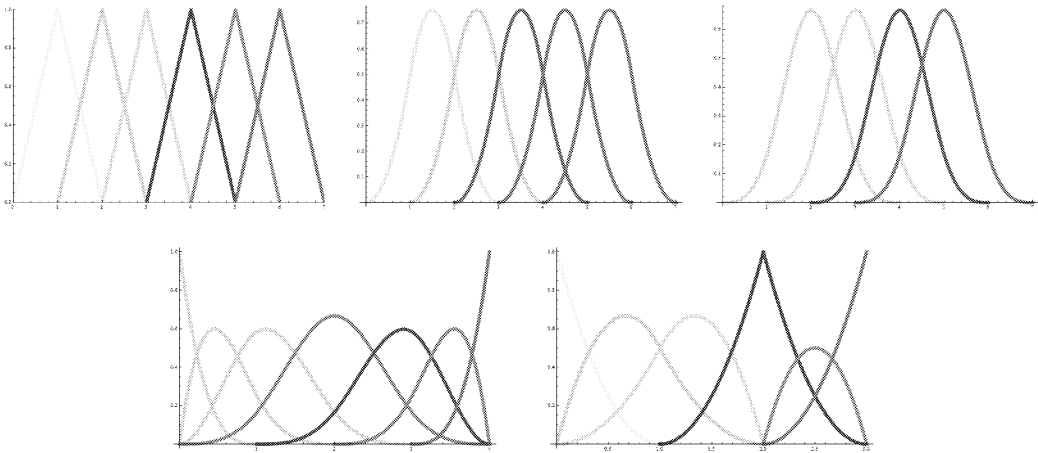


Fig.1: Top row: B-spline basis functions for the knot vector $\mathbf{T} = (0, 1, 2, 3, 4, 5, 6, 7)$ of degree p ($p = 1$ (left); $p = 2$ (middle); $p = 3$ (right)); Bottom row: B-spline basis functions for the knot vector $\mathbf{T} = (0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4)$, $p = 3$ (left) and for the knot vector $\mathbf{T} = (0, 0, 0, 1, 2, 2, 3, 3, 3)$, $p = 2$ (right)

Furthermore, we distinguish *uniform* (uniformly distributed knots) and *non-uniform* (non-uniformly distributed knots) knot vectors, where only internal knots are considered for non-periodic knot vectors.

One of the main properties of B-spline basis functions is a relation between the multiplicity of knots in the knot vector and the continuity of the corresponding B-spline basis functions – the B-spline basis functions are of continuity C^{p-k} , where k is the maximal multiplicity of inner knots in the knot vector. Among their other properties belong non-negativity, partition of unity and local support.

Remark 1 If all weights w_i of control points \mathbf{P}_i , $i = 0, \dots, n$, are equal, the denominator of (1) is a constant and a polynomial B-spline curve is obtained.

B-spline/NURBS curves possess a lot of important practical properties which follow from their definition and from the properties of B-spline basis functions, such as affine/projective invariance, convex hull property, local modification, variation diminishing property (see [10]).

As a straightforward extension, we can define NURBS surfaces. Similarly to the curve case, if weights of all control points are the same, the definition covers also polynomial B-spline surfaces (see Fig. 2).

Definition 2 A NURBS surface of degree (p, q) is determined by a control net of $(m + 1) \times (n + 1)$ control points \mathbf{P}_{ij} , their associated weights w_{ij} , $i = 0, \dots, m$, $j = 0, \dots, n$, and two knot vectors $\mathbf{U} = (u_0, \dots, u_{m+p+1})$, $\mathbf{V} = (v_0, \dots, v_{n+q+1})$. Then its parameterization is of the form

$$\mathbf{s}(u, v) = \frac{\sum_{i=0}^m \sum_{j=0}^n N_{i,p}(u) N_{j,q}(v) w_{ij} \mathbf{P}_{ij}}{\sum_{i=0}^m \sum_{j=0}^n N_{i,p}(u) N_{j,q}(v) w_{ij}}, \quad (2)$$

where $N_{i,p}(u)$ and $N_{j,q}(v)$ are B-spline basis functions of degree p and q corresponding to the knot vectors \mathbf{U} and \mathbf{V} , respectively.

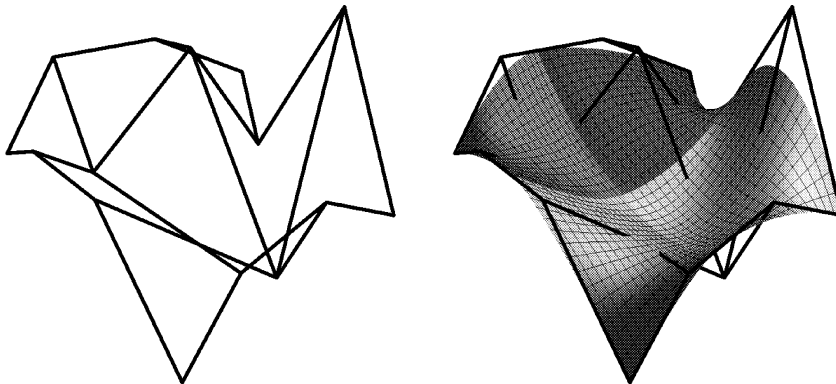


Fig.2: Left: Control net of a NURBS surface; Right: Control net and the corresponding NURBS surface composed of 4 patches

NURBS surfaces share most of the properties with NURBS curves (see [10]). Finally, we can analogously define B-spline/NURBS volumes.

Definition 3 A NURBS volume of degree (p, q, r) is determined by a control net of $(m + 1) \times (n + 1) \times (l + 1)$ control points \mathbf{P}_{ijk} , their associated weights w_{ijk} , $i = 0, \dots, m$, $j = 0, \dots, n$, $k = 0, \dots, l$ and three knot vectors $\mathbf{U} = (u_0, \dots, u_{m+p+1})$, $\mathbf{V} = (v_0, \dots, v_{n+q+1})$, $\mathbf{W} = (w_0, \dots, w_{l+r+1})$. Then its parameterization is of the form

$$\mathbf{v}(u, v, w) = \frac{\sum_{i=0}^m \sum_{j=0}^n \sum_{k=0}^l N_{i,p}(u) N_{j,q}(v) N_{k,r}(w) w_{ijk} \mathbf{P}_{ijk}}{\sum_{i=0}^m \sum_{j=0}^n \sum_{k=0}^l N_{i,p}(u) N_{j,q}(v) N_{k,r}(w) w_{ijk}}, \quad (3)$$

where $N_{i,p}(u)$, $N_{j,q}(v)$ and $N_{k,r}(w)$ are B-spline basis functions of degree p , q and r corresponding to the knot vectors \mathbf{U} , \mathbf{V} and \mathbf{W} , respectively.

To close this section, let us point out that the algorithms for finding parameterizations of planar and spatial NURBS domains presented in the following sections are mainly motivated by the needs of isogeometric analysis to have the same representation for the geometry and physical solutions. This leads us to the fact that ‘good’ parameterizations of domains for isogeometric analysis are described by injective mappings. One of the sufficient conditions is that a parameterization is injective if it defines a regular boundary and its Jacobian does not vanish on the parameter domain, see [12] for more details.

3. Parameterizations of planar NURBS domains

In this section we will describe a method for finding a NURBS description of a planar domain bounded by four connected NURBS curves or by one closed NURBS curve (see [2]).

At the end, we will apply these methods to obtain different parameterizations of a disc, which we will need in the remaining sections.

Let Ω be a simply connected domain given by four boundary NURBS curves $\mathbf{b}_1(u)$, $\mathbf{b}_2(v)$, $\mathbf{b}_3(u)$ and $\mathbf{b}_4(v)$, $u \in [0, 1]$, $v \in [0, 1]$, fulfilling the compatibility conditions

$$\mathbf{b}_1(1) = \mathbf{b}_2(0) , \quad \mathbf{b}_2(1) = \mathbf{b}_3(1) , \quad \mathbf{b}_3(0) = \mathbf{b}_4(1) , \quad \mathbf{b}_4(0) = \mathbf{b}_1(0) . \quad (4)$$

Let us assume that each boundary NURBS curve \mathbf{b}_i is determined by control points \mathbf{P}_j^i , their weights w_j^i , $j = 0, \dots, n_i$, a non-periodic knot vector \mathbf{U}_i and a degree p_i . We want to find a control net and corresponding knot vectors for planar NURBS surface describing Ω .

In the first step of the algorithm, it is necessary to modify opposite boundary curves $\mathbf{b}_1(u)$, $\mathbf{b}_3(u)$ and $\mathbf{b}_2(v)$, $\mathbf{b}_4(v)$ such that they have the same knot vectors and, consequently, also the same number of control points. This can be done in two steps:

1. If $p_1 \neq p_3$ or $p_2 \neq p_4$, the standard algorithm for degree elevation (see [10]) is applied on the curve with the lower degree. After this step, we obtain new knot vectors $\widehat{\mathbf{U}}_i$, $i = 1, \dots, 4$.
2. If the degrees are the same, but the inner knots in the knot vectors $\widehat{\mathbf{U}}_1$, $\widehat{\mathbf{U}}_3$ or $\widehat{\mathbf{U}}_2$, $\widehat{\mathbf{U}}_4$ are still different, the algorithm for knot insertion (see [10]) is used. For instance, if $\widehat{\mathbf{U}}_1 \neq \widehat{\mathbf{U}}_3$, we insert all inner knots in $\widehat{\mathbf{U}}_1$, which are not in $\widehat{\mathbf{U}}_3$, into $\widehat{\mathbf{U}}_3$ and vice versa. Let us denote new knot vectors $\bar{\mathbf{U}}_i$ and new control points $\bar{\mathbf{P}}_j^i$, $i = 1, 3$, $j = 0, \dots, m_1$ and $\bar{\mathbf{P}}_j^i$, $i = 2, 4$, $j = 0, \dots, m_2$.

The next step is to find inner control points \mathbf{Q}_j^i , $i = 1, \dots, m_2 - 1$, $j = 1, \dots, m_1 - 1$, of the final control net

$$\begin{pmatrix} \bar{\mathbf{P}}_0^1 = \bar{\mathbf{P}}_0^4 & \dots & \dots & \dots & \bar{\mathbf{P}}_{m_1}^1 = \bar{\mathbf{P}}_0^2 \\ \vdots & \mathbf{Q}_1^1 & \dots & \mathbf{Q}_{m_1-1}^1 & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \mathbf{Q}_{m_2-1}^{m_2-1} & \dots & \mathbf{Q}_{m_1-1}^{m_2-1} & \vdots \\ \bar{\mathbf{P}}_{m_2}^4 = \bar{\mathbf{P}}_0^3 & \dots & \dots & \dots & \bar{\mathbf{P}}_{m_2}^2 = \bar{\mathbf{P}}_{m_1}^3 \end{pmatrix} .$$

One of the possibilities is to compute the control points \mathbf{A}_j^i , $i = 0, \dots, m_2$, $j = 0, \dots, m_1$, and \mathbf{B}_j^i , $i = 0, \dots, m_2$, $j = 0, \dots, m_1$, of the ruled surfaces given by $\bar{\mathbf{P}}_j^1$, $\bar{\mathbf{P}}_j^3$, $j = 0, \dots, m_1$ and $\bar{\mathbf{P}}_j^2$, $\bar{\mathbf{P}}_j^4$, $j = 0, \dots, m_2$, respectively. Thus, we can write

$$\mathbf{A}_j^i = \bar{\mathbf{P}}_j^1 + \frac{i}{m_2} (\bar{\mathbf{P}}_j^3 - \bar{\mathbf{P}}_j^1) , \quad \mathbf{B}_j^i = \bar{\mathbf{P}}_j^4 + \frac{j}{m_1} (\bar{\mathbf{P}}_j^2 - \bar{\mathbf{P}}_j^4) , \quad i = 0, \dots, m_2, \quad j = 0, \dots, m_1 .$$

Then the inner control points \mathbf{Q}_j^i , $i = 1, \dots, m_2 - 1$, $j = 1, \dots, m_1 - 1$, can be found as centres of the corresponding line segments $\mathbf{A}_j^i \mathbf{B}_j^i$, i.e.,

$$\mathbf{Q}_j^i = \frac{1}{2} (\mathbf{A}_j^i + \mathbf{B}_j^i) , \quad i = 1, \dots, m_2 - 1 , \quad j = 1, \dots, m_1 - 1 .$$

The weights of control points \mathbf{Q}_j^i , $i = 1, \dots, m_2 - 1$, $j = 1, \dots, m_1 - 1$, can be set all equal to 1, or a similar method as described above can be used also for finding these weights.

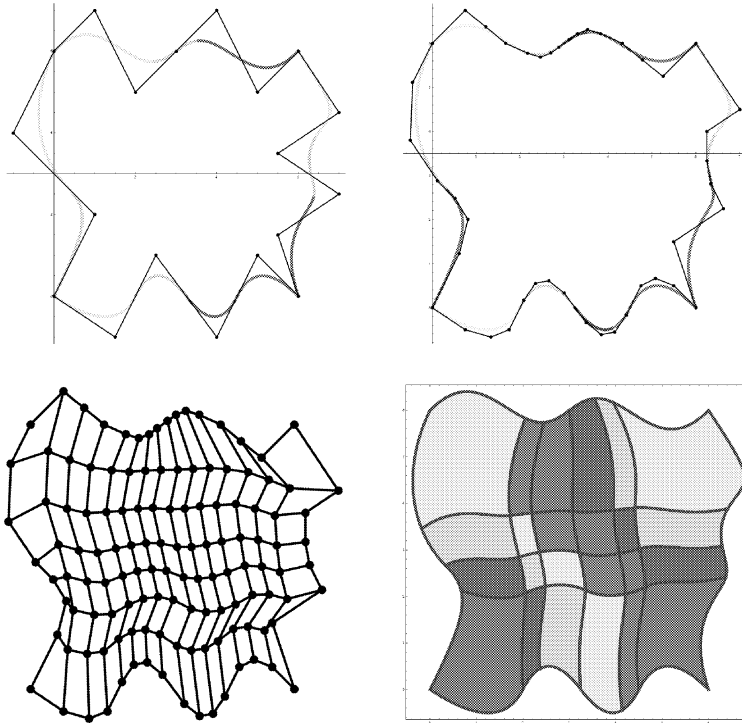


Fig.3: Top left: Initial boundary curves; Top right: Boundary curves after degree elevation and knot insertion; Bottom left: The control net of the planar domain; Bottom right: Resulting planar domain as the NURBS surface

Example 1 Let the boundary curves $\mathbf{b}_1(u)$, $\mathbf{b}_2(v)$, $\mathbf{b}_3(u)$ and $\mathbf{b}_4(v)$, $u \in [0, 1]$, $v \in [0, 1]$, be given by the control points and weights

$$\begin{aligned} \mathbf{P}^1 &= \{(0, 6, 1), (1, 7, 1), (2, 5, 1), (3, 6, 2), (4, 7, 1), (5, 5, 1), (6, 6, 1)\} , \\ \mathbf{P}^2 &= \{(6, 6, 1), (7, 9/2, 3), (11/2, 7/2, 1), (7, 5/2, 1), (11/2, 3/2, 1), (6, 0, 1)\} , \\ \mathbf{P}^3 &= \{(0, 0, 1), (3/2, -1, 1), (5/2, 1, 1), (4, -1, 1), (5, 1, 1), (6, 0, 1)\} , \\ \mathbf{P}^4 &= \{(0, 6, 1), (-1, 4, 1), (1, 2, 2), (0, 0, 1)\} , \end{aligned}$$

where the last ‘coordinate’ represents the weight of the control point, and the knot vectors

$$\begin{aligned} \mathbf{U}^1 &= (0, 0, 0, 0, 0, 1/3, 2/3, 1, 1, 1, 1, 1) , & \mathbf{U}^2 &= (0, 0, 0, 0, 0, 1/3, 2/3, 1, 1, 1, 1, 1) , \\ \mathbf{U}^3 &= (0, 0, 0, 0, 1/4, 1/2, 3/4, 1, 1, 1, 1) , & \mathbf{U}^4 &= (0, 0, 0, 0, 1/2, 1, 1, 1, 1) . \end{aligned}$$

Intermediate steps of the algorithm and the final result are shown in Fig. 3.

This method can be directly modified if only one closed NURBS curve, determining a simply connected domain, is given. Let $\mathbf{b}(u)$, $u \in [0, 1]$, be a closed boundary NURBS curve determined by its control points \mathbf{P}_i , their weights w_i , $i = 0, \dots, n$, a non-periodic knot vector \mathbf{U} and a degree p .

In the first step, we need to subdivide $\mathbf{b}(u)$ into four parts. Firstly, new knots are inserted such that \mathbf{U} contains $2r$ internal knot spans of non-zero length. Further, we denote $s = \lfloor \frac{r}{2} \rfloor$ and $t = r - s$. Then, we select internal knots which correspond to end knots of s -th, $(s+t)$ -th

and $(2s + t)$ -th knot span of non-zero length and we increase the multiplicity of these knots to $p + 1$ with the help of knot insertion. For each knot u_i of multiplicity $p + 1$, the control polygon contains as a control point directly the point $\mathbf{b}(u_i)$ on the curve, i.e., the boundary curve is subdivided at these points into four NURBS curves. Similarly, the knot vector can be subdivided at all inner knots of multiplicity $p + 1$ to obtain a full description of four boundary curves determining the same domain.

Then, we can directly apply the above mentioned algorithm, only the degree elevation is not necessary because all boundary curves have the same degree now.

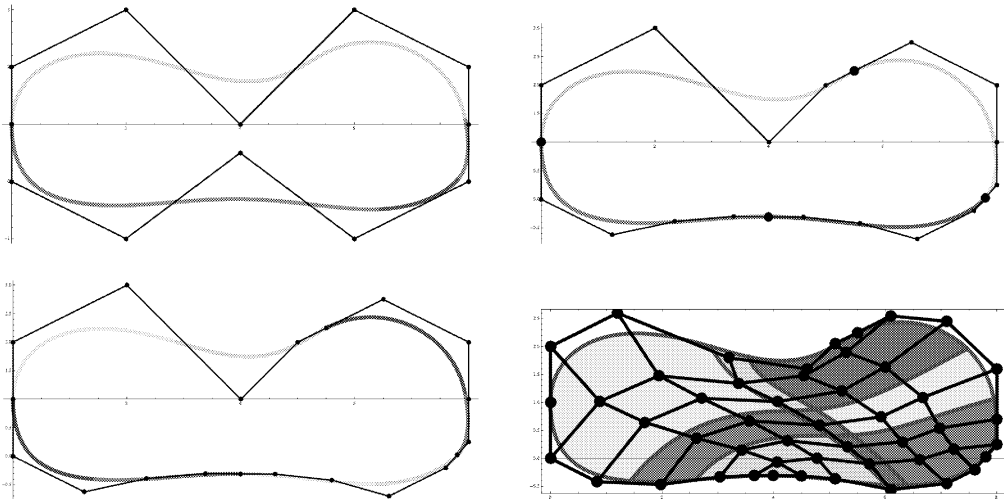


Fig.4: Top left: Initial closed boundary curve; Top right: Curve subdivided into four parts; Bottom left: Boundary curves after knot insertion; Bottom right: The control net and the resulting planar domain as the NURBS surface

Example 2 Let the closed boundary curve $\mathbf{b}(u)$, $u \in [0, 1]$, be given by the control points and weights

$$\mathbf{P} = \{(0, 1, 1), (0, 2, 1), (2, 3, 1), (4, 1, 1), (6, 3, 1), (8, 2, 1), (8, 1, 1), (8, 0, 1), (6, -1, 1), (4, 1/2, 1), (2, -1, 1), (0, 0, 1), (0, 1, 1)\},$$

where the last ‘coordinate’ represents the weight of the control point, and the knot vector

$$\mathbf{U} = (0, 0, 0, 0, 0, 1/10, 1/5, 1/5, 2/5, 1/2, 1/2, 3/5, 9/10, 1, 1, 1, 1, 1).$$

The subdivision into four boundary curves and other intermediate steps of the algorithm, together with the final result, are shown in Fig. 4.

In this example, the curve $\mathbf{b}(u)$, $u \in [0, 1]$, is C^1 continuous everywhere, as follow from the special position of the first two and the last two control points of \mathbf{P} . Nevertheless, the above mentioned method can be applied to an arbitrary C^0 continuous closed NURBS curve.

In the rest of this section, we will focus on different parameterizations of a disc, which we will need in the next sections, and how they can be obtained with the help of algorithms described above.

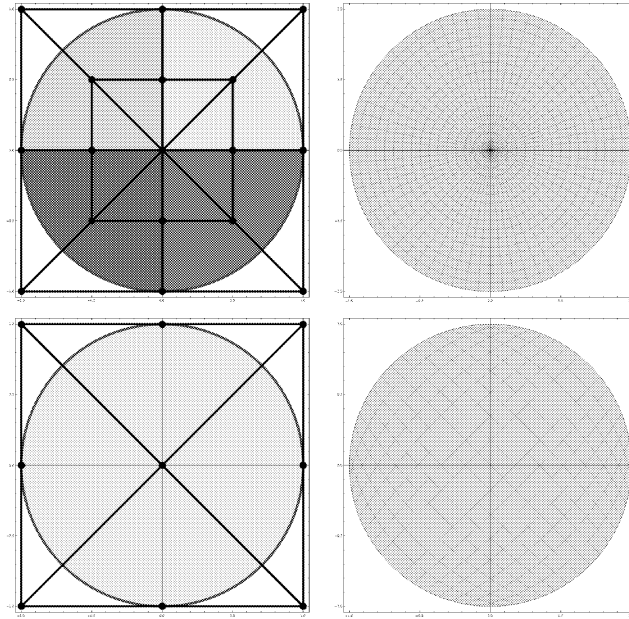


Fig.5: Top left: NURBS description of the unit disc obtained by the method (a); Top right: Distribution of parameter curves for the case (a); Bottom left: NURBS description of the unit disc obtained by the method (b); Bottom right: Distribution of parameter curves for the case (b)

Example 3 The unit circle with the center at the origin can be described by the control polygon

$$\mathbf{P} = \left\{ (1, 0, 1), (1, 1, \sqrt{2}/2), (0, 1, 1), (-1, 1, \sqrt{2}/2), (-1, 0, 1), (-1, -1, \sqrt{2}/2), \right. \\ \left. (0, -1, 1), (1, -1, \sqrt{2}/2), (1, 0, 1) \right\},$$

where the last ‘coordinate’ represents the weight, and the knot vector

$$\mathbf{U} = (0, 0, 0, 1/4, 1/4, 1/2, 1/2, 3/4, 3/4, 1, 1, 1).$$

In this example, we will show two possibilities of NURBS description of the unit disc, which will be used in the next sections:

- (a) The first possibility is to add the circle center to the control net and connect it with all boundary control points – moreover, it is possible to choose the second knot vector \mathbf{V} (see Fig. 5 (top row)). From the computational point of view, the problem is that the circle center becomes a singular point of the parameterization, i.e., this center corresponds to the whole line segment in a parameter domain.
- (b) The second possibility exploits the chosen NURBS description of the unit circle. Since the unit circle described by these control points and the knot vector is composed of four arcs, we can directly apply the above mentioned algorithm and obtain the parameterization with better distribution of parametric curves for the follow-up computations (see Fig. 5 (bottom row)).

Remark 2 The algorithm presented in this section can be directly used also for spatial NURBS curves, i.e., either for one closed spatial NURBS curve, or for four spatial NURBS curves fulfilling the compatibility conditions (4).

4. Parameterizations of domains bounded by NURBS surfaces of revolution

In this section we will demonstrate two different approaches for finding a NURBS volume parameterization of a solid bounded by a NURBS surface of revolution.

Let Σ be a volume bounded by a NURBS surface of revolution $\mathbf{s}(u, v)$, $u \in [0, 1]$, $v \in [0, 1]$, which is determined by a NURBS curve $\mathbf{c}(u)$, $u \in [0, 1]$, rotating around an axis o . Let $\mathbf{c}(u)$, $u \in [0, 1]$, be given by control points \mathbf{P}_i , associated weights w_i , $i = 0, \dots, n$ and a knot vector \mathbf{R} .

Input: Control points \mathbf{P}_i , weights w_i , $i = 0, \dots, n$ of a NURBS curve $\mathbf{c}(u)$, $u \in [0, 1]$, knot vector \mathbf{R} , axis $o : \mathbf{x} = \mathbf{O}_0 + \alpha(\mathbf{O}_1 - \mathbf{O}_0)$

Output: NURBS volume Σ determined by control net $\mathbf{P}_{i,j,k}$, weights $W_{i,j,k}$, $i = 0, \dots, n$, $j = 0, 1, 2$, $k = 0, 1, 2$, knot vectors \mathbf{R} , \mathbf{S} and \mathbf{T}

1: $\mathbf{S} = (0, 0, 0, 1, 1, 1)$, $\mathbf{T} = (0, 0, 0, 1, 1, 1)$ {Set the knot vectors \mathbf{S} , \mathbf{T} .}

2: $\mathbf{P}'_i \in \mathbf{O}_0 \mathbf{O}_1 \cdot (\mathbf{x} - \mathbf{P}_i) \cap o$ {Determine the orthogonal projections \mathbf{P}'_i of \mathbf{P}_i to axis o .}

3: **if** $\bigwedge_{i=0}^{n-1} (\|\mathbf{O}_0 \mathbf{P}'_i\| < \|\mathbf{O}_0 \mathbf{P}'_{i+1}\|)$

4: **for** $i = 0, \dots, n$

5: $d_i = \|\mathbf{P}'_i \mathbf{P}_i\|$

6: $\mathbf{q} = d_i \frac{\mathbf{O}_0 \mathbf{P}_i \times \mathbf{P}_i \mathbf{P}'_i}{\|\mathbf{O}_0 \mathbf{P}_i \times \mathbf{P}_i \mathbf{P}'_i\|}$

7: Compute control points of the disc centred at \mathbf{P}'_i and lying in $\mathbf{x} = \mathbf{P}'_i + \alpha \mathbf{P}'_i \mathbf{P}_i + \beta \mathbf{q}$

$$\mathbf{P}_{rs} = \begin{cases} \mathbf{P}'_i + s \mathbf{q} & \text{for } r = 0, s \neq 0, \\ \mathbf{P}'_i + r \mathbf{P}'_i \mathbf{P}_i & \text{for } s = 0, r \neq 0, \\ \mathbf{P}'_i + r \mathbf{P}'_i \mathbf{P}_i + s \mathbf{q} & \text{for } r \neq 0, s \neq 0, \end{cases} \quad \text{where } r = -1, 0, 1, \quad s = -1, 0, 1$$

8: NURBS description of the disc

$$\mathbf{P}_{ijk} = \begin{pmatrix} \mathbf{P}_i & \mathbf{P}_{1,1} & \mathbf{P}_{0,1} \\ \mathbf{P}_{1,-1} & \mathbf{P}'_i & \mathbf{P}_{-1,1} \\ \mathbf{P}_{0,-1} & \mathbf{P}_{-1,-1} & \mathbf{P}_{-1,0} \end{pmatrix}, \quad j = 0, 1, 2, \quad k = 0, 1, 2$$

9: Associated weights of the control points \mathbf{P}_{ijk}

$$W_{ijk} = \begin{pmatrix} 2w_i & w_i & w_i \\ w_i & 2w_i & w_i \\ w_i & w_i & 2w_i \end{pmatrix}, \quad j = 0, 1, 2, \quad k = 0, 1, 2$$

10: **end for**

11: **end if**

12: **return** $\mathbf{P}_{i,j,k}$, $W_{i,j,k}$, \mathbf{R} , \mathbf{S} , \mathbf{T}

Algorithm 1: Parameterization of NURBS volume of revolution

Remark 3 Let a rotation axis o be given by two points \mathbf{O}_0 and \mathbf{O}_1 and let $\mathbf{c}(u)$, $u \in [0, 1]$, be given by control points \mathbf{P}_i , associated weights w_i , $i = 0, \dots, n$ and a knot vector \mathbf{R} . Further, let \mathbf{P}'_i be an orthogonal projection of \mathbf{P}_i to the axis o and let $\mathbf{v}(u, v, w)$ be a NURBS volume

bounded by $s(u, v)$ and obtained by Algorithm 1. If

$$\bigwedge_{i=0}^{n-1} (\|O_0 P'_i\| < \|O_0 P'_{i+1}\|)$$

holds, then all u -parametric curves of $\mathbf{v}(u, v, w)$ do not mutually intersect (and similarly for v - and w -parametric curves).

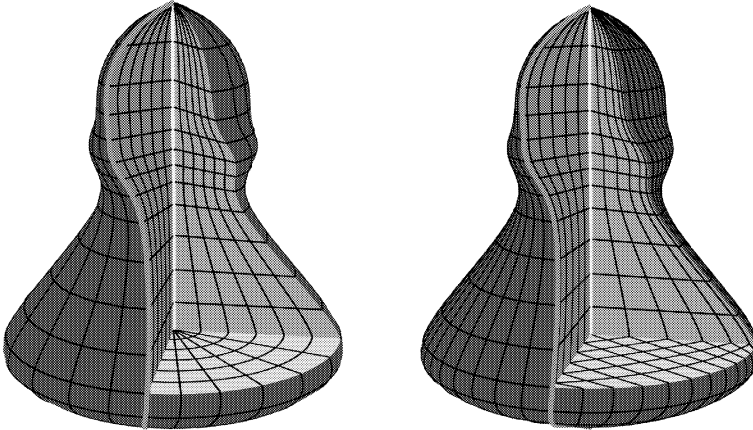


Fig.6: The final NURBS volumes for Example 4 obtained by Algorithm 1 (right) and by the modified approach (left)

If the condition mentioned in Remark 3 is fulfilled, we can use different parameterizations of a disc (see Section 3) to construct a control net of the corresponding NURBS volume. For each control point P_i determining $\mathbf{c}(u)$ we need to determine a plane perpendicular to the rotation axis o and containing P_i . In this plane, we construct the disc centred at P'_i and with the radius $\|P_i P'_i\|$, i.e., we can directly modify control points and weights obtained by the method mentioned in Example 3 (b) (see Steps 7–9 of Algorithm 1). The whole algorithm is summarized in Algorithm 1.

The approach used in Algorithm 1 can be modified by taking another NURBS description of a disc (see Example 3 (a)). In particular, we can replace Steps 7–9 by the following expressions

$$7': P_{rs} = \begin{cases} P'_i + r P'_i P_i & \text{for } s = 0, \\ P'_i + r P'_i P_i + s \mathbf{q} & \text{for } s \neq 0, \end{cases} \quad \text{where } r = -1, 1, s = -1, 0, 1$$

$$8': P_{ijk} = \begin{pmatrix} P_{1,0} & P_{1,1} & P_{-1,1} & P_{-1,0} & P_{-1,-1} & P_{1,-1} & P_{1,0} \\ P'_i & P'_i & P'_i & P'_i & P'_i & P'_i & P'_i \end{pmatrix} \quad j = 0, 1, k = 0, \dots, 6$$

$$9': W_{ijk} = \begin{pmatrix} w_i & w_i/2 & w_i/2 & w_i & w_i/2 & w_i/2 & w_i \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad j = 0, 1, k = 0, \dots, 6$$

where another NURBS description of a circle is used in Step 7' and each control point P_{rs} , $r = -1, 1, s = -1, 0, 1$, determining the circle is connected with P'_i .

Example 4 Let $\mathbf{c}(u)$ be given by the control points

$$P = \{(1, 0, 7, 1), (0, 3, 6, 1), (0, 2, 4, 2), (1, 3, 2, 2), (1, 1, 3/2, 2), (5, 6, -5, 2), (1, 2, -6, 1/2), (1, 0, -7, 1/2)\},$$

where the last ‘coordinate’ represents the weight, and the knot vector

$$\mathbf{R} = (0, 0, 0, 0, 1/4, 1/4, 1/2, 3/4, 1, 1, 1, 1).$$

Using Algorithm 1 we obtain the result shown in Fig. 6 (right), the result obtained by the modified approach is shown in Fig. 6 (left).

5. Parameterizations of domains bounded by generalized NURBS surfaces of revolution

In this section, two methods for finding a NURBS volume parameterization of the so-called generalized NURBS volume of revolution are presented.

Input: Boundary NURBS surface determined by a control net

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_{0,0} & \cdots & \mathbf{P}_{0,m} \\ \vdots & \ddots & \vdots \\ \mathbf{P}_{n,0} & \cdots & \mathbf{P}_{n,m} \end{pmatrix},$$

where $\mathbf{P}_{0,0} = \mathbf{P}_{0,1} = \cdots = \mathbf{P}_{0,m}$, $\mathbf{P}_{n,0} = \mathbf{P}_{n,1} = \cdots = \mathbf{P}_{n,m}$ and $\mathbf{P}_{i,0} = \mathbf{P}_{i,m}, \forall i$, weights $w_{i,j}, i = 0, \dots, n, j = 0, \dots, m$, and knot vectors \mathbf{R} and \mathbf{S}

Output: NURBS volume Σ determined by control net $\mathbf{P}_{i,j,k}$, weights $W_{i,j,k}$, knot vectors \mathbf{R}, \mathbf{S} and \mathbf{T}

- 1: $o : \mathbf{x} = \mathbf{P}_{0,0} + \alpha (\mathbf{P}_{n,0} - \mathbf{P}_{0,0})$ {The axis determined by the points $\mathbf{P}_{0,0}, \mathbf{P}_{n,0}$ }
- 2: $\mathbf{P}'_{i,j} \in \mathbf{P}_{0,0} \mathbf{P}_{n,0} \cdot (\mathbf{x} - \mathbf{P}_{i,j}) \cap o$ {Determine the orthogonal projections $\mathbf{P}'_{i,j}$ of $\mathbf{P}_{i,j}$ to o .}
- 3: **if** $\bigwedge_{j=0}^m \bigwedge_{i=0}^{n-1} (\|\mathbf{P}_{0,0} \mathbf{P}'_{i,j}\| < \|\mathbf{P}_{0,0} \mathbf{P}'_{i+1,j}\|)$
- 4: **for** $i = 1, \dots, n - 1$
- 5: Compute control points, weights and knot vectors of a NURBS surface with boundary NURBS curve determined by i -th row of \mathbf{P} with the help of algorithm in Section 3

$$\{\mathbf{P}_{i,j,k}, W_{i,j,k}, \mathbf{S}_1, \mathbf{S}_2\} = \text{NURBSSurface}(\mathbf{P}_{i,0}, \dots, \mathbf{P}_{i,m}, w_{i,0}, \dots, w_{i,m}, \mathbf{S})$$

- 6: **end for**
 - 7: **end if**
 - 8: **return** $\mathbf{P}_{i,j,k}, W_{i,j,k}, \mathbf{R}, \mathbf{S}_1, \mathbf{S}_2$
-

Algorithm 2: Parameterization of generalized NURBS volume of revolution

Definition 4 Let a NURBS surface $\mathbf{s}(u, v), u \in [0, 1], v \in [0, 1]$, be determined by a control net

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_{0,0} & \cdots & \mathbf{P}_{0,m} \\ \vdots & \ddots & \vdots \\ \mathbf{P}_{n,0} & \cdots & \mathbf{P}_{n,m} \end{pmatrix}, \tag{5}$$

where $\mathbf{P}_{0,0} = \mathbf{P}_{0,1} = \cdots = \mathbf{P}_{0,m}$, $\mathbf{P}_{n,0} = \mathbf{P}_{n,1} = \cdots = \mathbf{P}_{n,m}$ and $\mathbf{P}_{i,0} = \mathbf{P}_{i,m}, \forall i$, weights $w_{i,j}, i = 0, \dots, n, j = 0, \dots, m$, and knot vectors \mathbf{R} and \mathbf{S} . Further, let o be an axis determined by points $\mathbf{P}_{0,0}$ and $\mathbf{P}_{n,0}$ and $\mathbf{P}'_{i,j}$ be an orthogonal projection of $\mathbf{P}_{i,j}$ to the axis o . If

$$\bigwedge_{j=0}^m \bigwedge_{i=0}^{n-1} (\|\mathbf{P}_{0,0} \mathbf{P}'_{i,j}\| < \|\mathbf{P}_{0,0} \mathbf{P}'_{i+1,j}\|) \tag{6}$$

holds, then the volume bounded by $\mathbf{s}(u, v)$ is called a generalized NURBS volume of revolution.

Let $\mathbf{s}(u, v)$, $u \in [0, 1]$, $v \in [0, 1]$, be a NURBS surface given by a control net (5) fulfilling $\mathbf{P}_{0,0} = \mathbf{P}_{0,1} = \dots = \mathbf{P}_{0,m}$, $\mathbf{P}_{n,0} = \mathbf{P}_{n,1} = \dots = \mathbf{P}_{n,m}$ and $\mathbf{P}_{i,0} = \mathbf{P}_{i,m}, \forall i$, weights $w_{i,j}$, $i = 0, \dots, n$, $j = 0, \dots, m$ and knot vectors \mathbf{R} and \mathbf{S} . Let us denote by $\mathbf{c}_i(u)$, $i = 1, \dots, n - 1$, a NURBS curve determined by control points $\mathbf{P}_{i,0}, \dots, \mathbf{P}_{i,m}$, associated weights $w_{i,0}, \dots, w_{i,m}$ and a knot vector \mathbf{S} .

In the first step of the algorithm, we check whether the condition (6) is satisfied and $\mathbf{s}(u, v)$ bounds a generalized NURBS volume of revolution.

Further, we take each $\mathbf{c}_i(u)$ and the algorithm presented in Section 3 is applied to this curve (see Remark 2). We obtain control points with associated weights describing a NURBS surface bounded by $\mathbf{c}_i(u)$, together with new knot vectors $\mathbf{S}_1, \mathbf{S}_2$. The knot vectors $\mathbf{S}_1, \mathbf{S}_2$ arise from the subdivision of the given knot vector \mathbf{S} and depend on the knots inserted to \mathbf{S} in order to subdivide $\mathbf{c}_i(u)$ into four parts. Let us emphasize that it is necessary to insert the same knots into \mathbf{S} during subdividing $\mathbf{c}_i(u)$ for all $i = 1, \dots, n - 1$ in order to ensure that the knot vectors \mathbf{S}_1 and \mathbf{S}_2 are common for all $\mathbf{c}_i(u)$ (see Step 5 of Algorithm 2). The final NURBS volume description is only a composition of NURBS descriptions obtained for each $\mathbf{c}_i(u)$, $i = 1, \dots, n - 1$. The whole algorithm is summarized in Algorithm 2.

The method used in Algorithm 2 can be again modified. In particular, it is possible to replace Step 5 by the following steps

$$\begin{aligned}
 5': \mathbf{T} &= (0, 0, 1, 1) \\
 6': \mathbf{K}_i &= \frac{1}{m+1} \sum_{j=0}^m \mathbf{P}'_{i,j} \\
 7': \mathbf{P}_{ijk} &= \begin{pmatrix} \mathbf{P}_{i,0} & \dots & \mathbf{P}_{i,m} \\ \mathbf{K}_i & \dots & \mathbf{K}_i \end{pmatrix}, \quad j = 0, \dots, m, \quad k = 0, 1 \\
 8': \mathbf{W}_{ijk} &= \begin{pmatrix} w_{i,0} & \dots & w_{i,m} \\ 1 & \dots & 1 \end{pmatrix}, \quad j = 0, \dots, m, \quad k = 0, 1
 \end{aligned}$$

where this approach is based on the NURBS description of a disc presented in Example 3 (a). Each control point $\mathbf{P}_{i,j}$, $j = 0, \dots, m$ is connected with the point \mathbf{K}_i on the axis o by a line segment, i.e., the third knot vector is chosen to be $\mathbf{T} = (0, 0, 1, 1)$.

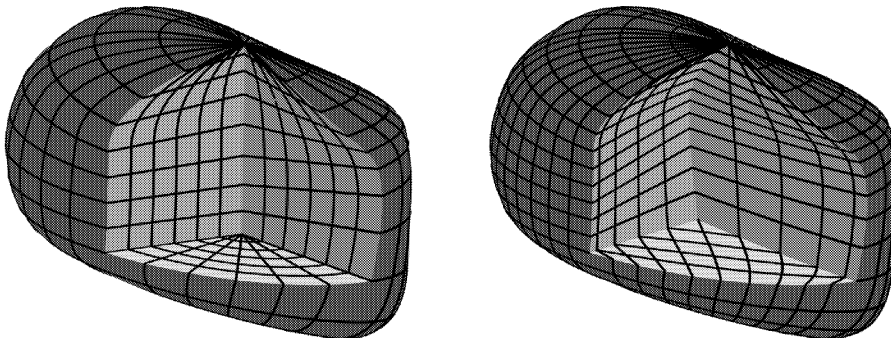


Fig.7: The final NURBS volumes for Example 5 obtained by Algorithm 2 (right) and by the modified approach (left)

Example 5 Let $s(u, v)$ be given by the control points

$$\mathbf{P}_{1,j} = \{(0, 2, 3), (1, 2, 3), (1/2, -1, 5/2), (0, -2, 3), (-2, 0, 5/2), (-1, 2, 3), (0, 2, 3)\},$$

$$\mathbf{P}_{2,j} = \{(0, 3, 1), (3, 3, 1), (2, 1, 1), (1, -2, 2), (-2, -1, 2), (-1, 3, 1), (0, 3, 1)\},$$

$$\mathbf{P}_{3,j} = \{(0, 2, 0), (1, 1, 0), (0, -2, 0), (-2, -3, 1), (-1, 0, 0), (-1, 3, 0), (0, 2, 0)\},$$

$\mathbf{P}_{0,j} = (0, 0, 3), \forall j$, $\mathbf{P}_{4,j} = (0, 0, 0), \forall j$, and weights of all control points are equal to 1. The knot vectors are $\mathbf{R} = (0, 0, 0, 1/3, 2/3, 1, 1, 1)$ and $\mathbf{S} = (0, 0, 0, 0, 1/4, 1/2, 1/2, 1, 1, 1, 1)$. Using Algorithm 2 we obtain the result shown in Fig. 7 (right), the result obtained by the modified approach is shown in Fig. 7 (left).

6. Conclusion

In this paper, we presented several algorithms for finding parameterizations of NURBS domains (surfaces and volumes). The motivation for this study arose from the recent extensive research in isogeometric analysis. We proposed methods for finding parameterizations of NURBS domains bounded by either a one closed NURBS curve, or four NURBS curves fulfilling compatibility conditions. Further, parameterizations of NURBS volumes of revolution based on two different parameterizations of a disc were demonstrated. Finally, we formulated an algorithm for computation of parameterizations of generalized NURBS volumes of revolution which is the main result of the paper.

Acknowledgement

The second author was supported by GAČR P101/12/2315.

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Received in editor's office: April 15, 2012

Approved for publishing: November 27, 2012