Network Formation under Multiple Sources of Externalities*

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Abstract

The architecture of social and economic networks is often explained in terms of the externalities shaping the link-forming incentives of players. We make two contributions to this literature. First, we bring into its ambit the linear-quadratic utility model. Since players' utilities are now a function of their network centralities, this permits endogenizing their locational incentives in a network. Second, we show that the *mode of transmission* of externalities can be crucial in dictating the topology of equilibrium networks. We consider two alternative modes for channeling externalities. Both have the same primary source (direct links) but a different secondary source (global effects versus indirect links). We characterize equilibrium networks for different positive-negative combinations of primary-secondary externalities for both modes. We show that the mode of transmission influences the equilibrium architecture when externalities from the *primary* source are *positive*; when these externalities are negative, the equilibrium network is empty.

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1 Introduction

The equilibrium configuration of interconnections among players that emerges endogenously as players evaluate the benefits and costs of establishing bilateral links has occupied an important strand in the network literature.[1] The existing literature has attempted to explain the equilibrium network architecture in terms of the *externalities* that bear upon players by virtue of their position in the network architecture. Our objective in this paper is two-fold. First, we bring into the ambit of the endogenous network formation literature the linear-quadratic specification of utility which now forms the bedrock for many games on an exogenously fixed network.[2] Second, exploiting the tractability offered by the linear-quadratic model, we demonstrate that the *mode of transmission* of externalities is an equally important determinant in shaping the topology of equilibrium networks. We now elaborate on each of these objectives.

The linear-quadratic specification of utility has an important feature that explains its popularity for Nash games on a fixed network: the Nash action and reduced utility of players is a function of their Katz-Bonacich centrality in a network.[3] In some games (e.g. the oligopoly game of example 3.1 below), Nash actions and reduced utility are increasing in network centrality. In other games (e.g. provision of public goods in Bramoullé et al 2014), higher Nash action levels are no longer associated with the most central players; in fact players with greater centrality can free ride on the actions of those less central (Ballester and Calvó-Armengol 2010, Example 11). These results lend special impetus towards endogenizing the network formation process and systematically examining the motivation of players to assume, or not assume, a central position in the network with an eye towards the subsequent game to be played on the ensuing network. Accordingly we posit a two-stage game where in the first stage players form a network and in the second stage they engage in a Nash game contingent on the network. Inducting backwards from the Nash game in actions, players weigh the benefits and

[1] Please see Goyal (2007) and Jackson (2008) for a survey of this literature in the context of both directed and undirected networks.

[2] This utility specification was introduced by Ballester et al (2006) and has found numerous applications in the literature on games on networks. These include education (Calvó-Armengol et al, 2009), crime (Ballester et al, 2010), conformity (Patacchini and Zenou, 2012), and public goods provision (Bramoullé et al, 2014) . Please see Jackson and Zenou (2012) for a survey and Ballester and Calvó-Armengol (2010) for extensions.

[3] Please see section 3 below for the Katz-Bonacich measure, or Bonacich (1987) and Jackson (2008, Section 2.2.4).

costs of linking with each other. Our first objective is to characterize the network architecture that ensues with particular emphasis on the centrality of the locations occupied by players in the network.

Our second objective is to investigate within the context of the linear-quadratic utility model two alternative modes of channeling externalities. Our main thesis is that the mode of transmission of externalities can dictate the architecture of equilibrium networks. The two modes that we consider have a primary source and a secondary source for transmitting externalities. We present examples to show that externalities from either source can be positive or negative. We characterize the equilibrium networks for different positive-negative combinations of primary-secondary externalities for both modes. In the process we demonstrate that the mode of transmission has definite implications for the equilibrium network topology when externalities from the *primary* source are *positive.* When these externalities are negative, the equilibrium architecture is always empty and thus robust to the mode of transmission.

The first mechanism for transmitting externalities is based on Ballester et al (2006). The primary source is *local* interaction (flow of externalities through direct links or neighbors within a network) while the secondary source is *global* interaction (externalities transmitted without the aegis of the network). We say that strategic local externalities exist and are negative (resp. positive) when an increase in the actions of neighbors in a network decreases (resp. increases) the utility of a player. Similarly, strategic global externalities are negative (resp. positive) if a player's utility is uniformly and negatively (resp. positively) influenced by the actions of *all* players. Examples 3.1 and 3.2 below provide two instances of the possible configurations that can ensue. For brevity we refer to this mechanism as the local-global mode.

The second mode of transmitting externalities has the same primary source as the first mode – viz. the direct links of players. The secondary source however is constituted by the *indirect* links of a player. Positive and negative externalities from these sources are defined similar to the first mode. There are three ways in which this direct-indirect mode differs from the local-global mode of transmission. (i) It explicitly accounts for indirect links in the primitive utility function itself.[4] This is the case for instance in friendship

[4] In contrast the local-global mode of transmission only considers direct links (as captured in the adjacency matrix) in the primitive utility function. It is only *reduced* utility corresponding to the Nash equilibrium in actions that aggregates the feedback effects from all direct and indirect links in the network.

networks (e.g. Brueckner 2003) where players derive benefits from friends and friends of friends. In the educational sphere (e.g. Basu and Foster 1998) informal learning occurs through direct links (immediate family) in addition to indirect links (neighbors and friends of friends). (ii) Externalities are only transmitted within a network and players retain the choice of absolving themselves from any external effects by dissociating from the network. (iii) It permits indirect links influence a player differentially from direct links. In fact, it allows indirect links to be more prominent in exerting externalities as compared to direct links (for e.g., if there are large spillovers from indirect links as in Example 4.1 below); it also permits externalities from direct and indirect link externalities to be of opposite signs (for e.g. synergy from direct links but competition from indirect links as in Example 4.2 below). For brevity we refer to this mechanism as the direct-indirect mode.

The equilibrium notion that we employ to study the network formation game is the *pairwise stable equilibrium* from Goyal and Joshi (2006). A network is a pairwise stable equilibrium if no player has an incentive to unilaterally delete any subset of links and no unlinked pair of players have an incentive to form a link. Table 1 lists the equilibrium networks for the two modes. It shows that when the primary source of externalities confers positive benefits, then the secondary mode through which externalities are transported has implications for the equilibrium network architecture. In order to clearly delineate the differences, the architectures that differ for any given configuration of externalities is highlighted in bold.

Local Externalities: Columns
Global Externalities: Rows

	(+)	(-)
(+)	Empty, Complete, Dominant Group, **Nested Split Graphs**.	Empty
(-)	Empty, Complete, Dominant Group, **Nested Split Graphs**, **Regular**.	Empty

Direct Links Externalities: Columns
Indirect Links Externalities: Rows

	(+)	(-)
(+)	Empty, Complete, Dominant Group, **Minimally Connected.**	Empty
(-)	Empty, Complete, **Exclusive Groups**.	Empty

Table 1: Equilibrium Networks

While the equilibrium characterization results are discussed in depth in the main body of the paper, we provide here some intuition for the observed differences in equilibrium architectures. When both primary and secondary impart positive externalities, then the local-global mode can generate nested split graphs (NSG) while the direct-indirect mode can generate minimally connected networks. In the local-global mode, marginal utility from links is increasing in the centrality of players involved; thus, if some player i finds it profitable to link to some player j, then player i also finds it profitable to link to all players who are more central in the network than player j. This leads to the NSG architecture in which the neighborhood of players with fewer links is nested in the neighborhoods of those with more links. In contrast, the direct-indirect mode induces a *non-monotonicity* in reduced utility with respect to links, a feature which does not obtain in the local-global model. When an isolated player i forms a link with some player j who is part of a non-singleton component, then player i realizes a large marginal gain both from the direct link as well as the indirect links of player j; all subsequent links formed by player i within the component generate marginal gains due to direct links only but no further utility increments due to indirect links. Thus players in a minimally connected network have no incentive to delete links (given the large utility loss that is involved from being excised from the component) but will also not find it profitable to form additional links (since there are no additional indirect links to access within the component).

Now suppose that externalities from the primary (resp. secondary) source are positive (resp. negative). In the local-global mode, the incentives of central players to form additional links is suitably tempered from the negative global effect. Additional links raise the Nash action levels of centrally located players due to positive strategic local externalities but adversely impacts marginal gains from links due to negative strategic global externalities. This leads players to form regular networks in which they all have the same number of links. In contrast, in the direct-indirect mode players are led to form exclusive groups. Within each component, players have an incentive to connect completely because they harness positive externalities from direct links while remaining insulated from negative externalities due to indirect links; links across components though expose the participating players to negative externalities from the indirect links of their potential partners thereby vitiating their formation.

We now place our paper in the context of the existing literature on the endogenous formation of games. Our paper is closest to Goyal and Joshi

(2006) in that one of our preoccupations is to obtain characterization of equilibrium networks in a fairly large class of games. Goyal and Joshi characterize pairwise stable equilibrium in two broad class of games: "playing-the-field" and "local spillovers" games. The two models with local-global and direct-indirect modes of transmission of externalities do not belong to the class of games they consider. Therefore our paper is an attempt to extend their analysis to a class of linear-quadratic utility games that only recently have been introduced by Ballester et al (2006). Our paper also has a second preoccupation – viz. to illustrate that the mode of conveying externalities also matters. Since the reduced utility of players in the linear-quadratic utility model is a function of their Katz-Bonacich centrality, our paper also shares the same concerns as Galeotti and Goyal (2010) who examine network formation from the point of view of strategically accessing and disseminating information, and Goyal and Vega-Redondo (2007) who examine the incentives of players to occupy central positions in a network (more precisely, to bridge structural holes in a network) to extract intermediation rents. Hiller (2012) considers a game that corresponds to positive strategic local and global externalities and obtains the core-periphery architecture in equilibrium which is subsumed under NSG. Baetz (2013) considers positive strategic local and negative global externalities and obtains regular networks as one of the equilibrium similar to our result. While our paper considers pairwise stable equilibrium, the results we derive are consistent with the literature that examines stochastically stable networks arising from a dynamic evolutionary process of network formation. In a framework of strategic positive local externalities and negative global externalities, Dawid and Hellman (2012) and König et al (2013) show respectively that stochastically stable networks are dominant groups and NSG similar to our analysis.

The paper is organized as follows. The model is described in section 2. The local-global mode and the direct-indirect mode of transmitting externalities are examined in sections 3 and 4 respectively. The conclusions are contained in section 5.

2 The Model

We consider a model with two stages. In the first stage players form a network according to a link announcement game. In the second stage players engage in a non-cooperative game contingent on the network formed in the first stage. We now elaborate on the various elements of the model.

Let $\mathcal{N} = \{1, 2, ..., N\}$ denote the set of ex-ante symmetric players. We consider a link announcement game based on Dutta et al (1995). Each player $i \in \mathcal{N}$ makes an announcement of links given by the strategy $s_i = (s_{ij})_{j \neq i}$ where $s_{ij} = 1$ when player i intends to form a link with j and $s_{ij} = 0$ when no such link is intended. S_i is the strategy set of player i consisting of all possible announcements that can be made and $S = \times_{i \in \mathcal{N}} S_i$ the set of all strategy profiles. A bilateral link between players i and j is formed if and only if $s_{ij} = s_{ji} = 1$. A strategy profile $s \in S$ therefore induces a network which is defined next.

A *network* is a tuple $(\mathcal{N}, \mathbf{G}(s))$ where the nodes correspond to the players and $\mathbf{G}(s)$ records the bilateral links that exist between the players induced by $s \in S$. For notational simplicity the dependence of $\mathbf{G}$ on s is suppressed. When the set of players is unambiguous, we refer to $\mathbf{G}$ as the network and represent it in two alternative ways. The first is by letting $\mathbf{G}$ denote the collection of all pairwise links and $ij \in \mathbf{G}$ indicate that players i and j are linked in the network. The second is by letting $\mathbf{G} = [g_{ij}]$ denote the adjacency matrix such that $g_{ij} = g_{ji} = 1$ if i and j are linked and $g_{ij} = 0$ otherwise (where $g_{ii} = 0 \ \forall i \in \mathcal{N}$). We will use both representations interchangeably and let $\mathcal{G}$ denote the set of all networks.

The set $\mathbf{N}_i(\mathbf{G}) = \{j \in \mathcal{N} \backslash \{i\} : ij \in \mathbf{G}\}$ denotes the *neighbors* of player i in $\mathbf{G}$ and $d_i(\mathbf{G}) = |\mathbf{N}_i(\mathbf{G})|$ the cardinality of this set, also referred to as player i's *degree*, d_i. A *walk* in $\mathbf{G}$ connecting i and j is a set of nodes $\{i_1, \ldots, i_n\}$ such that $ii_1, i_1i_2, ..., i_{n-1}i_n, i_nj \in \mathbf{G}$; if $i = j$. A *path* is a walk in which all nodes are distinct. A network is *connected* if there exists a path between any pair $i, j \in \mathcal{N}$; otherwise the network is *unconnected*. A sub-network, $C(\mathbf{G}) \equiv (\mathcal{N}', \mathbf{G}')$, $\mathcal{N}' \subset \mathcal{N}$, $\mathbf{G}' \subset \mathbf{G}$, is a *component* of the network $(\mathcal{N}, \mathbf{G})$ if it is connected and if $ij \in \mathbf{G}$ for $i \in \mathcal{N}'$, $j \in \mathcal{N}$, implies $j \in \mathcal{N}'$ and $ij \in \mathbf{G}'$. Let $\mathbf{G} - ij$ (resp. $\mathbf{G} + ij$) denote the network obtained from $\mathbf{G}$ by deleting (resp. adding) the link ij. A link $ij \in \mathbf{G}$ is *critical* if $\mathbf{G} - ij$ has more components than $\mathbf{G}$; otherwise the link is non-critical. A network is *minimally connected* if all links are critical.

In *regular* networks all players have the same degree which is also the degree of the network. For example, the complete network, $\mathbf{G}^c$, has degree $N - 1$, the wheel network, $\mathbf{G}^{wheel}$, has degree 2, and the empty network, $\mathbf{G}^e$, has degree 0. Next we define a *nested split graph* (NSG) along the lines of König et al (2013). Consider a network $\mathbf{G}$ and let its distinct positive degrees be $d_{(1)} < d_{(2)} < \cdots < d_{(m)}$; let $d_{(0)} = 0$ even though there may not exist an isolated player in $\mathbf{G}$. Letting $P_k(\mathbf{G}) = \{i \in \mathcal{N} : |\mathbf{N}_i(\mathbf{G})| = d_{(k)}\}$, $k = 0, 1, ..., m$, define the *degree partition* of $\mathbf{G}$ as $\mathcal{P}(\mathbf{G}) = \{P_0(\mathbf{G}), P_1(\mathbf{G}), ..., P_m(\mathbf{G})\}$. Letting $\lfloor x \rfloor$ denote the largest integer smaller than or equal to x, a network G is a NSG if for each player $i \in P_h(\mathbf{G})$, $h = 1, 2, .., m$:

$$\mathbf{N}_i(\mathbf{G}) = \begin{cases} \cup_{l=1}^{h} P_{m+1-l}(\mathbf{G}), & h = 1, 2, ..., \lfloor \frac{m}{2} \rfloor \\ \cup_{l=1}^{h} P_{m+1-l}(\mathbf{G}) \backslash \{i\}, & h = \lfloor \frac{m}{2} \rfloor + 1, ..., m \end{cases}$$

A *star* network is a special case of a NSG where $|P_1(\mathbf{G})| = N - 1$ and $|P_m(\mathbf{G})| = 1$. Another network is one of *exclusive groups* where $ij \in \mathbf{G}$ if $i, j \in P_h(\mathbf{G})$, and $ij \notin \mathbf{G}$ if $i \in P_h(\mathbf{G})$, $j \in P_{h'}(\mathbf{G})$, $h \neq h'$, i.e. $\mathbf{G}$ is composed of complete components; a *dominant group* network has at most one non-singleton complete component.

In the first stage the network $\mathbf{G}$ is formed. In the second stage, players play a Nash game contingent on $\mathbf{G}$. Let $X_i \subset \mathbb{R}_+$ denote the action set of player i for this second stage Nash game. Let $X = \times_{j=1}^{n} X_j$ denote the set of action profiles and $u_i : X \times \mathbf{L} \to \mathbb{R}_+$ the utility function of player i where $\mathbf{L} = \mathbf{G} + \mathbf{H}$. We will consider two sets of preferences, the first in which $\mathbf{H}$ is a zero matrix, and the second in which $\mathbf{H}$ is a matrix tracking indirect connections in the network $\mathbf{G}$. Since $\mathbf{H}$ is induced from the network $\mathbf{G}$, for the sake of consistency we will write the dependence of the second stage Nash game on $\mathbf{G}$ only. An action profile $\mathbf{x}^*(\mathbf{G})$ is a Nash equilibrium of the second stage game if $\forall x_i \in X_i$, $\forall i \in \mathcal{N}$:

$$u_i(x_i^*(\mathbf{G}), \mathbf{x}_{-i}^*(\mathbf{G}), \mathbf{G}) \geq u_i(x_i, \mathbf{x}_{-i}^*(\mathbf{G}), \mathbf{G}) \tag{1}$$

where $\mathbf{x}_{-i}^*(\mathbf{G})$ is the Nash action profile of players other than i. The first stage reduced form utility function of player i is given by:

$$U_i(\mathbf{G}) = u_i(x_i^*(\mathbf{G}), \mathbf{x}_{-i}^*(\mathbf{G}), \mathbf{G}) \tag{2}$$

Let $c \geq 0$ denote the cost to a player of forming a bilateral link. Given the strategy profile $s \in S$ of link announcements from the first stage, the net utility function of player i in the network $\mathbf{G}(s)$ is given by $U_i(\mathbf{G}(s)) - d_i(\mathbf{G}(s))c$. A strategy profile $s^* \in S$ is a Nash equilibrium of the first stage link formation game if $\forall i \in \mathcal{N}$, $\forall s_i \in S_i$:

$$U_i(\mathbf{G}(s_i^*, \mathbf{s}_{-i}^*)) - d_i(\mathbf{G}(s_i^*, \mathbf{s}_{-i}^*))c \geq U_i(\mathbf{G}(s_i, \mathbf{s}_{-i}^*)) - d_i(\mathbf{G}(s_i, \mathbf{s}_{-i}^*))c \tag{3}$$

Since the Nash criterion is not discriminating enough to address multiplicity of equilibria, we follow Goyal and Joshi (2006) and consider a refinement based on the pairwise stability notion of Jackson and Wolinsky

(1996). Therefore we will say that $\mathbf{G}$ is a *pairwise stable equilibrium* (or *pws-equilibrium*) if there is a Nash strategy profile $s^* \in S$ which induces $\mathbf{G}$ and for any $ij \notin \mathbf{G}$:

$$U_i(\mathbf{G}+ij) - U_i(\mathbf{G}) > c \;\Rightarrow\; U_j(\mathbf{G}+ij) - U_j(\mathbf{G}) < c \tag{4}$$

It is important to distinguish between the two sets of games involved. Therefore pws-equilibrium will always refer to the equilibrium of the first-stage link formation game. The equilibrium of the second stage game, which is conditional on $\mathbf{G}$, will be referred to as a Nash equilibrium in actions.

We will say that $\mathbf{G}' = [g'_{ij}] > \mathbf{G} = [g_{ij}]$ if $g'_{ij} \geq g_{ij}$ $\forall i, j$ and $g'_{ij} > g_{ij}$ for at least one pair i, j. The reduced utility of player i is strictly increasing (resp. decreasing) if $U_i(\mathbf{G}') >$(resp. $<$) $U_i(\mathbf{G})$ for all $\mathbf{G}' > \mathbf{G}$.

3 Global versus Local Interactions

We will follow Ballester et al (2006) in specifying a linear-quadratic specification of the utility function of the second stage Nash game:

$$u_i(\mathbf{x}, \mathbf{G}) = \left[x_i - \frac{1}{2}x_i^2\right] + \left[\gamma x_i \sum_{j=1}^{N} x_j\right] + \left[\lambda x_i \sum_{j=1}^{N} g_{ij} x_j\right] \tag{5}$$

The second term in square parentheses represents global interaction while the third term represents local interaction. The utility function is said to exhibit *positive* (resp. *negative*) *strategic global externalities* if $\gamma > 0$ (resp. $\gamma < 0$). Similarly the utility function is said to exhibit *positive* (resp. *negative*) *strategic local externalities* if $\lambda > 0$ (resp. $\lambda < 0$). We now provide two examples of strategic local-global externalities.[5]

Example 3.1:[6] Consider a homogeneous product oligopoly with demand function:

$$P = \alpha - \frac{1}{2}x_i - \frac{1}{2}\sum_{j \neq i} x_j, \quad \alpha > 0$$

[5] Other applications of this model include among others social networks in education (Calvó-Armengol et al 2009), provision of public goods (Bramoullé et al 2014) and crime (Ballester et al 2010, Calvó-Armengol and Zenou 2004).

[6] This example is based on König (2013).

where x_i denotes the output of firm i. Assume that a link $ij \in \mathbf{G}$ corresponds to a collaborative alliance between firms i and j. The marginal cost of firm i is decreasing in the number of its collaborative alliances as well as the output level of its collaborator. Hence marginal cost is of the form:

$$c_i = c_0 - \lambda \sum_{j \neq i} g_{ij} x_j, \quad c_0, \lambda > 0$$

The gross profit of firm i is of the form given by (5) with positive strategic local complementarities and negative global substitutabilities. ■

Example 3.2:[7] Consider a population of N banks, where bank i provides quantity $x_i \geq 0$ of loans and banks compete in the quantity of lending. Letting p_i denote the interest rate, the demand function for loans for bank i is given by:

$$p_i = \alpha - \frac{1}{2} x_i - \frac{1}{2} \sum_{j \neq i} x_j, \quad \alpha, \beta > 0$$

There is positive probability that a bank's loan may turn out to be "bad" in which case the bank turns to those it is linked to in a network $\mathbf{G}$ to cover its obligations to depositors. The marginal cost of bank i is given by:

$$c_i = c_0 + \lambda \sum_{j \neq i} g_{ij} x_j, \quad c_0, \lambda > 0$$

The second term is the "contagion" effect: a larger number of links, or greater lending by banks to whom bank i is linked, increases the exposure of bank i to any bad shocks of its partners in the network $\mathbf{G}$ and increases the (marginal) cost of servicing its loans. The gross profit of bank i takes the form given by (5) exhibiting strategic negative local complementarities and negative global substitutabilities. ■

When the Nash equilibrium in actions exists and is interior, it satisfies the first order condition:

$$\frac{\partial u_i}{\partial x_i} = 1 - x_i^*(\mathbf{G}) + \gamma \sum_{j=1}^{N} x_j^*(\mathbf{G}) + \gamma x_i^*(\mathbf{G}) + \lambda \sum_{j=1}^{N} g_{ij} x_j^*(\mathbf{G}) = 0 \qquad (6)$$

Since $\frac{\partial u_i}{\partial x_i}(\mathbf{0}, \mathbf{G}) = 1 > 0$, it follows that $\mathbf{0}$ cannot be a Nash equilibrium. The characterization of the second stage Nash equilibrium requires the notion of

[7] This example is inspired by an example on networks of banks provided in König et al (2013, Appendix D).

Katz-Bonacich (henceforth KB) centrality. Letting $\mathbf{I}$ denote the identity matrix and $\xi > 0$ a (sufficiently small) attenuation parameter, consider the matrix:

$$\mathbf{M}(\mathbf{G}, \boldsymbol{\xi}) = [\mathbf{I} - \boldsymbol{\xi}\mathbf{G}]^{-1} = \sum_{m=0}^{\infty} \xi^m \mathbf{G}^m \tag{7}$$

where $\mathbf{G}^0 = \mathbf{I}$. It is well known that since $\mathbf{G}$ is symmetric, all its eigenvalues are real and sum to zero, and thus the largest eigenvalue $\mu_{\max}(\mathbf{G})$ is positive. Further $[\mathbf{I}-\xi\mathbf{G}]^{-1}$ is well-defined and non-negative if $\xi\mu_{\max}(\mathbf{G}) < 1$. Letting $\mathbf{1}$ denote the column vector of 1's, the vector of unweighted *KB centralities* of the players is given by:

$$\mathbf{b}(\mathbf{G}, \boldsymbol{\xi}) = \mathbf{M}(\mathbf{G}, \boldsymbol{\xi})\mathbf{1} = \sum_{m=0}^{\infty} \xi^m \mathbf{G}^m \mathbf{1} \tag{8}$$

The i^{th}-component of the vector $\mathbf{b}(\mathbf{G}, \boldsymbol{\xi})$, $b_i(\mathbf{G}, \boldsymbol{\xi})$, measures the number of weighted walks in the network $\mathbf{G}$ originating from node i where the weights on the paths fall exponentially with their length. Note that $b_i(\mathbf{G}, \boldsymbol{\xi}) \geq 1$ since there is at least one walk from a node to itself. Also note that $\mathbf{G}^1\mathbf{1}$ is a column vector whose i^{th}-row is $\sum_{j=1}^{N} g_{ij} = d_i(\mathbf{G})$ and thus for sufficiently small $\xi > 0$ the KB centrality of player i is directly proportional to i's degree. If ξ is large, then in addition to a player's degree, the connections of partners, and partners' partners etcetera, also matter. The following lemma, whose proof is similar to Jackson and Zenou (2012, Section 4) and therefore omitted, proves the existence of a Nash equilibrium in actions.

Lemma 1 *Suppose the utility function is given by* (5). *Letting* $\mu_{\min}(\mathbf{G})$ *and* $\mu_{\max}(\mathbf{G})$ *denote the smallest and largest eigenvalues of* $\mathbf{G}$ *respectively, a unique Nash equilibrium in actions exists if:*

a. $\gamma(1 + N) < 1 - \lambda\mu_{\max}(\mathbf{G})$ *for* $\lambda > 0$ *and* $\gamma > 0$.

b. $\gamma < 1 - \lambda\mu_{\max}(\mathbf{G})$ *for* $\lambda > 0$ *and* $\gamma < 0$.

c. $\gamma(1 + N) < 1 - \lambda|\mu_{\min}(\mathbf{G})|$ *for* $\lambda < 0$ *and* $\gamma > 0$.

d. $\gamma < 1 - \lambda|\mu_{\min}(\mathbf{G})|$ *for* $\lambda < 0$ *and* $\gamma < 0$.

Since the set $\mathcal{G}$ is finite, we will assume that the parametric restrictions of Lemma 1 apply to all $\mathbf{G} \in \mathcal{G}$ for each combination of strategic local and global externalities. The attenuation parameter $\xi = \lambda/(1-\gamma)$ below accounts for the feedback effects stemming from both local and global interactions. Lemma 1 implicitly restricts ξ in order to bound these feedback loops. Given a set $S \subset \mathcal{N}$, and matrix $\mathbf{A}$, let $\mathbf{A}_S$ denote the submatrix of $\mathbf{A}$ whose rows and columns correspond to S; similarly given vector $\mathbf{x}$, the subvector $\mathbf{x}_S$ includes only the rows in S. Note that $\mu_{\max}(\mathbf{A}) \geq \mu_{\max}(\mathbf{A}_S)$. We offer without proof the following result based on Ballester et al (2006, Theorem 1) and Bramoullé et al (2014, Proposition 1).

Proposition 1 *Suppose the utility function is given by* (5), *and* $\mathbf{G}$ *is the underlying network.*

(a) *Let* $\lambda > 0$. *Then the unique Nash equilibrium in actions is interior and is given by*:

$$\mathbf{x}^*(\mathbf{G}) = \frac{1}{1-\gamma\left[1+b\left(\mathbf{G},\xi\right)\right]}\mathbf{b}\left(\mathbf{G},\xi\right), \quad b\left(\mathbf{G},\xi\right) = \sum_{i=1}^{N} b_i\left(\mathbf{G},\xi\right) \tag{9}$$

(b) *Let* $\lambda < 0$. *If* $|\xi b\left(\mathbf{G},\xi\right)| < 1$, *and* $b\left(\mathbf{G},\xi\right) > 0$, *then the unique Nash equilibrium in actions is interior and given by* (9). *If* $|\xi b_S\left(\mathbf{G},\xi\right)| \geq 1$ *for some* $S \subset \mathcal{N}$, *then the Nash equilibrium may involve inactive players whose Nash action is* 0. *Letting* $A \subset \mathcal{N}$ *denote the set of active players in the Nash equilibrium*:

$$\mathbf{x}_A^*(\mathbf{G}) = \frac{1}{1-\gamma\left[1+b_A\left(\mathbf{G}_A,\xi\right)\right]}\mathbf{b}\left(\mathbf{G}_A,\xi\right), \quad b_A\left(\mathbf{G}_A,\xi\right) = \sum_{i\in A} b_i\left(\mathbf{G}_A,\xi\right) \tag{10}$$

Let us define:

$$\alpha\left(\mathbf{G},\xi\right) = \frac{1}{1-\gamma\left[1+b\left(\mathbf{G},\xi\right)\right]}, \quad B_i\left(\mathbf{G},\xi\right) = \alpha\left(\mathbf{G},\xi\right) b_i\left(\mathbf{G},\xi\right), \quad i \in \mathcal{N} \tag{11}$$

The reduced form utility of player i can be written as:

$$U_i(\mathbf{G}) = x_i^*(\mathbf{G})\left[1+\gamma\sum_{j=1}^{N} x_j^*(\mathbf{G}) + \lambda\sum_{j=1}^{N} g_{ij}x_j^*(\mathbf{G})\right] - \frac{1}{2}\left(x_i^*(\mathbf{G})\right)^2 \tag{12}$$

and substituting from (6) we have:

$$U_i(\mathbf{G}) = \left(\frac{1}{2} - \gamma\right)(x_i^*(\mathbf{G}))^2 = \left(\frac{1}{2} - \gamma\right) B_i^2(\mathbf{G}, \xi) \tag{13}$$

Therefore the reduced form utility of each player is quadratic in own rescaled KB centrality. In the subsequent discussion we will explore the implications of relatively "small" and "large" values of ξ and it is understood that it is within the bounds imposed by Lemma 1.

Positive Strategic Local and Global Externalities

We begin with the case of strategic global and local positive externalities, i.e. where $0 < \gamma < 1/2$ and $\lambda > 0$. Suppose the attenuation parameter $\xi > 0$ is sufficiently small so that KB centrality of a player is proportional to her degree. In this case reduced utility of player $i \in \mathcal{N}$ is convex in the sense that in a network G in which $ij, ik \notin \mathbf{G}$:

$$U_i(\mathbf{G} + ij + ik) - U_i(\mathbf{G} + ik) > U_i(\mathbf{G} + ik) - U_i(\mathbf{G}) \tag{14}$$

This is verified as follows. Let $\mathbf{G}'' = \mathbf{G} + ij + ik$ and $\mathbf{G}' = \mathbf{G} + ik$. Since $\gamma > 0$, $\alpha(\mathbf{G}'', \xi) > \alpha(\mathbf{G}', \xi) > \alpha(\mathbf{G}, \xi)$ and therefore $B_i(\mathbf{G}'', \xi) > B_i(\mathbf{G}', \xi) > B_i(\mathbf{G}, \xi)$. For small ξ, $b(\mathbf{G}'', \xi) - b(\mathbf{G}', \xi) > b(\mathbf{G}', \xi) - b(\mathbf{G}, \xi)$ and thus $\alpha(\mathbf{G}'', \xi) - \alpha(\mathbf{G}', \xi) > \alpha(\mathbf{G}', \xi) - \alpha(\mathbf{G}, \xi)$. It now follows that:

$$\begin{aligned}
&B_i\left(\mathbf{G}'', \xi\right) - B_i\left(\mathbf{G}', \xi\right) = \alpha\left(\mathbf{G}'', \xi\right) b_i\left(\mathbf{G}'', \xi\right) - \alpha\left(\mathbf{G}', \xi\right) b_i\left(\mathbf{G}', \xi\right) \\
&> \left[\alpha\left(\mathbf{G}', \xi\right) + \alpha\left(\mathbf{G}', \xi\right) - \alpha(\mathbf{G}, \xi)\right] b_i\left(\mathbf{G}'', \xi\right) - \alpha\left(\mathbf{G}', \xi\right) b_i\left(\mathbf{G}', \xi\right) \\
&= \alpha\left(\mathbf{G}', \xi\right)\left[b_i\left(\mathbf{G}'', \xi\right) - b_i\left(\mathbf{G}', \xi\right)\right] + b_i\left(\mathbf{G}'', \xi\right)\left[\alpha\left(\mathbf{G}', \xi\right) - \alpha(\mathbf{G}, \xi)\right] \\
&> \alpha(\mathbf{G}, \xi)\left[b_i\left(\mathbf{G}', \xi\right) - b_i(\mathbf{G}, \xi)\right] + b_i\left(\mathbf{G}', \xi\right)\left[\alpha\left(\mathbf{G}', \xi\right) - \alpha(\mathbf{G}, \xi)\right] \\
&= \alpha\left(\mathbf{G}', \xi\right) b_i\left(\mathbf{G}', \xi\right) - \alpha(\mathbf{G}, \xi) b_i(\mathbf{G}, \xi) = B_i\left(\mathbf{G}', \xi\right) - B_i(\mathbf{G}, \xi)
\end{aligned} \tag{15}$$

Therefore $U_i(\mathbf{G}'') - U_i(\mathbf{G}') > U_i(\mathbf{G}') - U_i(\mathbf{G})$ follows.

Returning to (14), any pair of unlinked players with at least one profitable link will have an incentive to connect with each other. This is because with positive strategic local externalities, an additional link induces players who are connected in a network to raise their (second stage) action levels. Increased actions further enhance the marginal utility from forming a link for these connected players and therefore a dominant group emerges. There can be at most one non-singleton component because otherwise players in two separate components will have an incentive to connect. Due to positive strategic global externalities, isolated players also have an incentive to increase their action levels. However, since the parameter γ is restricted to be small to ensure existence of a Nash equilibrium in actions, the marginal benefit from the first link of isolated players may not be sufficiently high to cover the cost of link formation. Therefore it is possible for isolated players to co-exist alongside a dominant group in a pws-equilibrium.

If ξ is large, then higher degree may not always translate into greater centrality and reduced utility may not satisfy (14). In order to examine how marginal reduced utility from a link varies with the centralities of the players involved, let us partition the players with at least one link in $\mathbf{G}$ according to their KB centralities, $\mathcal{C}(\mathbf{G}) = \{C_1(\mathbf{G}), C_2(\mathbf{G}), ..., C_m(\mathbf{G})\}$. Under positive strategic global and local externalities, a more central player realizes a larger utility gain than a less central one from a link, and reciprocally a player gets greater marginal utility from linking with a more central player. This is because a more central player is able to extract greater positive externalities from the network through an additional link than a less central player. A link with a more central partner also permits a player to reciprocally harness greater positive externalities from the network. Therefore reduced marginal utility satisfies the following:

Property (P1): Consider $\mathbf{G}$ in which $ij \in \mathbf{G}$ but $jk \notin \mathbf{G}$. If $b_k(\mathbf{G}, \xi) > b_i(\mathbf{G} - ij, \xi)$, then:

$$\begin{aligned} U_k(\mathbf{G}+jk) - U_k(\mathbf{G}) &> U_i(\mathbf{G}) - U_i(\mathbf{G}-ij) \\ U_j(\mathbf{G}+jk) - U_j(\mathbf{G}) &> U_j(\mathbf{G}) - U_j(\mathbf{G}-ij) \end{aligned} \tag{P1}$$

This property is established as follows. Letting $\mathbf{G}'' = \mathbf{G} + jk$ and $\mathbf{G}' = \mathbf{G} - ij$, the greater centrality of player k relative to i implies that $B_k(\mathbf{G}'', \xi) > B_k(\mathbf{G}, \xi) \geq B_i(\mathbf{G}, \xi) > B_i(\mathbf{G}', \xi)$. Further $b_k(\mathbf{G}'', \xi) - b_k(\mathbf{G}, \xi) > b_i(\mathbf{G}, \xi) - b_i(\mathbf{G}', \xi)$. This is because in transitioning from $\mathbf{G}'$ to $\mathbf{G}$, player i adds one walk of length 1 (to j) and additional walks of lengths 2 and greater. Compare this with player k's transition from $\mathbf{G}$ to $\mathbf{G}''$ which also adds one walk of length 1 (with j) but more walks of lengths 2 and greater than player i by virtue of k's greater KB centrality than i. This yields $B_k(\mathbf{G}'', \xi) - B_k(\mathbf{G}, \xi) > B_i(\mathbf{G}, \xi) - B_i(\mathbf{G}', \xi)$ and the first part of (P1) follows. The second part follows similarly. With property (P1) replacing (14), a different architecture of pws-equilibrium emerges. In particular, we argue that pws-equilibrium are NSG.

Suppose $\mathbf{G} \neq \mathbf{G}^e, \mathbf{G}^c$ is a pws-equilibrium. Consider player i with the lowest KB centrality, $b_i(\mathbf{G}, \xi) = \underline{b}(\mathbf{G}, \xi)$ who is linked to some player j. Then player j must have the highest KB centrality in $\mathbf{G}$, $\overline{b}(\mathbf{G}, \xi)$. To see why this must be true, suppose instead that $b_j(\mathbf{G}, \xi) < \overline{b}(\mathbf{G}, \xi)$. Then there exists a player k with at least one link such that $jk \notin \mathbf{G}$. Note that by virtue of the pws-equilibrium property, $U_i(\mathbf{G}) - U_i(\mathbf{G} - ij) \geq c$ and $U_j(\mathbf{G}) - U_j(\mathbf{G} - ij) \geq c$. Since $b_k(\mathbf{G}, \xi) \geq \underline{b}(\mathbf{G}, \xi) > b_i(\mathbf{G} - ij, \xi)$, it follows from (P1) that $U_j(\mathbf{G} + jk) - U_j(\mathbf{G}) > U_j(\mathbf{G}) - U_j(\mathbf{G} - ij) \geq c$ and $U_k(\mathbf{G} + jk) - U_k(\mathbf{G}) > U_i(\mathbf{G}) - U_i(\mathbf{G} - ij) \geq c$. Therefore players j and k have a mutually profitable link. But then player j will be linked to all players with non-zero links in $\mathbf{G}$ and thus $b_j(\mathbf{G}, \xi) \geq \overline{b}(\mathbf{G}, \xi)$, a contradiction. Therefore players with the lowest centrality are only connected to those with the highest. This argument also shows that players with the highest KB centrality, by virtue of (P1) must be linked to all players who have at least one link.

Now consider players $l_2, l_3, .., l_{m-1}$ representing $C_2(\mathbf{G}), C_3(\mathbf{G}), ..., C_{m-1}(\mathbf{G})$ respectively such that $\underline{b}(\mathbf{G}, \xi) < b_{l_2}(\mathbf{G}, \xi) < b_{l_3}(\mathbf{G}, \xi) < \cdot\cdot < b_{l_{m-1}}(\mathbf{G}, \xi) < \overline{b}(\mathbf{G}, \xi)$. Consider $l_2 \in C_2(\mathbf{G})$. Since $\underline{b}(\mathbf{G}, \xi) < b_{l_2}(\mathbf{G}, \xi)$, and each player in $C_1(\mathbf{G})$ is directly linked to all players in $C_m(\mathbf{G})$, there must exist a player $k \notin C_m(\mathbf{G})$ such that $kl_2 \in \mathbf{G}$.We now show that $kl_2 \in \mathbf{G}$ for all $k \in C_{m-1}(\mathbf{G}) \cup C_m(\mathbf{G})$. Suppose not and let $kl_2 \in \mathbf{G}$ but $k \notin C_{m-1}(\mathbf{G}) \cup C_m(\mathbf{G})$, i.e. $b_k(\mathbf{G}, \xi) < b_{l_{m-1}}(\mathbf{G}, \xi)$. From the pws-equilibrium property, $U_k(\mathbf{G}) - U_k(\mathbf{G} - kl_2) \geq c$ and $U_{l_2}(\mathbf{G}) - U_{l_2}(\mathbf{G} - kl_2) \geq c$. Each $l \in C_2(\mathbf{G}) \cup \cdots \cup C_m(\mathbf{G})$ will satisfy $b_l(\mathbf{G}, \xi) > b_{l_2}(\mathbf{G} - kl_2, \xi)$. Therefore from (P1), $U_l(\mathbf{G} + kl) - U_l(\mathbf{G}) > U_{l_2}(\mathbf{G}) - U_{l_2}(\mathbf{G} - kl_2) \geq c$ and $U_k(\mathbf{G} + kl) - U_k(\mathbf{G}) > U_k(\mathbf{G}) - U_k(\mathbf{G} - kl_2) \geq c$. In other words, player k will form profitable links with all players implying that $b_k(\mathbf{G}, \xi) \geq b_{l_{m-1}}(\mathbf{G}, \xi)$. But then $k \in C_{m-1}(\mathbf{G}) \cup C_m(\mathbf{G})$, a contradiction. Continuing inductively in this manner generates a NSG with the neighborhoods of low degree players nested in the neighborhoods of high degree players. Culling together the above arguments we have proved:

Proposition 2 *Suppose the utility function of each player exhibits positive strategic local and global externalities. If ξ is small, then a pws-equilibrium is either empty, complete, or a dominant group. If ξ is large, then a pws-equilibrium is either empty, complete, or consists of at most one non-singleton component which is complete or a NSG.*

Remark: Since the Nash actions are directly proportional to KB centralities, both dominant group and NSG support different positive Nash action levels in equilibrium. The dominant group architecture displays two distinct levels of actions with players in the complete component choosing a higher action level than the isolated players. In a NSG, Nash actions are increasing in the player's degree.

Positive Strategic Local and Negative Strategic Global Externalities

Next let us consider the case of positive strategic local externalities and negative global externalities. If $ij \notin \mathbf{G}$ then $b_k(\mathbf{G}+ij,\xi) \geq b_k(\mathbf{G},\xi)$ for $\forall k \in \mathcal{N}$ and strictly for $k = i, j$. On the other hand, $\gamma < 0$ implies that $\alpha(\mathbf{G}+ij,\xi) < \alpha(\mathbf{G},\xi)$. Therefore we cannot unambiguously characterize $U_i(\mathbf{G})$ since the product $\alpha(\mathbf{G},\xi)\, b_i(\mathbf{G},\xi)$ will be a function of the relative magnitudes of strategic local and global externalities.

Suppose $b_k(\mathbf{G},\xi) > b_i(\mathbf{G},\xi)$ in a network $\mathbf{G}$ in which $jk \in \mathbf{G}$ but $ij \notin \mathbf{G}$ and let $\mathbf{G}'' = \mathbf{G} + ij$ and $\mathbf{G}' = \mathbf{G} - jk$. We have already argued that a more central player k records a larger gain in KB centrality than a less central player i from linking with player j; reciprocally, player j records a larger gain in KB centrality by linking to k rather than i. Further, the link jk increases aggregate KB centrality, $b(.,\xi) = \sum_{l=1}^{N} b_l(.,\xi)$, more than the link ij. Therefore the multiplicative term $\alpha(.,\xi)$ that is common across all players falls more due to link jk than ij. We consider two possible cases. The first case is where for each network $\mathbf{G} \in \mathcal{G}$, the fall in $\alpha(.,\xi)$ from an additional link is relatively modest. Therefore the increase in KB centrality of a player from a link exceeds the decrease in $\alpha(.,\xi)$ and consequently Property (P1) obtains. The second case is where for each $\mathbf{G} \in \mathcal{G}$, the fall in $\alpha(.,\xi)$ is relatively substantive. Therefore the larger gain in KB centrality afforded to players from the link jk is outweighed by the decrease in $\alpha(.,\xi)$. Consequently a more central player k receives lower marginal utility from link jk than a less central player i from link ij, and reciprocally a player j receives lower marginal utility from linking with k than with i. We refer to this as:

Property (P2): Consider $\mathbf{G}$ in which $jk \in \mathbf{G}$ but $ij \notin \mathbf{G}$. If $b_k(\mathbf{G},\xi) > b_i(\mathbf{G} - jk,\xi)$, then:

$$\begin{aligned} U_k(\mathbf{G}) - U_k(\mathbf{G}-jk) &< U_i(\mathbf{G}+ij) - U_i(\mathbf{G}) \\ U_j(\mathbf{G}) - U_j(\mathbf{G}-jk) &< U_j(\mathbf{G}+ij) - U_j(\mathbf{G}) \end{aligned} \tag{P2}$$

In the first case where Property (P1) obtains, the analysis is identical to Proposition 2. We will therefore consider Property (P2) and show that for small ξ, pws-equilibrium are regular (i.e. all players have the same degree). When ξ increases, then centrality matters and it is possible to construct regular networks that have players with different KB centralities. For large ξ we argue that pws-equilibrium are a subset of regular networks in which all players have the same KB centrality (for e.g. the wheel network).

Suppose ξ is large and consider a pws-equilibrium $\mathbf{G}$ in which there are players with at least two distinct KB centralities. Then we can identify a player i (resp. player k) with the lowest (resp. highest) KB centrality $\underline{b}(\mathbf{G}, \xi)$ (resp. $\overline{b}(\mathbf{G}, \xi)$) and some player $j \neq i, k$ such that $jk \in \mathbf{G}$ but $ij \notin \mathbf{G}$. In a pws-equilibrium, $U_j(\mathbf{G}) - U_j(\mathbf{G} - jk) \geq c$ and $U_k(\mathbf{G}) - U_k(\mathbf{G} - jk) \geq c$. It now follows by virtue of (P2) that:

$$\begin{aligned} U_i(\mathbf{G} + ij) - U_i(\mathbf{G}) &> U_k(\mathbf{G}) - U_k(\mathbf{G} - jk) \geq c \\ U_j(\mathbf{G} + ij) - U_j(\mathbf{G}) &> U_j(\mathbf{G}) - U_j(\mathbf{G} - jk) \geq c \end{aligned}$$

But then players i and j have an incentive to form a link contradicting that $\mathbf{G}$ is a pws-equilibrium. Therefore all players must have the same KB centrality in a pws-equilibrium. When ξ is small, then KB centrality is proportional to degree and all regular networks are a pws-equilibrium.

Let us now consider an increase in the attenuation parameter. Let $\mathbf{G}^{\delta}$ denote a regular network of degree δ. Note that when ξ increases, then reduced utility increases as added weight is put on walks of all lengths. To simplify the notation let us write reduced utility as $U_i(\mathbf{G}, \xi)$. Consider some linking cost $c > 0$. In this case the degree of the pws-equilibrium is uniquely determined as $\delta(c, \xi)$. We can show this as follows. Suppose regular networks of degrees $\delta'(c, \xi)$ and $\delta''(c, \xi)$ are pws-equilibria where $\delta'(c, \xi) < \delta''(c, \xi)$. Let $ik \in \mathbf{G}^{\delta''(c,\xi)}$ and $ij \notin \mathbf{G}^{\delta'(c,\xi)}$. Then $c \leq U_i\left(\mathbf{G}^{\delta''(c,\xi)}, \xi\right) - U_i\left(\mathbf{G}^{\delta''(c,\xi)} - ik, \xi\right) \leq U_i\left(\mathbf{G}^{\delta'(c,\xi)} + ij, \xi\right) - U_i\left(\mathbf{G}^{\delta'(c,\xi)}, \xi\right) < c$. The first inequality is due to no player in $\mathbf{G}^{\delta''(c,\xi)}$ having an incentive to delete a link. The second inequality follows from (P2) and that player k is more central in $\mathbf{G}^{\delta''(c,\xi)}$ than player j in $\mathbf{G}^{\delta'(c,\xi)}$. The third inequality follows from the fact from no pair of unlinked players in $\mathbf{G}^{\delta'(c,\xi)}$ have an incentive to form a link. We therefore have a contradiction that establishes the uniqueness of the degree of a regular pws-equilibrium.

Now suppose marginal reduced utility is increasing in ξ and $\mathbf{G}^{\delta}$ is a pws-equilibrium for linking costs $c > 0$ when $\xi = \xi'$. Therefore players in

$\mathbf{G}^{\delta}$ have no incentive to delete their existing links and no incentive to form an additional link. Now if the attenuation parameter is raised to ξ'', then the marginal utility from each link increases and thus players continue to retain the incentive not to delete existing links. Then for $ij \in \mathbf{G}^{\delta(c,\xi')}$, $c \leq U_i\left(\mathbf{G}^{\delta(c,\xi')}, \xi'\right) - U_i\left(\mathbf{G}^{\delta(c,\xi')} - ij, \xi'\right) \leq U_i\left(\mathbf{G}^{\delta(c,\xi')}, \xi''\right) - U_i\left(\mathbf{G}^{\delta(c,\xi')} - ij, \xi''\right)$, and therefore $\delta(c,\xi') \leq \delta(c,\xi'')$. Now for $ik \notin \mathbf{G}^{\delta(c,\xi')}$, suppose $U_i\left(\mathbf{G}^{\delta(c,\xi')} + ik, \xi'\right) - U_i\left(\mathbf{G}^{\delta(c,\xi')}, \xi'\right) < c < U_i\left(\mathbf{G}^{\delta(c,\xi')} + ik, \xi''\right) - U_i\left(\mathbf{G}^{\delta(c,\xi')}, \xi''\right)$. Then each pair of unlinked players will have an incentive to add at least one additional link thereby increasing the degree of the regular network by 1. Therefore $\delta(c,\xi') < \delta(c,\xi'')$ and the degree of pws-equilibrium is increasing in ξ.

Now consider the case where marginal reduced utility is decreasing in ξ. Since players had no incentive to add links in $\mathbf{G}^{\delta}$ for linking costs $c > 0$ when $\xi = \xi'$, they will certainly not have any incentive to do so when ξ is raised to ξ'' given the reduction in their marginal gain from an additional link. On the other hand it is possible that at least one of their existing links now becomes unprofitable inducing them to delete these links thereby reducing the degree of the network. Therefore if marginal reduced utility is increasing (resp. decreasing) in the attenuation parameter, then an increase in ξ increases (resp. decreases) the degree of the regular pws-equilibrium network. We have therefore proved that:

Proposition 3 *Suppose the utility function of each player exhibits positive strategic local externalities and negative strategic global externalities.*

(a) *Suppose reduced utility satisfies (P1). If ξ is small, then a pws-equilibrium is empty, complete, or a dominant group. If ξ is high, then a pws-equilibrium is either empty, complete, or consists of at most one non-singleton component which is complete or a NSG.*

(b) *Suppose reduced utility satisfies (P2). If ξ is small, then pws-equilibrium networks are regular. If ξ is high, then pws-equilibrium is regular and in addition all players have the same KB centrality. The degree of the pws-equilibrium is increasing (resp. decreasing) in ξ if marginal reduced utility is increasing (resp. decreasing) in ξ.*

Remark: When property (P2) obtains, then we have a setting similar to Baetz (2013) in which players receive positive externalities from local connections but are subject to overall diminishing returns. We therefore recover a similar result that regular networks obtain in equilibrium in which all players choose a single positive Nash action level.

Negative Strategic Local Externalities; Negative or Positive Strategic Global Externalities

Next we consider the case of negative strategic local externalities. If we have positive global externalities, then Lemma 1 requires that the parameter γ should be relatively small. But then the attenuation parameter is *negative*, a case of special importance in Bonacich (1987) where a larger number of walks from a node does not translate into a higher centrality measure. Therefore (7) places negative weight on all walks of odd length and positive weight on walks of even length. In particular, for small $\xi < 0$, a player's reduced utility is negatively related to her degree. Therefore deleting links raises reduced utility. Further it allows saving the costs of link formation. It therefore follows that $\mathbf{G}^e$ is the unique pws-equilibrium. As ξ increases in magnitude, walks of even length become more prominent in reduced utility. However, since the greatest weight is still on the degree of the player, the empty network continues to be a pws-equilibrium. If we have negative global externalities, then $\gamma < 0$ and therefore $\xi < 0$. The same argument applies and hence $\mathbf{G}^e$ continues to be a pws-equilibrium.

Proposition 4 *Suppose the utility function of each player exhibits negative local externalities. Then the unique pws-equilibrium network is the empty network.*

Remark: In $\mathbf{G}^e$ we have $b_i(\mathbf{G}, \boldsymbol{\xi}) = 1 \ \forall i \in \mathcal{N}$. All players choose an identical positive Nash action equal to $\frac{1}{1-\gamma}$.

4 Direct versus Indirect Links

In this section we examine a framework in which players exert externalities on others through direct and indirect links. There are no global interactions and the externality is transmitted entirely within the network. Consider the following payoff function:

$$u_i(\mathbf{x}, \mathbf{G}) = \left[x_i - \frac{1}{2}x_i^2\right] + \left[\lambda x_i \sum_{j=1}^{N} g_{ij} x_j\right] + \left[\psi x_i \sum_{k=1}^{N} h_{ik} x_k\right] \tag{16}$$

The utility of player i is differentially affected from direct links and indirect links. Recall that $\mathbf{G}$ is the adjacency matrix describing direct links among nodes. We now define another "adjacency matrix" $\mathbf{H}$ that keeps track of only indirect links. As long as i and k do not have a direct link but belong to the same component, $h_{ik} = 1$; otherwise, $h_{ik} = 0$.[8] The matrix $\mathbf{L} = \mathbf{G} + \mathbf{H}$ keeps track of both direct and indirect links. The utility function is said to exhibit positive (resp. negative) externalities from direct links if $\lambda > 0$ (resp. $\lambda < 0$). Similarly the utility function is said to exhibit positive (resp. negative) externalities from indirect links if $\psi > 0$ (resp. $\psi < 0$).

Example 4.1: Consider N firms that are local monopolies. Each firm i has the demand function, $P = \alpha - \frac{1}{2}x_i$, $\alpha > 0$. A link $ij \in \mathbf{G}$ corresponds to a collaborative alliance between firms i and j. The marginal cost of firm i is decreasing both due to its direct links as well as due to indirect links stemming from spillovers:

$$c_i = c_0 - \lambda \sum_{j \neq i} g_{ij} x_j - \psi \sum_{k \neq i} h_{ik} x_k, \quad c_0, \lambda, \psi > 0$$

The gross profit of firm i can be written as (16) and exhibits positive externalities from direct and indirect links. If spillovers from indirect links are sufficiently large, then $\psi > \lambda$ and indirect links are more effective than direct links. ■

Example 4.2: Consider N firms who are competing to be the first to innovate a new product that will generate a monopoly profit of V. Each firm i can invest an effort level $x_i \in [0, \overline{x}]$ towards innovation. The outcome of this "patent race" is stochastic. Given an effort vector $\mathbf{x} = (x_1, x_2, ..., x_N)$, firm i is the first to successfully innovate the product with probability $p_i(\mathbf{x})$ where:

$$p_i(\mathbf{x}) = \frac{\lambda x_i \sum_{j \neq i} g_{ij} x_j - \psi x_i \sum_{k \neq i} h_{ik} x_k}{(N-1)\overline{x}^2}, \quad \lambda, \psi > 0$$

[8] Let g_{ik}^m denote the entry in the i^{th}-row and j^{th}-column of $\mathbf{G}^m$. Then $h_{ik} = 1$ if $g_{ik} = 0$ and for some $m = 2, 3, ..., g_{ik}^m \geq 1$; otherwise $h_{ik} = 0$.

According to this specification, firm i's probability of success is increasing with the synergy between own effort and the effort exerted by the direct partners. However, direct partners are also the source of information from own effort leaking to other players. Therefore higher effort level by indirect partners, combined with the information spillovers from own effort that accrues to indirect partners, lowers the probability of success. A firm makes zero profits if it is unsuccessful. The cost of exerting effort x_i is $\frac{1}{2}x_i^2$ for firm i. The gross profit of firm i is:

$$u_i(\mathbf{x}, \mathbf{G}) = \frac{V}{(N-1)\overline{x}^2}\left[\lambda x_i \sum_{j \neq i} g_{ij} x_j\right] - \frac{V}{(N-1)\overline{x}^2}\left[\psi x_i \sum_{k \neq i} h_{ik} x_k\right] - \frac{1}{2}x_i^2$$

This model exhibits positive externalities from direct links and negative externalities from indirect links. ■

A Nash equilibrium in actions, $\mathbf{x}^*(\mathbf{G})$, is characterized by:

$$\frac{\partial u_i}{\partial x_i} = 1 - x_i^*(\mathbf{G}) + \lambda \sum_{j=1}^{N} g_{ij} x_j^*(\mathbf{G}) + \psi \sum_{k=1}^{N} h_{ik} x_k^*(\mathbf{G}) = 0 \qquad (17)$$

Note that $\mathbf{0}$ cannot be a Nash equilibrium. Let $\lambda\mathbf{G}+\psi\mathbf{H} = (\lambda-\psi)\mathbf{G}+\psi\mathbf{L} = (\lambda-\psi)\left(\mathbf{G} + \frac{\psi}{\lambda-\psi}\mathbf{L}\right) \equiv (\lambda-\psi)\mathbf{W}$. The following existence result follows Ballester et al (2006, Theorem 1) and therefore the proof is omitted.

Proposition 5 *Suppose the utility function is given by* (16)*, and* $\mathbf{G}$ *is the underlying network. If* $(\lambda - \psi)\mu_{\max}(\mathbf{W}) < 1$ *for* $\lambda - \psi > 0$ *and* $1 - |(\lambda - \psi)\mu_{\min}(\mathbf{W})| > 0$ *for* $\lambda - \psi < 0$*, then a unique interior Nash equilibrium in actions exists and is given by*:

$$\mathbf{x}^*(\mathbf{G}) = \left[\mathbf{I} - (\lambda - \psi)\mathbf{W}\right]^{-1}\mathbf{1} \equiv \mathbf{B}(\mathbf{W}, \lambda - \psi) \qquad (18)$$

Similar to (8), $\mathbf{B}_i(\mathbf{W}, \lambda - \psi)$ accounts for the number of weighted walks of player i in a network with the “adjacency” matrix $\mathbf{W}$ with attenuation parameter $\lambda - \psi$. Substituting into (16), and using (17)-(18), the reduced form utility of player i is given by:

$$U_i(\mathbf{G}) = \frac{1}{2}\mathbf{B}_i^2(\mathbf{W}, \lambda - \psi) = \frac{1}{2}\mathbf{B}_i^2\left(\mathbf{G} + \frac{\psi}{\lambda - \psi}\mathbf{L}, \lambda - \psi\right) \qquad (19)$$

To reduce the notational burden, in the following discussion we will drop reference to the attenuation parameter $\lambda - \mu$ and will write $\mathbf{B}_i(\mathbf{G}, \mathbf{L})$ instead of $\mathbf{B}_i\left(\mathbf{G} + \frac{\psi}{\lambda-\psi}\mathbf{L}, \lambda - \psi\right)$.

Positive Externalities from Direct and Indirect Links

Suppose the externalities transmitted via direct and indirect links are positive. Consider first the case where $\lambda > \psi > 0$, i.e. direct links are more effective transmitters of externalities than indirect links. Note that if $\mathbf{G}' > \mathbf{G}$ and $\mathbf{H}' \geq \mathbf{H}$ then $\lambda\mathbf{G}' + \psi\mathbf{H}' > \lambda\mathbf{G} + \psi\mathbf{H}$, and thus $U_i(\mathbf{G}') > U_i(\mathbf{G})$ for each i. Therefore, for small (resp. large) costs of link formation, $\mathbf{G}^c$ (resp. $\mathbf{G}^e$) will be the unique pws-equilibrium. We now turn to the more interesting case of intermediate costs of link formation.

Consider an incomplete component in $\mathbf{G}$ in which $ij \in \mathbf{G}$ but $ik \notin \mathbf{G}$. Let $\mathbf{G}' = \mathbf{G} + ik$ and $\mathbf{G}'' = \mathbf{G} - ij$, with corresponding matrices $\mathbf{H}'$ and $\mathbf{H}''$ of indirect links. Note that $\mathbf{G}' > \mathbf{G} > \mathbf{G}''$ and $\mathbf{H}' < \mathbf{H} < \mathbf{H}''$, but $\mathbf{L}'' = \mathbf{L}' = \mathbf{L}$. Proposition 5 restricts $(\lambda - \psi)$ to be sufficiently small. In this case reduced utility satisfies the convexity property that $U_i(\mathbf{G}') - U_i(\mathbf{G}) > U_i(\mathbf{G}) - U_i(\mathbf{G}'')$ for players in the same component. Thus if the link ij is profitable for player i, then the link ik will be profitable as well. Similar consideration applies to player k. Therefore in a pws-equilibrium the component must be complete.

Now suppose $\mathbf{G}$ has two or more complete components and suppose players i and k belong to different components. Consider a link $ij \in \mathbf{G}$ and note that by forming this link in $\mathbf{G}'' = \mathbf{G} - ij$ player i adds a direct link in her component but does not influence $\mathbf{L}$. On the other hand, by connecting to player k who belongs to a different component, player i benefits from an additional link in $\mathbf{G}' = \mathbf{G} + ik$ as well as additional indirect links to players in k's component. In other words, player i benefits from the fact that $\mathbf{G}' > \mathbf{G}$ and $\mathbf{L}' > \mathbf{L}$. Therefore $\mathbf{B}_i(\mathbf{G}', \mathbf{L}') - \mathbf{B}_i(\mathbf{G}, \mathbf{L}) > \mathbf{B}_i((\mathbf{G}, \mathbf{L}) - \mathbf{B}_i(\mathbf{G}'', \mathbf{L})$. Therefore, if link ij is profitable for player i, then link ik is profitable as well. The same consideration applies to player k. It follows that in a pws-equilibrium, there can be at most one non-singleton component.

What is particularly striking is the emergence of minimally connected networks in equilibrium even when direct links are relatively more effective. This is a consequence of the *non-monotonicity* in reduced utility with respect to increasing links: an outside player linking to a player within a component realizes a greater marginal utility due to additional indirect connections in contrast to two players within a component linking up. To illustrate, suppose that neither $\mathbf{G}^e$ or $\mathbf{G}^c$ are a pws-equilibrium, i.e. for $c > 0$:

$$U_i(\mathbf{G}^e + ij) - U_i(\mathbf{G}^e) > c > U_i(\mathbf{G}^c) - U_i(\mathbf{G}^c - ij), \quad \forall i, j \in \mathcal{N}$$

Now consider a line network, $\mathbf{G}^{lin}$, in which players are arranged in ascending order of their index. Thus players 1 and N have only one direct link respectively to players 2 and $N-1$, while all remaining players have two links. Note that by virtue of positive externalities from indirect links, $U_1(\mathbf{G}^{lin}) - U_1(\mathbf{G}^{lin} - 12) > U_1(\mathbf{G}^e + 12) - U_1(\mathbf{G}^e) > c$ and therefore players 1 and N will not delete their link. The same argument establishes that player 2 will not delete the link with player 3 thereby dissociating from the larger network. Player 2 will also not delete the link with player 1 because $U_2(\mathbf{G}^{lin}) - U_2(\mathbf{G}^{lin} - 12) \geq U_2(\mathbf{G}^e + 12) - U_2(\mathbf{G}^e) > c$. This is because adding the link with player 1 does not confer any positive indirect externalities (given that player 1 is isolated); however player 2 can harness at least the same positive externalities from a direct link with an isolated player as in $\mathbf{G}^e$. Thus all players in $\mathbf{G}^{lin}$ will maintain their existing links. Next note that no two players in $\mathbf{G}^{lin}$ have an incentive to forge a link. This can be seen by checking the incentives of players 1 and N who realize the largest marginal utility from connecting in $\mathbf{G}^{lin}$. Since these two players are already a part of a component, they will not reap any additional positive externalities from indirect links; thus $c > U_i(\mathbf{G}^c) - U_i(\mathbf{G}^c - ij) > U_1(\mathbf{G}^{lin} + 1N) - U_1(\mathbf{G}^{lin})$. Hence $\mathbf{G}^{lin}$ is a pws-equilibrium by virtue of the fact that an isolated player derives the highest marginal utility from the first link with a non-singleton component but a lower marginal utility from the second link because there are no further externalities that are being transmitted from indirect links. This also establishes that there cannot be any isolated players.

Similar considerations apply for a star network, $\mathbf{G}^{star}$. The deletion of a direct link gives the center of the star, say player i, a reduction in utility of at least $U_i(\mathbf{G}^e + ij) - U_i(\mathbf{G}^e)$. For the spoke, the reduction in utility is even greater because it is now isolated with no direct or indirect links to the remaining network. Any other direct link from one spoke, say player j, to another, say player k, will yield marginal utility that is less than $U_j(\mathbf{G}^c) - U_j(\mathbf{G}^c - jk)$. Therefore the star emerges as a pws-equilibrium.

Now consider the case where indirect links are more effective in transmitting externalities than direct ones, i.e. $0 < \lambda < \psi$. Then $\mathbf{G}^e$ is a pws-equilibrium for high linking costs satisfying $U_i(\mathbf{G}^e + ij) - U_i(\mathbf{G}^e) \leq c$. Therefore let $U_i(\mathbf{G}^e + ij) - U_i(\mathbf{G}^e) > c$. Note as a consequence that a player cannot be isolated in a pws-equilibrium. We argue that a pws-equilibrium must be minimally connected. Suppose $\mathbf{G}$ is not minimally connected with a non-critical link $ij \in \mathbf{G}$. Then players i and j continue to remain connected to the same component after deleting their link. This implies $\mathbf{G}' = \mathbf{G} - ij < \mathbf{G}$, $\mathbf{H}' > \mathbf{H}$, and $\mathbf{L} = \mathbf{G} + \mathbf{H} = \mathbf{G}' + \mathbf{H}'$. Since $\lambda < \psi$, it follows that $(\lambda - \psi)\,\mathbf{G}' > (\lambda - \psi)\,\mathbf{G}$ and thus $U_i(\mathbf{G} - ij) = \frac{1}{2}B_i^2\,(\mathbf{G}', \mathbf{L}) > \frac{1}{2}B_i^2\,(\mathbf{G}, \mathbf{L}) = U_i(\mathbf{G})$. Therefore players i and j have an incentive to delete their non-critical link. Minimally connected networks consequently emerge as pws-equilibrium because players wish to remain connected to receive positive externalities but with minimal connections. We have proved that:

Proposition 6 *Suppose the utility function displays positive externalities from both direct and indirect links.* (*a*) *Suppose $\lambda > \psi > 0$. The pws-equilibrium are empty, complete, dominant group or minimally connected* (*with no isolated players*). (*b*) *Suppose $\psi > \lambda > 0$. The pws-equilibrium are empty or minimally connected* (*with no isolated players*).

Remark: Similar to the local-global mode, we observe that the direct-indirect mode also generates pws-equilibrium supporting distinct levels of Nash actions. Therefore multiple Nash actions will be observed in the equilibrium corresponding to positive externalities from primary and secondary sources.

Positive Externalities from Direct Links and Negative Externalities from Indirect Links

Now suppose direct links yield positive externalities and indirect ones negative. Let $\psi = -\psi'$, where $\psi' > 0$. Then $U_i(\mathbf{G}) = \frac{1}{2}B_i^2\left(\mathbf{G} - \frac{\psi'}{\lambda+\psi'}\mathbf{L}, \lambda + \psi'\right)$ and, since Proposition 5 restricts $\lambda + \psi'$ to be sufficiently small, each player has an incentive to add direct links (i.e. increase $\mathbf{G}$) in a manner that does not change $\mathbf{L}$. As before, $\mathbf{G}^e$ (resp. $\mathbf{G}^c$)is a pws-equilibrium for sufficiently low (resp. high) cost of link formation. Moreover a network with incomplete components cannot be a pws-equilibrium.

Next we argue that pws-equilibrium are exclusive groups. Consider $\mathbf{G}$ composed of multiple complete components. Suppose player i belongs to one group, and players j, k are part of another exclusive group. Suppose player i contemplates a link with player j. This will generate negative externalities for player i from being indirectly connected to players such as k in j's component. In other words, player i is adversely impacted by the fact that $\mathbf{L}$ increases (to say $\mathbf{L}'$) due to additional indirect links created in the network $\mathbf{G}' = \mathbf{G} + ij$. However, a subsequent link with player k will yield greater marginal utility because ik increases $\mathbf{G}'$ to $\mathbf{G}' + ik$ but does not change $\mathbf{L}'$. Therefore if an outside player i can profit from the first link to a player j in another component, then player i will have an incentive to connect to all players in j's component. The same argument establishes that all players in j's component will reciprocate. Therefore we end up with a network where one exclusive group is strictly greater in size and the number of exclusive groups has decreased by one. This argument either culminates with $\mathbf{G}^c$ or some exclusive groups network as a pws-equilibrium. We have therefore established that:

Proposition 7 *Suppose the utility function displays positive externalities from direct links and negative externalities from indirect links. Then pws-equilibrium are either empty, complete, or exclusive groups.*

Remark: Note that Nash actions can differ if the exclusive groups are asymmetrically-sized. However, if all exclusive groups are of the same size, and there are no isolated players, then such a network is regular and all players will choose the same Nash action. Therefore positive (resp. negative) externalities from the primary (resp. secondary) source will generally support equilibrium with similar Nash action levels.

Negative Externalities from Direct Links; Positive or Negative Externalities from Indirect Links

We now consider the case where direct links yield negative externalities. Accordingly let $\lambda = -\lambda'$ where $\lambda' > 0$. If indirect links confer positive externalities then $\psi > 0$ and $U_i(\mathbf{G}) = \frac{1}{2}B_i^2\left(\left(-\lambda' - \psi\right)\mathbf{G} + \psi\mathbf{L}\right)$. Thus each player has an incentive to delete direct links (i.e. decrease $\mathbf{G}$) in a manner that does not change $\mathbf{L}$. Clearly $\mathbf{G}^e$ is a pws-equilibrium because adding a direct link reduces payoffs and, in addition, connecting to an isolated player yields no positive externalities because there are no indirect links to exploit. Formally, adding a link ij to an empty network increases both $\mathbf{g}_i^e\mathbf{1}$ and $\mathbf{l}_i^e\mathbf{1}$ by 1, where $\mathbf{g}_i$ and $\mathbf{l}_i$ are the i^{th}-row of $\mathbf{G}$ and $\mathbf{L}$ respectively, and thus $U_i(\mathbf{G}^e + ij) < U_i(\mathbf{G}^e)$. We now argue that $\mathbf{G}^e$ is in fact the only pws-equilibrium. Consider therefore a network $\mathbf{G} \neq \mathbf{G}^e$ with a non-empty component in which there is a non-critical link ij. Let $\mathbf{G}' = \mathbf{G} - ij$ and note that the deletion of ij leaves $\mathbf{L}$ unaffected. Since player i can reduce negative externalities by deleting ij but harness positive externalities from the same number of indirect links, $U_i(\mathbf{G}') > U_i(\mathbf{G})$. Therefore player i will delete all non-critical links. Therefore pws-equilibrium cannot have components with non-critical links.

Now consider networks in which all links are critical, i.e. the class of minimally connected networks. We already know from the above argument that two-player components cannot be a pws-equilibrium. Therefore consider minimally connected networks in which components have at least 3 players. In such networks there exists at least one peripheral player, say j, who becomes isolated if some link ij with a player i is severed. But then player i has an incentive to delete the link ij with a peripheral player because it permits i to eliminate negative externalities from the direct link but continue to access the remaining component. Formally, deleting the link with player j reduces both $\mathbf{g}_i\mathbf{1}$ and $\mathbf{l}_i\mathbf{1}$ by 1 thus yielding $U_j(\mathbf{G} - ij) > U_j(\mathbf{G})$. This only leaves the empty network which we know is a pws-equilibrium. When indirect links also yield negative externalities, i.e. $\psi < 0$, then the incentive to delete direct links is intensified and $\mathbf{G}^e$ continues to be the unique pws-equilibrium. Thus we have proved that:

Proposition 8 *Suppose the utility function displays negative externalities from direct links. Then the empty network is the unique pws-equilibrium.*

Remark: We observe that when externalities from the primary source are negative, then both modes of transmitting externalities induce the empty network in equilibrium in which all players choose an identical positive Nash action level.

5 Conclusion

This paper argued that the nature of externalities as well as their mode of transmission are both important in dictating the architecture of equilibrium networks. However the existence of externalities, both in forming links and in choosing actions, are not taken into account by players in their equilibrium calculations. It is therefore an interesting issue to explore the *efficient* network architecture that maximizes aggregate utility and identify potential conflict between efficient and equilibrium outcomes. Consider the model of local-global externalities. Given a network $\mathbf{G}$, in the second stage the planner chooses an action vector that maximizes aggregate utility. Following Jackson and Zenou (2012) this is given by:

$$\mathbf{x}^e(\mathbf{G}) = \frac{1}{1-\gamma\left[1+b\left(\mathbf{G},2\xi\right)\right]}\mathbf{b}\left(\mathbf{G},\mathbf{2}\xi\right) \equiv \mathbf{B}\left(\mathbf{G},\mathbf{2}\xi\right)$$

In the first stage, the planner maximizes aggregate utility which is given by:

$$W(\mathbf{G}) = \left(\frac{1}{2}-\gamma\right)\mathbf{B}^\top\left(\mathbf{G},\mathbf{2}\xi\right)\mathbf{B}\left(\mathbf{G},\mathbf{2}\xi\right)$$

where $\mathbf{B}^\top\left(\mathbf{G},\mathbf{2}\xi\right)$ denotes the transpose of the KB-centrality vector. Consider positive strategic local and global externalities. If reduced utility satisfies convexity in links, then depending on the cost of link formation either $\mathbf{G}^e$ or $\mathbf{G}^c$ will be efficient. If on the other hand property (P1) obtains, then the analysis of Billand et al (2013) and Westbrock (2010) would suggest that NSG is efficient. With negative strategic local and global externalities, links reduce KB-centrality and we would expect $\mathbf{G}^e$ to be efficient. In other cases though the characterization of efficient networks is not clear cut and is an important avenue for further research. Next consider externalities from direct-indirect links. If both yield positive externalities, with direct links a more effective transmitter, then $\mathbf{G}^e$, $\mathbf{G}^c$, or minimally connected networks can be efficient depending on linking costs. If on the other hand indirect links are more effective, minimally connected networks are efficient. If direct (resp. indirect) links yield positive (resp. negative) externalities, then either $\mathbf{G}^e$ or $\mathbf{G}^c$ will be efficient. These results seem to suggest that the mode of transmission of externalities has a bearing on the topology of efficient networks. A systematic study of this issue remains an important open question. Efficiency in a model where links and actions are chosen simultaneously (e.g. Cabrales et al 2011) remains another issue warranting further research.

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