Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 460530, 12 pages
http://dx.doi.org/10.1155/2014/460530

Research Article

Convergence Analysis of Semi-Implicit Euler Methods for Solving Stochastic Age-Dependent Capital System with Variable Delays and Random Jump Magnitudes

Qinghui Du[1] and Chaoli Wang[2]

[1] Business School, University of Shanghai for Science and Technology, Shanghai 200093, China
[2] Department of Control Science and Engineering, University of Shanghai for Science and Technology, Shanghai 200093, China

Correspondence should be addressed to Chaoli Wang; clclwang@126.com

Received 29 July 2014; Revised 14 October 2014; Accepted 14 October 2014; Published 13 November 2014

Academic Editor: Hari M. Srivastava

We consider semi-implicit Euler methods for stochastic age-dependent capital system with variable delays and random jump magnitudes, and investigate the convergence of the numerical approximation. It is proved that the numerical approximate solutions converge to the analytical solutions in the mean-square sense under given conditions.

1. Introduction

Stochastic differential equations have been widely used to model the phenomena arising in many branches of science and industry fields such as biology, economic, finance, and ecology [1–4]. Recently, the numerical construction of stochastic age-dependent capital system has received a great deal of attention [5–9]. In [8], exponential stability of numerical solutions for a class stochastic age-dependent capital system with Poisson jumps was studied by Zhang et al. in the case of deterministic magnitude. Zhang and Rathinasamy [9] studied convergence of numerical solutions for a class of stochastic age-dependent capital system with random jump magnitudes, which extended the analysis in [8] to the case where the jump magnitudes are random. However, in many real problems, the capital systems can be modeled by stochastic dynamical systems whose evolutions depend not only on the current states, but also on their historical states. So we need a stochastic delay model including an extra term, which is called time delay, to simulate them. In this paper, we consider stochastic age-dependent capital system with variable delays and random jump magnitudes:

$$\begin{aligned} &dK(a,t) \\ &\quad = -\mu(a,t)K(a,t)\,dt + f(t,K(a,t),K(a,t-\tau(t)))\,dt \\ &\qquad + g(t,K(a,t),K(a,t-\tau(t)))\,dW(t) \\ &\qquad + h(t,K(a,t),K(a,t-\tau(t)),\gamma_{N(t)+1})\,dN(t), \\ &\qquad\qquad \text{in } D, \\ &K(a,t) = \psi(a,t), \quad \text{in } [-r,0], \\ &K(0,t) = \varphi(t) = \alpha(t)B(t)F\left(L(t),\int_0^A K(a,t)\,da\right), \\ &\qquad\qquad \text{in } t\in[0,T], \\ &K(a,0) = K_0(a), \quad \text{in } a\in[0,A], \\ &N(t) = \int_0^A K(a,t)\,da, \quad \text{in } t\in[0,T], \end{aligned} \tag{1}$$

where $dK(a,t) = (\partial K(a,t)/\partial t+\partial K(a,t)/\partial a)dt$, $\varphi(0) = K_0(0)$, and $D = [0,A]\times[0,T]$. The stock of capital goods of age a at time t is denoted by $K(a,t)$. This makes that total output produced at time t defined as $N(t)$; also a is the age of the capital, the investment $\varphi(t)$ in the new capital. $f(t,K(a,t),$

$K(a,t-\tau(t))) + g(t, K(a,t), K(a,t-\tau(t)))dW(t) + h(t, K(a,t), K(a,t-\tau(t)), \gamma_{N(t)+1})dN(t)$ denote effects of the external environment for capital system, such as innovations in techniques, natural disasters, introduction of new products, and changes in laws and government policies, and so on. $\tau(t)$ is a variable delay, and the f is the appreciation (when $f \geq 0$) or depreciation (when $f \leq 0$) of the production capacity, and g represents the volatility of the capital stock. $W(t)$ is a standard Wiener process. $N(t)$ is a Poisson process with mean λt and γ_i, $i = 1, 2, \ldots$, are independent, identically distributed random variables representing magnitudes for each jump. We assume that for some $p \geq 2$ there is a constant C such that $E[|\gamma_i|^p] \leq C$; that is, the pth moment of the jump magnitude is bounded. The maximum physical lifetime of capital A, the planning interval of calendar time $[0, T]$, the depreciation rate $\mu(a,t)$ of capital, and the capital density $K_0(a)$ (the initial distribution of capital over age) are given. The $\alpha(t)$ denotes the accumulative rate of capital at the moment of t, $0 < \alpha(t) < 1$, and $B(t)$ is the technical progress at the moment of t. Each sector of all the firms has an identical neoclassical technology and produces output using labor and capital. The production function $F(L(t), \int_0^A K(a,t)da)$ is neoclassical, where $\int_0^A K(a,t)da$ is the total sum of capital at time t and $L(t)$ is the labor force.

In general most equations of stochastic age-dependent capital system with variable delays and random jump magnitudes do not have explicit solution. Thus, numerical approximation schemes are invaluable tools for exploring its properties. There is a significant amount of literature that has been published concerning approximate schemes for stochastic differential equations with jumps [9–11] or stochastic differential delay equations [12–14]. In [15], Chalmers and Higham gave the semi-implicit Euler approximate solutions and proved the convergence of semi-implicit Euler methods under the Lipschitz condition. However, in many situations, the coefficients f, g, and h are only locally Lipschitz continuous. It is therefore useful to establish the strong convergence of the semi-implicit Euler method under the local Lipschitz condition. In this paper, we relax the global Lipschitz condition on the coefficients which was imposed in [15] and prove that the semi-implicit Euler approximate solutions converge to the exact solutions of (1) in the mean-square sense under the local Lipschitz condition.

In Section 2, we will collect some notation and hypotheses concerning (1), and the semi-implicit Euler method is used to produce numerical solutions. In Section 3, we give the useful lemmas which are essential to prove our main results, that is, Theorem 17.

2. Preliminaries and the Semi-Implicit Euler Approximation

Throughout this paper, it will be denoted by $L^2([0, A])$ the space of functions that are square-integrable over the domain $[0, A]$. Let

$$V = H^1([0, A]) \equiv \left\{ \varphi \mid \varphi \in L^2([0, A]), \frac{\partial \varphi}{\partial a} \in L^2([0, A]), \right. \\ \left. \text{where } \frac{\partial \varphi}{\partial a} \text{ are generalized partial derivatives} \right\}. \tag{2}$$

V is the Sobolev space. $H = L^2([0, A])$ such that $V \to H \equiv H' \to V'$. $V' = H^{-1}([0, A])$ is the dual space of V. We denote by $\|\cdot\|$ and $|\cdot|$ the norms in V and V', respectively; by $(\cdot, \cdot)$ the scalar product in H. $\langle \cdot, \cdot \rangle$, the duality product between V and V', is defined by

$$\langle \cdot, \cdot \rangle = \int_0^A u \cdot v \, da, \quad u \in V, \; v \in V'. \tag{3}$$

Let $W(t)$ be a Wiener process defined on complete probability space (Ω, F, P) with covariance operator G and taking its values in the separable Hilbert space S:

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) e_i, \tag{4}$$

where $\{e_i\}_{i \geq 0}$ is an orthonormal set of eigenvectors of G, $\beta_i(t)$ that are mutually independent real Wiener processes with incremental covariance $\lambda_i > 0$, $Ge_i = \lambda_i e_i$, and $\operatorname{tr} G = \sum_{i=1}^{\infty} \lambda_i$ (tr denotes the trace of an operator). For an operator $A \in \phi(S, H)$ is the space of all bounded linear operators from S into H; it is denoted by $\|A\|_2$; its denotes the Hilbert-Schmidt norm; that is,

$$\|A\|_2^2 = \operatorname{tr}\left(AGA^T\right). \tag{5}$$

Let $C = C([0, T]; H)$ be the space of all continuous function from $[0, T]$ into H with sup-norm $\|\varphi\|_c = \sup_{0 \leq s \leq T} |\varphi(s)|$, $L_V^p = L^p([0, T]; V)$, and $L_H^p = L^p([0, T]; H)$.

Let (Ω, F, P) be a complete probability space with a filtration $\{F_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous with a left-hand side limit, and F_0 contains all P-null sets). Let $N(t)$ be a scalar Poisson process with intensity λ which is independent from Wiener process $W(t)$. Denote by $D_{F_0}^p([-r, 0]; H)$ the family of all bounded, F_0-measurable, and $D([-r, 0]; H)$-valued random variables. Let $p > 0$, $t \geq 0$, and $D_{F_0}^p([-r, 0]; H)$ denote the family of all F_0-measurable, $D([-r, 0]; H)$-valued variables ψ which satisfies $\sup_{-r \leq s \leq 0} E|\psi(s)|^p < \infty$. In this paper, $f(t, \cdot, \cdot)$, $g(t, \cdot, \cdot)$, and $h(t, \cdot, \cdot)$ are family of nonlinear operators, F_t-measurable almost surely in t.

The integral version of (1) is given by

$$\begin{aligned} K_t = K_0 &- \int_0^t \frac{\partial K_s}{\partial a} ds - \int_0^t \mu(a, s) K_s ds \\ &+ \int_0^t f(t, K_s, K_{s-\tau(s)}) ds + \int_0^t g(t, K_s, K_{s-\tau(s)}) dW(s) \\ &+ \int_0^t h(t, K_s, K_{s-\tau(s)}, \gamma(s)) dN(s). \end{aligned} \tag{6}$$

For system (6), the discrete semi-implicit Euler approximation on $t \in \{0, h, 2h, \ldots\}$ is given by the iterative scheme

$$\begin{aligned} Q_{n+1} &= Q_n - \frac{\partial Q_{n+1}}{\partial a} h + (1-\theta) \\ &\quad \times \left[-\mu(t_n, a) Q_n + f\left(Q_n, Q_{[(nh-\tau(nh))/h]}\right)\right] h \\ &\quad + \theta\left[-\mu(t_n, a) Q_{n+1} + f\left(Q_{n+1}, Q_{[((n+1)h-\tau((n+1)h))/h]}\right)\right] h \\ &\quad + g\left(Q_n, Q_{[(nh-\tau(nh))/h]}\right) \Delta W_n \\ &\quad + h\left(Q_n, Q_{[(nh-\tau(nh))/h]}, \gamma_{N(t_n)+1}\right) \Delta N_n, \end{aligned} \tag{7}$$

with initial value $Q_0 = K(0, a) = K_0$; $0 \le \theta \le 1$; and $[u]$ represents the integer part of u. Here the time increment is $h = T/m$, for some sufficiently large integer m such that $h \ll 1$ and $Q_n \approx K(a, t_n)$, for $h = t_{n+1} - t_n$, and $\Delta W_n = W(t_{n+1}) - W(t_n)$ and $\Delta N_n = N(t_{n+1}) - N(t_n)$, $n = 0, 1, 2, \ldots, m$, are the Wiener and Poisson increments, respectively.

Define the step functions:

$$\begin{aligned} \overline{Y}_t &= \overline{Y}(t, a) = \sum_{n=0}^{m-1} Q_n I_{[nh,(n+1)h)}(t), \\ \widehat{Y}_t &= \widehat{Y}(t, a) = \sum_{n=0}^{m-1} Q_{[(nh-\tau(nh))/h]} I_{[nh,(n+1)h)}(t), \\ \overline{Z}_t &= \overline{Z}(t, a) = \sum_{n=0}^{m-1} Q_{n+1} I_{[nh,(n+1)h)}(t), \\ \widehat{Z}_t &= \widehat{Z}(t, a) = \sum_{n=0}^{m-1} Q_{[((n+1)h-\tau((n+1)h))/h]} I_{[nh,(n+1)h)}(t), \\ \overline{\gamma}(t) &= \sum_{n=0}^{m-1} \gamma_{N(t_n)+1} I_{[nh,(n+1)h)}(t), \end{aligned} \tag{8}$$

where I_A is the indicator function for set A. Then we define the continuous semi-implicit Euler approximation:

$$\begin{aligned} Q_t &= K_0 - \int_0^t \frac{\partial Q_s}{\partial a} ds \\ &\quad + \int_0^t (1-\theta)\left[-\mu(s, a) \overline{Y}_s + f\left(\overline{Y}_s, \widehat{Y}_s\right)\right] ds \\ &\quad + \int_0^t \theta\left[-\mu(s, a) \overline{Z}_s + f\left(\overline{Z}_s, \widehat{Z}_s\right)\right] ds \\ &\quad + \int_0^t g\left(\overline{Y}_s, \widehat{Y}_s\right) dW(s) \\ &\quad + \int_0^t h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right) dN(s), \end{aligned} \tag{9}$$

with $Q_0 = K(0, a)$, $Q_t = Q(a, t)$ for fixed a.

Remark 1. If the parameter $\theta = 0$ in (7), then the semi-implicit Euler methods become the Euler methods, which have been studied in [10, 12, 14].

In order to establish the convergence theorem, we propose the following assumptions.

Assumption 2. The function $\tau : [0, \infty] \to R$ is the time delay which satisfies

$$\tau(t) \ge 0, \quad t - \tau(t) \ge -r. \tag{10}$$

For $t, s \ge 0$, there exists a constant $k < 1$ such that

$$|\tau(t) - \tau(s)| \le k|t - s|. \tag{11}$$

Assumption 3 (local Lipschitz condition). There exist two positive constants K_d, K such that, for all $x_1, y_1, x_2, y_2, z_1, z_2 \in H$ and $t \in [0, T]$,

$$\begin{aligned} &\left|f(x_1, y_1) - f(x_2, y_2)\right|^2 \vee \left\|g(x_1, y_1) - g(x_2, y_2)\right\|_2^2 \\ &\quad \le K_d\left(\left|x_1 - x_2\right|^2 + \left|y_1 - y_2\right|^2\right), \\ &\left|h(x_1, y_1, z_1) - h(x_2, y_2, z_2)\right|^2 \\ &\quad \le K_d\left(\left|x_1 - x_2\right|^2 + \left|y_1 - y_2\right|^2\right) + K\left(\left|z_1 - z_2\right|^2\right). \end{aligned} \tag{12}$$

Assumption 4. $\mu(t, a)$ is a nonnegative measurable function in D such that

$$0 \le \mu_0 \le \mu(t, a) \le \overline{\alpha} < \infty, \tag{13}$$

and $B(t)$ is a nonnegative continuous function in $[0, T]$ such that $\alpha(t)B(t) \le \eta$; η is a nonnegative constant in $[0, T]$.

Assumption 5. The initial function ψ is Holder-continuous with exponent γ; that is, there exists a positive constant K' such that, for $t, s \in [-\tau, 0]$,

$$E\left(\left|\psi(t) - \psi(s)\right|^2\right) \le K'|t - s|^{2\gamma}. \tag{14}$$

Assumption 6. Consider $f(t, 0) = 0$, $g(t, 0) = 0$, and $h(t, 0, \gamma_{N(t)+1}) = 0$, $t \in [0, T]$.

Assumption 7. Consider $F(L, N) \ge 0$, $(F(L, 0) = 0)$, $\partial F/\partial L > 0$, and $0 < \partial F/\partial N \le F_1$, where F_1 is a positive constant.

Remark 8. If the local Lipschitz condition holds, then there exists a positive constant K_d' such that, for $x, y, z \in R^n$,

$$\left|f(x, y)\right|^2 \vee \left\|g(x, y)\right\|_2^2 \vee \left|h(x, y, z)\right|^2 \le K_d'. \tag{15}$$

3. Convergence of the Semi-Implicit Euler Approximate Solution

In this section, several lemmas which are useful for the following main result are given.

Lemma 9. *Under Assumptions 2–7, there are constants $k \geq 2$ and $C_1 > 0$ such that*

$$E\left[\sup_{0\leq t\leq T}\left|K_t\right|^k\right] \leq C_1. \tag{16}$$

The proof is similar to that of in [16].

Let $\sigma_n = \inf\{t \geq 0 : |K_t| \geq n\}$ and $\rho_n = \inf\{t \geq 0 : |Q_t| \geq n\}$. Define the stopping time $\upsilon_n = \rho_n \wedge \sigma_n$.

Lemma 10. *Under Assumptions 2–7, there exists a constant $C_2 > 0$ such that*

$$E\left[\sup_{t\in[0,T]}\left|Q_{t\wedge\upsilon_n}\right|^2\right] \leq C_2. \tag{17}$$

Proof. From (9), applying Itô formula to $|Q_{t\wedge\upsilon_n}|^2$ yields

$$\begin{aligned}
&\left|Q_{t\wedge\upsilon_n}\right|^2 \\
&= \left|Q_0\right|^2 + 2\int_0^{t\wedge\upsilon_n}\left\langle -\frac{\partial Q_s}{\partial a}, Q_s\right\rangle ds \\
&\quad - 2\int_0^{t\wedge\upsilon_n}\left(Q_s, \mu(s,a)\left[(1-\theta)\overline{Y}_s + \theta\overline{Z}_s\right] ds\right) \\
&\quad + 2\int_0^{t\wedge\upsilon_n}\left(Q_s, \left[(1-\theta) f\left(\overline{Y}_s, \widehat{Y}_s\right) + \theta f\left(\overline{Z}_s, \widehat{Z}_s\right)\right] ds\right) \\
&\quad + \int_0^{t\wedge\upsilon_n}\left\|g\left(\overline{Y}_s, \widehat{Y}_s\right)\right\|_2^2 ds + 2\int_0^{t\wedge\upsilon_n}\left(Q_s, g\left(\overline{Y}_s, \widehat{Y}_s\right) dW_s\right) \\
&\quad + 2\int_0^{t\wedge\upsilon_n}\left(Q_s, h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right) dN(s) \\
&\quad + \int_0^{t\wedge\upsilon_n}\left|h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right|^2 dN(s) \\
&\leq \left|Q_0\right|^2 + 2\int_0^{t\wedge\upsilon_n}\left\langle -\frac{\partial Q_s}{\partial a}, Q_s\right\rangle ds \\
&\quad - 2\int_0^{t\wedge\upsilon_n}\left(Q_s, \mu(s,a)\left[(1-\theta)\overline{Y}_s + \theta\overline{Z}_s\right] ds\right) \\
&\quad + 2\int_0^{t\wedge\upsilon_n}\left(Q_s, \left[(1-\theta) f\left(\overline{Y}_s, \widehat{Y}_s\right) + \theta f\left(\overline{Z}_s, \widehat{Z}_s\right)\right] ds\right) \\
&\quad + \int_0^{t\wedge\upsilon_n}\left\|g\left(\overline{Y}_s, \widehat{Y}_s\right)\right\|_2^2 ds + 2\int_0^{t\wedge\upsilon_n}\left(Q_s, g\left(\overline{Y}_s, \widehat{Y}_s\right) dW_s\right) \\
&\quad + 2\int_0^{t\wedge\upsilon_n}\left(Q_s, h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right) d\overline{N}(s)\right) \\
&\quad + 2\lambda\int_0^{t\wedge\upsilon_n}\left(Q_s, h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right) ds\right) \\
&\quad + \lambda\int_0^{t\wedge\upsilon_n}\left|h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right|^2 ds \\
&\quad + \int_0^{t\wedge\upsilon_n}\left|h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right|^2 d\overline{N}(s),
\end{aligned} \tag{18}$$

where $N(t) = \overline{N}(t) - \lambda t$ is a compensated Poisson process. Note

$$\begin{aligned}
&\left\langle -\frac{\partial Q_s}{\partial a}, Q_s\right\rangle \\
&= -\int_0^A Q_s \cdot \frac{\partial Q_s}{\partial a} da \\
&= \frac{1}{2}\alpha^2(s) B^2(s)\left[F\left(L(s), \int_0^A Q_s da\right) - F(L(s), 0)\right]^2 \\
&\leq \frac{1}{2}\eta^2\left(\frac{\partial F(L,N)}{\partial N}\bigg|_y\right)^2\left(\int_0^A Q_s da\right) \\
&\leq \frac{1}{2}AF_1^2\eta^2\left|Q_s\right|^2,
\end{aligned} \tag{19}$$

where $y \in (0, \int_0^A Q_s da)$.

Therefore, by Assumption 4, (15), and a variant of Cauchy-Schwarz inequality for any $t \in [0,T]$, along with (18), we get that

$$\begin{aligned}
&\left|Q_{t\wedge\upsilon_n}\right|^2 \\
&\quad \leq \left|Q_0\right|^2 + \left(AF_1^2\eta^2 + 1 + \overline{\alpha} + \lambda\right) \\
&\qquad \times \int_0^{t\wedge\upsilon_n}\left|Q_s\right|^2 ds + 2\overline{\alpha}\int_0^{t\wedge\upsilon_n}\left[\left|\overline{Y}_s\right|^2 + \left|\overline{Z}_s\right|^2\right] ds \\
&\qquad + (5+\lambda) K_d' T + 2\int_0^{t\wedge\upsilon_n}\left(Q_s, g\left(\overline{Y}_s, \widehat{Y}_s\right) dW_s\right) \\
&\qquad + 2\int_0^{t\wedge\upsilon_n}\left(Q_s, h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right) d\overline{N}(s)\right) \\
&\qquad + \lambda\int_0^{t\wedge\upsilon_n}\left|h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right|^2 ds \\
&\qquad + \int_0^{t\wedge\upsilon_n}\left|h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right|^2 d\overline{N}(s).
\end{aligned} \tag{20}$$

Let $k_1 = AF_1^2\eta^2 + 1 + \overline{\alpha} + \lambda$, $k_2 = 2\overline{\alpha}$; we have

$$\begin{aligned}
&E\left[\sup_{s\in[0,t]}\left|Q_s\right|^2\right] \\
&\quad \leq E\left|Q_0\right|^2 + (k_1 + 2k_2) \\
&\qquad \times \int_0^{t\wedge\upsilon_n} E\left[\sup_{s\in[0,t]}\left|Q_s\right|^2\right] ds + (5+\lambda) K_d' T \\
&\qquad + 2E\left[\sup_{s\in[0,t]}\int_0^{t\wedge\upsilon_n}\left(Q_u, g\left(\overline{Y}_u, \widehat{Y}_u\right) dW(u)\right)\right]
\end{aligned}$$

$$+ 2E\left[\sup_{s\in[0,t]}\int_0^{t\wedge\upsilon_n}\left(Q_u, h\left(\overline{Y}_u, \widehat{Y}_u, \overline{\gamma}(u)\right)d\overline{N}(u)\right)\right]$$
$$+ E\left[\sup_{s\in[0,t]}\int_0^{t\wedge\upsilon_n}\left|h\left(\overline{Y}_u, \widehat{Y}_u, \overline{\gamma}(u)\right)\right|^2 d\overline{N}(u)\right]. \tag{21}$$

By Burkholder-Davis-Gundy inequality, we have

$$E\left[\sup_{s\in[0,t]}\int_0^{t\wedge\upsilon_n}\left(Q_u, g\left(\overline{Y}_u, \widehat{Y}_u\right)dW(u)\right)\right] \le \frac{1}{6}E\left[\sup_{s\in[0,t]}\left|Q_s\right|^2\right] + k_3K_d'T,$$
$$E\left[\sup_{s\in[0,t]}\int_0^{t\wedge\upsilon_n}\left(Q_u, h\left(\overline{Y}_u, \widehat{Y}_u, \overline{\gamma}(u)\right)d\overline{N}(u)\right)\right] \le \frac{1}{6}E\left[\sup_{s\in[0,t]}\left|Q_s\right|^2\right] + k_4K_d'T,$$
$$E\left[\sup_{s\in[0,t]}\int_0^{t\wedge\upsilon_n}\left|h\left(\overline{Y}_u, \widehat{Y}_u, \overline{\gamma}(u)\right)\right|^2 d\overline{N}(u)\right] \le k_5K_d'T, \tag{22}$$

where k_3, k_4, and k_5 are positive constants. Substituting (22) into (21) yields, again for a possibly different k_6 and k_7,

$$E\left[\sup_{s\in[0,t]}\left|Q_s\right|^2\right] \le 3E\left|Q_0\right|^2 + k_7\int_0^{t\wedge\upsilon_n}E\left[\sup_{s\in[0,t]}\left|Q_s\right|^2\right]ds + k_6K_d'T, \tag{23}$$

for every $t \in [0, T]$. Now, Gronwall lemma obviously implies the required result. The proof is complete. □

Lemma 11. *Under Assumptions 2–7, there exist constants $k \ge 2$ and $C_3 > 0$ such that*

$$E\left[\sup_{t\in[0,T]}\left|Q_t^k\right|\right] \le C_3. \tag{24}$$

The proof is similar to that of Lemma 9.

Lemma 12. *Under Assumption 3 and $E|\partial Q_s/\partial a|^2 \le \infty$, for any $t \in [0, T]$,*

$$E\left[\sup_{t\in[0,T]}\left|Q_t - \overline{Y}_t\right|^2\right] \le C_1(d)h,$$
$$E\left[\sup_{t\in[0,T]}\left|Q_t - \overline{Z}_t\right|^2\right] \le C_3(d)h, \tag{25}$$

where $C_1(d)$, $C_3(d)$ are positive constants dependent only on θ, λ, and K_d' and independent of h.

Proof. For any $t \in [0, T]$, there exists an integer n such that $t \in [nh, (n+1)h)$; then

$$\begin{aligned} Q_t - \overline{Y}_t &= Q_t - Q_n \\ &= -\int_{nh}^t \frac{\partial Q_s}{\partial a}ds - \int_{nh}^t \mu(s,a)\left[(1-\theta)\overline{Y}_s + \theta\overline{Z}_s\right]ds \\ &\quad + \int_{nh}^t\left[(1-\theta)f\left(\overline{Y}_s, \widehat{Y}_s\right) + \theta f\left(\overline{Z}_s, \widehat{Z}_s\right)\right]ds \\ &\quad + \int_{nh}^t g\left(\overline{Y}_s, \widehat{Y}_s\right)dW(s) \\ &\quad + \int_{nh}^t h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)dN(s). \end{aligned} \tag{26}$$

Applying the basic inequality $|a+b+c+d+e|^2 \le 5|a|^2 + 5|b|^2 + 5|c|^2 + 5|d|^2 + 5|e|^2$ to 5 terms on the above right-hand side, we have

$$\begin{aligned} &E\left|Q_t - \overline{Y}_t\right|^2 \\ &\le 5E\left|\int_{kh}^t \frac{\partial Q_s}{\partial a}ds\right|^2 + 5E\left|\int_{nh}^t \mu(s,a)\left[(1-\theta)\overline{Y}_s + \theta\overline{Z}_s\right]ds\right|^2 \\ &\quad + 5E\left|\int_{nh}^t\left[(1-\theta)f\left(\overline{Y}_s, \widehat{Y}_s\right) + \theta f\left(\overline{Z}_s, \widehat{Z}_s\right)\right]ds\right|^2 \\ &\quad + 5E\left|\int_{nh}^t g\left(\overline{Y}_s, \widehat{Y}_s\right)dW(s)\right|^2 \\ &\quad + 5E\left|\int_{nh}^t h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)dN(s)\right|^2. \end{aligned} \tag{27}$$

Now, the Cauchy-Schwarz inequality and Assumptions 2–7 give

$$\begin{aligned} &E\left[\sup_{t\in[0,T]}\left|Q_t - \overline{Y}_t\right|^2\right] \\ &\le 5hE\left[\sup_{t\in[0,T]}\int_{kh}^t\left|\frac{\partial Q_s}{\partial a}\right|^2 ds\right] \\ &\quad + 10h\overline{\alpha}^2E\left[\sup_{t\in[0,T]}\int_{nh}^t\left(\left|\overline{Y}_s\right|^2 + \left|\overline{Z}_s\right|^2\right)ds\right] \\ &\quad + 10hE\left[\sup_{t\in[0,T]}\int_{nh}^t\left(\left|f\left(\overline{Y}_s, \widehat{Y}_s\right)\right|^2 + \left|f\left(\overline{Z}_s, \widehat{Z}_s\right)\right|^2\right)ds\right] \end{aligned}$$

$$+ 5E\left[\sup_{t\in[0,T]}\left|\int_{nh}^{t} g\left(\overline{Y}_s,\widehat{Y}_s\right)dW(s)\right|^2\right]$$
$$+ 10E\left[\sup_{t\in[0,T]}\left|\int_{nh}^{t} h\left(\overline{Y}_s,\widehat{Y}_s,\overline{\gamma}(s)\right)d\overline{N}(s)\right|^2\right]$$
$$+ 10E\left[\sup_{t\in[0,T]}\left|\lambda\int_{nh}^{t} h\left(\overline{Y}_s,\widehat{Y}_s,\overline{\gamma}(s)\right)ds\right|^2\right]$$
$$\le 5hE\left[\sup_{t\in[0,T]}\int_{kh}^{t}\left|\frac{\partial Q_s}{\partial a}\right|^2 ds\right]$$
$$+ 10h\overline{\alpha}^2E\left[\sup_{t\in[0,T]}\int_{nh}^{t}\left(\left|\overline{Y}_s\right|^2+\left|\overline{Z}_s\right|^2\right)ds\right]$$
$$+ 5\left(4h^2+h+2h^2\lambda^2+2k_8h\right)K_d', \tag{28}$$

where k_8 is a constant. Because the differential operator $\partial/\partial a$ is a bounded linear operator, we obtain

$$E\left[\sup_{t\in[0,T]}\left|Q_t-\overline{Y}_t\right|^2\right]$$
$$\le 5h^2E\left|\frac{\partial Q_s}{\partial a}\right|^2 + 20\overline{\alpha}^2h^2\sup_{t\in[0,T]}E\left|Q_t\right|^2$$
$$+ 5\left(4h+1+2h\lambda^2+2k_8\right)hK_d'$$
$$\le C_1(d)\,h. \tag{29}$$

Similarly, the second part of (25) can be obtained. □

Lemma 13. *Under Assumptions 2–7, for any $t\in[0,T]$,*

$$E\left[\sup_{t\in[0,T]}\left|Q_{t-\tau(t)}-\widehat{Y}_t\right|^2\right]\le C_2(d)\,h^{1\wedge 2\gamma},$$
$$E\left[\sup_{t\in[0,T]}\left|Q_{t-\tau(t)}-\widehat{Z}_t\right|^2\right]\le C_4(d)\,h^{1\wedge 2\gamma}, \tag{30}$$

where $C_2(d)$, $C_4(d)$ are positive constants dependent only on θ, λ, and K_d' and independent of h.

Proof. For any $t\in[0,T]$, there exists an integer n such that $t\in[nh,(n+1)h)$; then $Q_{t-\tau(t)}-\widehat{Y}_t = Q_{t-\tau(t)}-Q_{[(nh-\tau(nh))/h]h}$. To show the estimate $E[|Q_{t-\tau(t)}-\widehat{Y}_t|^2]$, let us consider the following five possible cases.

(1) If $t-\tau(t)\ge[(nh-\tau(nh))/h]h\ge 0$, then $t-\tau(t)-[(nh-\tau(nh))/h]h\le t-\tau(t)+\tau(nh)-(n-1)h\le(k+2)h$.

Using the Cauchy-Schwarz inequality, martingale isometries, and (15), we have

$$\left|Q_{t-\tau(t)}-\widehat{Y}_t\right|^2$$
$$\le 5\left|\int_{[(nh-\tau(nh))/h]h}^{t-\tau(t)}\frac{\partial Q_s}{\partial a}ds\right|^2$$
$$+ 5\left|\int_{[(nh-\tau(nh))/h]h}^{t-\tau(t)}\mu(s,a)\left[(1-\theta)\overline{Y}_s+\theta\overline{Z}_s\right]ds\right|^2$$
$$+ 5\left|\int_{[(nh-\tau(nh))/h]h}^{t-\tau(t)}\left[(1-\theta)f\left(\overline{Y}_s,\widehat{Y}_s\right)+\theta f\left(\overline{Z}_s,\widehat{Z}_s\right)\right]ds\right|^2$$
$$+ 5\left|\int_{[(nh-\tau(nh))/h]h}^{t-\tau(t)}g\left(\overline{Y}_s,\widehat{Y}_s\right)dW(s)\right|^2$$
$$+ 5\left|\int_{[(nh-\tau(nh))/h]h}^{t-\tau(t)}h\left(\overline{Y}_s,\widehat{Y}_s,\overline{\gamma}(s)\right)dN(s)\right|^2$$
$$\le 5(k+2)^2h^2\left|\frac{\partial Q_s}{\partial a}\right|^2$$
$$+ 10\overline{\alpha}^2(k+2)h\int_{[(nh-\tau(nh))/h]h}^{t-\tau(t)}\left(\left|\overline{Y}_s\right|^2+\left|\overline{Z}_s\right|^2\right)ds$$
$$+ 10(k+2)h$$
$$\times\int_{[(nh-\tau(nh))/h]h}^{t-\tau(t)}\left(\left|f\left(\overline{Y}_s,\widehat{Y}_s\right)\right|^2+\left|f\left(\overline{Z}_s,\widehat{Z}_s\right)\right|^2\right)ds$$
$$+ 5\int_{[(nh-\tau(nh))/h]h}^{t-\tau(t)}\left|g\left(\overline{Y}_s,\widehat{Y}_s\right)\right|^2ds$$
$$+ \left(10\lambda+10\lambda^2(k+2)h\right)$$
$$\times\int_{[(nh-\tau(nh))/h]h}^{t-\tau(t)}\left|h\left(\overline{Y}_s,\widehat{Y}_s,\overline{\gamma}(s)\right)\right|^2ds. \tag{31}$$

Hence,

$$E\left[\sup_{t\in[0,T]}\left|Q_{t-\tau(t)}-\widehat{Y}_t\right|^2\right]$$
$$\le 5(k+2)^2h^2E\left|\frac{\partial Q_s}{\partial a}\right|^2+20\overline{\alpha}^2(k+2)^2h^2\sup_{t\in[0,T]}E\left|Q_t\right|^2$$
$$+ 5\left(2\lambda+2\lambda^2(k+2)h+1+4(k+2)h\right)(k+2)hK_d'$$
$$\le C_2(d)\,h. \tag{32}$$

(2) If $[(nh-\tau(nh))/h]h\ge t-\tau(t)\ge 0$, then $[(nh-\tau(nh))/h]h-(t-\tau(t))\le nh-\tau(nh)-(t-\tau(t))\le kh$.

Using the Cauchy-Schwarz inequality, martingale isometries, and (15), we have

$$\begin{aligned}
&\left|Q_{t-\tau(t)}-\widehat{Y}_t\right|^2\\
&\leq 5\left|\int_{t-\tau(t)}^{[(nh-\tau(nh))/h]h}\frac{\partial Q_s}{\partial a}ds\right|^2\\
&\quad+5\left|\int_{t-\tau(t)}^{[(nh-\tau(nh))/h]h}\mu(s,a)\left[(1-\theta)\overline{Y}_s+\theta\overline{Z}_s\right]ds\right|^2\\
&\quad+5\left|\int_{t-\tau(t)}^{[(nh-\tau(nh))/h]h}\left[(1-\theta)f\left(\overline{Y}_s,\widehat{Y}_s\right)+\theta f\left(\overline{Z}_s,\widehat{Z}_s\right)\right]ds\right|^2\\
&\quad+5\left|\int_{t-\tau(t)}^{[(nh-\tau(nh))/h]h}g\left(\overline{Y}_s,\widehat{Y}_s\right)dW(s)\right|^2\\
&\quad+5\left|\int_{t-\tau(t)}^{[(nh-\tau(nh))/h]h}h\left(\overline{Y}_s,\widehat{Y}_s,\overline{\gamma}(s)\right)dN(s)\right|^2\\
&\leq 5k^2h^2\left|\frac{\partial Q_s}{\partial a}\right|^2\\
&\quad+10\overline{\alpha}^2kh\int_{t-\tau(t)}^{[(nh-\tau(nh))/h]h}\left(\left|\overline{Y}_s\right|^2+\left|\overline{Z}_s\right|^2\right)ds\\
&\quad+10kh\int_{t-\tau(t)}^{[(nh-\tau(nh))/h]h}\left(\left|f\left(\overline{Y}_s,\widehat{Y}_s\right)\right|^2+\left|f\left(\overline{Z}_s,\widehat{Z}_s\right)\right|^2\right)ds\\
&\quad+5E\int_{t-\tau(t)}^{[(nh-\tau(nh))/h]h}\left|g\left(\overline{Y}_s,\widehat{Y}_s\right)\right|^2ds\\
&\quad+\left(10\lambda+10\lambda^2kh\right)\int_{[(nh-\tau(nh))/h]h}^{t-\tau(t)}\left|h\left(\overline{Y}_s,\widehat{Y}_s,\overline{\gamma}(s)\right)\right|^2ds.
\end{aligned}\tag{33}$$

Hence,

$$\begin{aligned}
&E\left[\sup_{t\in[0,T]}\left|Q_{t-\tau(t)}-\widehat{Y}_t\right|^2\right]\\
&\quad\leq 5k^2h^2E\left|\frac{\partial Q_s}{\partial a}\right|^2+20\overline{\alpha}^2k^2h^2\sup_{t\in[0,T]}E\left|Q_t\right|^2\\
&\qquad+5\left(2\lambda+2\lambda^2kh+1+4kh\right)khK'\\
&\quad\leq C_2(d)\,h.
\end{aligned}\tag{34}$$

(3) If $0\geq t-\tau(t)\geq[(nh-\tau(nh))/h]h$ or $0\geq[(nh-\tau(nh))/h]h\geq t-\tau(t)$, then $|t-\tau(t)-[(nh-\tau(nh))/h]h|\leq(k+2)h$. So we get, by Assumption 5 on ψ,

$$\begin{aligned}
&E\left[\left|Q_{t-\tau(t)}-\widehat{Y}_t\right|^2\right]\\
&\quad=E\left|\psi(t-\tau(t))-\psi([(nh-\tau(nh))/h]h)\right|^2\\
&\quad\leq K'\left|s-\tau(s)-\left[\frac{(nh-\tau(nh))}{h}\right]h\right|^2\\
&\quad\leq K'(k+2)^{2\gamma}h^{2\gamma}.
\end{aligned}\tag{35}$$

(4) If $t-\tau(t)\geq 0\geq[(nh-\tau(nh))/h]h$, then $t-\tau(t)\leq t-\tau(t)-[(nh-\tau(nh))/h]h\leq(k+2)h$ and $-[(nh-\tau(nh))/h]h\leq t-\tau(t)-[(nh-\tau(nh))/h]h\leq(k+2)h$. We have, by Assumption 5 on ψ,

$$\begin{aligned}
&\left|Q_{t-\tau(t)}-\widehat{Y}_t\right|^2\\
&=\left|Q_{t-\tau(t)}-\psi\left(\left[\frac{(nh-\tau(nh))}{h}\right]h\right)\right|^2\\
&=\left|Q_{t-\tau(t)}-\psi(0)+\psi(0)-\psi\left(\left[\frac{(nh-\tau(nh))}{h}\right]h\right)\right|^2\\
&\leq 2\left|Q_{t-\tau(t)}-\psi(0)\right|^2\\
&\quad+2\left|\psi(0)-\psi\left(\left[\frac{(nh-\tau(nh))}{h}\right]h\right)\right|^2\\
&=2\left|Q_{t-\tau(t)}-Q_0\right|^2+2\left|\psi(0)-\psi\left(\left[\frac{(nh-\tau(nh))}{h}\right]h\right)\right|^2\\
&\leq 10\left|\int_0^{t-\tau(t)}\frac{\partial Q_s}{\partial a}ds\right|^2\\
&\quad+10\left|(1-\theta)\int_0^{t-\tau(t)}\mu(s,a)\left[(1-\theta)\overline{Y}_s+\theta\overline{Z}_s\right]ds\right|^2\\
&\quad+10\left|\int_0^{t-\tau(t)}\left[(1-\theta)f\left(\overline{Y}_s,\widehat{Y}_s\right)+\theta f\left(\overline{Z}_s,\widehat{Z}_s\right)\right]ds\right|^2\\
&\quad+10\left|\int_0^{t-\tau(t)}g\left(\overline{Y}_s,\widehat{Y}_s\right)dW(s)\right|^2\\
&\quad+10\left|\int_0^{t-\tau(t)}h\left(\overline{Y}_s,\widehat{Y}_s,\overline{\gamma}(s)\right)dN(s)\right|^2\\
&\quad+2K'\left|-\left[\frac{(nh-\tau(nh))}{h}\right]h\right|^{2\gamma}.
\end{aligned}\tag{36}$$

Hence,

$$\begin{aligned}
&E\left[\sup_{t\in[0,T]}\left|Q_{t-\tau(t)}-\widehat{Y}_t\right|^2\right]\\
&\leq 10(k+2)^2h^2E\left|\frac{\partial Q_s}{\partial a}\right|^2+20\overline{\alpha}^2(k+2)^2h^2\sup_{t\in[0,T]}E\left|Q_t\right|^2
\end{aligned}$$

$$+5\left(2\lambda+2\lambda^2kh+1+4kh\right)(k+2)hK'$$
$$+2K'(k+2)^{2\gamma}h^{2\gamma}$$
$$\leq C_2(d)h^{1\wedge 2\gamma}. \tag{37}$$

(5) If $[(nh-\tau(nh))/h]h \geq 0 \geq t-\tau(t)$, then $[(nh-\tau(nh))/h]h \leq [(nh-\tau(nh))/h]h-(t-\tau(t)) \leq (k+2)h$ and $-(t-\tau(t)) \leq [(nh-\tau(nh))/h]h-(t-\tau(t)) \leq (k+2)h$. We have, by Assumption 5 on ψ,

$$\begin{aligned}
&\left|Q_{t-\tau(t)}-\widehat{Y}_t\right|^2\\
&=\left|\psi(t-\tau(t))-Q_{[(nh-\tau(nh))/h]h}\right|^2\\
&=\left|\psi(t-\tau(t))-\psi(0)+\psi(0)-Q_{[(nh-\tau(nh))/h]h}\right|^2\\
&\leq 2\left|\psi(t-\tau(t))-\psi(0)\right|^2+2\left|\psi(0)-Q_{[(nh-\tau(nh))/h]h}\right|^2\\
&=2K'\left|-(t-\tau(t))\right|^{2\gamma}+2\left|Q_0-Q_{[(nh-\tau(nh))/h]h}\right|^2\\
&\leq 2K'\left|-(t-\tau(t))\right|^{2\gamma}+10\left|\int_0^{[(nh-\tau(nh))/h]h}\frac{\partial Q_s}{\partial a}ds\right|^2\\
&\quad+10\left|\int_0^{[(nh-\tau(nh))/h]h}\mu(s,a)\left[(1-\theta)\overline{Y}_s+\theta\overline{Z}_s\right]ds\right|^2\\
&\quad+10\left|\int_0^{[(nh-\tau(nh))/h]h}\left[(1-\theta)f\left(\overline{Y}_s,\widehat{Y}_s\right)+\theta f\left(\overline{Z}_s,\widehat{Z}_s\right)\right]ds\right|^2\\
&\quad+10\left|\int_0^{[(nh-\tau(nh))/h]h}g\left(\overline{Y}_s,\widehat{Y}_s\right)dW(s)\right|^2\\
&\quad+10\left|\int_0^{[(nh-\tau(nh))/h]h}h\left(\overline{Y}_s,\widehat{Y}_s,\overline{\gamma}(s)\right)dN(s)\right|^2.
\end{aligned} \tag{38}$$

Hence,

$$\begin{aligned}
&E\left[\sup_{t\in[0,T]}\left|Q_{t-\tau(t)}-\widehat{Y}_t\right|^2\right]\\
&\leq 10(k+2)^2h^2E\left|\frac{\partial Q_s}{\partial a}\right|^2+20\overline{\alpha}^2(k+2)^2h^2\sup_{t\in[0,T]}E\left|Q_t\right|^2\\
&\quad+5\left(2\lambda+2\lambda^2kh+1+4kh\right)(k+2)hK'\\
&\quad+2K'(k+2)^{2\gamma}h^{2\gamma}\\
&\leq C_2(d)h^{1\wedge 2\gamma}.
\end{aligned} \tag{39}$$

Combining these different cases, we get

$$E\left[\sup_{t\in[0,T]}\left|Q_{t-\tau(t)}-\widehat{Y}_t\right|^2\right]\leq C_2(d)h^{1\wedge 2\gamma}. \tag{40}$$

Similarly, we have

$$E\left[\sup_{t\in[0,T]}\left|Q_{t-\tau(t)}-\widehat{Z}_t\right|^2\right]\leq C_4(d)h^{1\wedge 2\gamma}. \tag{41}$$

□

Lemma 14. *There exists a constant C for any $t\in[0,T]$ and $E[|\gamma_i|^p]\leq C$ such that*

$$E\left[\int_0^t\left|\gamma(s)-\overline{\gamma}(s)\right|^2ds\right]\leq Ch^{1-2/p}. \tag{42}$$

Proof. The proof is basically similar to that of Theorem 3.4 in [15], and we thus omit it here. □

We are now in a position to prove the main convergence results.

Theorem 15. *If Assumptions 2–7 hold, then the semi-implicit Euler approximate solutions converge to the exact solutions of (1) in the mean-square sense; that is,*

$$E\left[\sup_{0\leq t\leq T}\left|Q_{t\wedge v_n}-K_{t\wedge v_n}\right|^2\right]\leq C_dh^{(1-2/p)\wedge 2\gamma}, \tag{43}$$

where C_d is a positive constant dependent only on θ, λ, and K_d and T and independent of h.

Proof. Subtraction of (6) and (9) gives

$$\begin{aligned}
&K_t-Q_t\\
&=-\int_0^t\frac{\partial(K_s-Q_s)}{\partial a}ds\\
&\quad-\int_0^t\mu(s,a)\left[(1-\theta)\left(K_s-\overline{Y}_s\right)+\theta\left(K_s-\overline{Z}_s\right)\right]ds\\
&\quad+\int_0^t(1-\theta)\left[\left(f\left(K_s,K_{s-\tau(s)}\right)-f\left(\overline{Y}_s,\widehat{Y}_s\right)\right)\right.\\
&\qquad\left.+\theta\left(f\left(K_s,K_{s-\tau(s)}\right)-f\left(\overline{Z}_s,\widehat{Z}_s\right)\right)\right]ds\\
&\quad+\int_0^tg\left(K_s,K_{s-\tau(s)}\right)-g\left(\overline{Y}_s,\widehat{Y}_s\right)dW(s)\\
&\quad+\int_0^th\left(K_s,K_{s-\tau(s)},\gamma(s)\right)-h\left(\overline{Y}_s,\widehat{Y}_s,\overline{\gamma}(s)\right)dN(s).
\end{aligned} \tag{44}$$

Therefore using the generalized Itô formula, along with the Cauchy-Schwarz inequality and Assumptions 2–7, yields

$$\begin{aligned}
&\left|K_t - Q_t\right|^2 \\
&= -2\int_0^t \left\langle K_s - Q_s, \frac{\partial (K_s - Q_s)}{\partial a}\right\rangle ds \\
&\quad - 2\int_0^t \Big(K_s - Q_s, \\
&\qquad \mu(s,a)\left[(1-\theta)\left(K_s - \overline{Y}_s\right) + \theta\left(K_s - \overline{Z}_s\right)\right] ds\Big) \\
&\quad + 2\int_0^t \Big(K_s - Q_s, \Big[(1-\theta)\left(f\left(K_s, K_{s-\tau(s)}\right) - f\left(\overline{Y}_s, \widehat{Y}_s\right)\right) \\
&\qquad + \theta\left(f\left(K_s, K_{s-\tau(s)}\right) - f\left(\overline{Z}_s, \widehat{Z}_s\right)\right)\Big] ds\Big) \\
&\quad + 2\int_0^t \left(K_s - Q_s, g\left(K_s, K_{s-\tau(s)}\right) - g\left(\overline{Y}_s, \widehat{Y}_s\right) dW(s)\right) \\
&\quad + \int_0^t \left\|g\left(K_s, K_{s-\tau(s)}\right) - g\left(\overline{Y}_s, \widehat{Y}_s\right)\right\|_2^2 ds \\
&\quad + 2\int_0^t \Big(K_s - Q_s, \\
&\qquad \left(h\left(K_s, K_{s-\tau(s)}, \gamma(s)\right) - h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right)\Big) dN(s) \\
&\quad + \int_0^t \left|h\left(K_s, K_{s-\tau(s)}, \gamma(s)\right) - h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right|^2 dN(s) \\
&\le \left(A^2 F_1^2 \eta^2 + 9 + 9\overline{\alpha}\right)\int_0^t \left|K_s - Q_s\right|^2 ds \\
&\quad + 4\left(\overline{\alpha} + K_d\right)\int_0^t \left[\left|Q_s - \overline{Y}_s\right|^2 + \left|Q_s - \overline{Z}_s\right|^2\right] ds \\
&\quad + 4K_d \int_0^t \left|K_{s-\tau} - Q_{s-\tau}\right|^2 ds \\
&\quad + 4K_d \int_0^t \left[\left|Q_{s-\tau} - \widehat{Y}_s\right|^2 + \left|Q_{s-\tau} - \widehat{Z}_s\right|^2\right] ds \\
&\quad + 2\int_0^t \left(K_s - Q_s, g\left(K_s, K_{s-\tau(s)}\right) - g\left(\overline{Y}_s, \widehat{Y}_s\right) dW(s)\right) \\
&\quad + \int_0^t \left\|g\left(K_s, K_{s-\tau(s)}\right) - g\left(\overline{Y}_s, \widehat{Y}_s\right)\right\|_2^2 ds \\
&\quad + 2\lambda\int_0^t \Big(K_s - Q_s, \\
&\qquad \left(h\left(K_s, K_{s-\tau(s)}, \gamma(s)\right) - h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right) ds\Big) \\
&\quad + 2\int_0^t \Big(K_s - Q_s, \\
&\qquad \left(h\left(K_s, K_{s-\tau(s)}, \gamma(s)\right) - h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right) d\overline{N}(s)\Big) \\
&\quad + \lambda\int_0^t \left|h\left(K_s, K_{s-\tau(s)}, \gamma(s)\right) - h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right|^2 ds \\
&\quad + \int_0^t \left|h\left(K_s, K_{s-\tau(s)}, \gamma(s)\right) - h\left(\overline{Y}_s, \widehat{Y}_s, \overline{\gamma}(s)\right)\right|^2 d\overline{N}(s).
\end{aligned} \tag{45}$$

By Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
&2E\sup_{s\in[0,t]}\int_0^{s\wedge v_n}\left(K_u - Q_u, g\left(K_u, K_{u-\tau(u)}\right) - g\left(\overline{Y}_u, \widehat{Y}_u\right) dW(u)\right) \\
&\le 2k_9\int_0^{t\wedge v_n} E\Big(\left|K_s - Q_s\right|^2 + \left|Q_s - \overline{Y}_s\right|^2 \\
&\qquad + \left|K_{s-\tau(s)} - Q_{s-\tau(s)}\right|^2 + \left|Q_{s-\tau(s)} - \widehat{Y}_s\right|^2\Big) ds \\
&\quad + \frac{1}{6}E\left[\sup_{0\le s\le t}\left|K_{s\wedge v_n} - Q_{s\wedge v_n}\right|^2\right], \\
&2E\Big[\sup_{s\in[0,t]}\int_0^{s\wedge v_n}\Big(K_u - Q_u, \Big(h\left(K_u, K_{u-\tau(u)}, \gamma(u)\right) \\
&\qquad - h\left(\overline{Y}_u, \widehat{Y}_u, \overline{\gamma}(u)\right)\Big) d\overline{N}(u)\Big)\Big] \\
&\le \frac{1}{6}E\left[\sup_{0\le s\le t}\left|K_{s\wedge v_n} - Q_{s\wedge v_n}\right|^2\right] \\
&\quad + k_{11}\int_0^{t\wedge v_n} E\left(\left|\gamma(s) - \overline{\gamma}(s)\right|^2\right) ds \\
&\quad + 2k_{10}\int_0^{t\wedge v_n} E\Big(\left|K_s - Q_s\right|^2 \\
&\qquad + \left|Q_s - \overline{Y}_s\right|^2 + \left|K_{s-\tau(s)} - Q_{s-\tau(s)}\right|^2 \\
&\qquad + \left|Q_{s-\tau(s)} - \widehat{Y}_s\right|^2\Big) ds, \\
&E\left[\int_0^{s\wedge v_n}\left|h\left(K_u, K_{u-\tau(u)}, \gamma(u)\right) - h\left(\overline{Y}_u, \widehat{Y}_u, \overline{\gamma}(u)\right)\right|^2 d\overline{N}(u)\right] \\
&\le \frac{1}{6}E\left[\sup_{0\le s\le t}\left|K_{s\wedge v_n} - Q_{s\wedge v_n}\right|^2\right] \\
&\quad + \frac{1}{6}E\left[\sup_{0\le s\le t}\left|Q_{s\wedge v_n} - \overline{Y}_{s\wedge v_n}\right|^2\right] \\
&\quad + \frac{1}{6}E\left[\sup_{0\le u\le t}\left|K_{u\wedge v_n} - Q_{u\wedge v_n}\right|^2\right] \\
&\quad + \frac{1}{6}E\left[\sup_{0\le u\le t}\left|Q_{u\wedge v_n} - \widehat{Y}_s\right|^2\right] \\
&\quad + k_{13}\int_0^{t\wedge v} E\left(\left|\gamma(s) - \overline{\gamma}(s)\right|^2\right) ds \\
&\quad + 2k_{12}\int_0^{t\wedge v_n} E\Big(\left|K_s - Q_s\right|^2
\end{aligned}$$

$$+\left|Q_s-\overline{Y}_s\right|^2+\left|K_{s-\tau(s)}-Q_{s-\tau(s)}\right|^2$$
$$+\left|Q_{s-\tau(s)}-\widehat{Y}_s\right|^2\Big)\,ds,$$

$$2\lambda E\sup_{s\in[0,t]}\int_0^{s\wedge v_n}\Big(K_u-Q_u,$$
$$\left(h\left(K_u,K_{u-\tau(u)},\gamma(u)\right)-h\left(\overline{Y}_u,\widehat{Y}_u,\overline{\gamma}(u)\right)\right)du\Big)$$
$$\leq\lambda\int_0^{t\wedge v_n}E\left|K_s-Q_s\right|^2ds$$
$$+\lambda K\int_0^{t\wedge v_n}E\left(\left|\gamma(s)-\overline{\gamma}(s)\right|^2\right)ds$$
$$+2\lambda K_dE\int_0^{t\wedge v_n}\Big(\left|K_s-Q_s\right|^2$$
$$+\left|Q_s-\overline{Y}_s\right|^2+\left|K_{s-\tau(s)}-Q_{s-\tau(s)}\right|^2$$
$$+\left|Q_{s-\tau(s)}-\widehat{Y}_s\right|^2\Big)\,ds,$$

$$\lambda\int_0^{t\wedge v_n}\left|h\left(K_s,K_{s-\tau(s)},\gamma(s)\right)-h\left(\overline{Y}_s,\widehat{Y}_s,\overline{\gamma}(s)\right)\right|^2ds$$
$$\leq2\lambda K_d\int_0^{t\wedge v_n}\Big(\left|K_s-Q_s\right|^2$$
$$+\left|Q_s-\overline{Y}_s\right|^2+\left|K_{s-\tau(s)}-Q_{s-\tau(s)}\right|^2$$
$$+\left|Q_{s-\tau(s)}-\widehat{Y}_s\right|^2\Big)\,ds$$
$$+\lambda K\int_0^{t\wedge v_n}\left(\left|\gamma(s)-\overline{\gamma}(s)\right|^2\right)ds$$

$$\int_0^{t\wedge v_n}\left\|g\left(K_s,K_{s-\tau(s)}\right)-g\left(\overline{Y}_s,\widehat{Y}_s\right)\right\|_2^2ds$$
$$\leq K_d\int_0^{t\wedge v_n}\left(\left|K_s-\overline{Y}_s\right|^2+\left|K_{s-\tau(s)}-\widehat{Y}_s\right|^2\right)ds$$
$$\leq2K_d\int_0^{t\wedge v_n}\Big(\left|K_s-Q_s\right|^2$$
$$+\left|Q_s-\overline{Y}_s\right|^2+\left|K_{s-\tau(s)}-Q_{s-\tau(s)}\right|^2$$
$$+\left|Q_{s-\tau(s)}-\widehat{Y}_s\right|^2\Big)\,ds, \tag{46}$$

where $k_9, k_{10}, k_{11}, k_{12}$, and k_{13} are positive constants. Let $k=\max\{k_9,k_{10},k_{11},k_{12},k_{13}\}$; inserting (46) into (45) we obtain

$$E\left[\sup_{s\in[0,T]}\left|K_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\right]$$
$$\leq\left(A^2F_1^2\eta^2+9+\lambda+9\overline{\alpha}\right)$$
$$\times\int_0^{t\wedge v_n}E\sup_{u\in[0,s]}\left|K_{u\wedge v_n}-Q_{u\wedge v_n}\right|^2ds$$
$$+4\left(\overline{\alpha}+K_d\right)E\int_0^{t\wedge v_n}\left[\left|Q_s-\overline{Y}_s\right|^2+\left|Q_s-\overline{Z}_s\right|^2\right]ds$$
$$+4K_dE\int_0^{t\wedge v_n}\left|K_{s-\tau}-Q_{s-\tau}\right|^2ds$$
$$+4K_dE\int_0^{t\wedge v_n}\left[\left|Q_{s-\tau}-\widehat{Y}_s\right|^2+\left|Q_{s-\tau}-\widehat{Z}_s\right|^2\right]ds$$
$$+\frac{2}{3}E\left[\sup_{0\leq s\leq t}\left|K_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\right]$$
$$+\frac{1}{6}E\left[\sup_{0\leq s\leq t}\left|Q_{s\wedge v_n}-\overline{Y}_{s\wedge v_n}\right|^2\right]$$
$$+\frac{1}{6}E\left[\sup_{0\leq u\leq t}\left|Q_{u\wedge v_n}-\widehat{Y}_s\right|^2\right]$$
$$+\left(2K_d+4\lambda K_d+6k\right)$$
$$\times\int_0^{t\wedge v_n}E\Big(\left|K_s-Q_s\right|^2$$
$$+\left|Q_s-\overline{Y}_s\right|^2+\left|K_{s-\tau(s)}-Q_{s-\tau(s)}\right|^2$$
$$+\left|Q_{s-\tau(s)}-\widehat{Y}_s\right|^2\Big)\,ds,$$
$$+\left(2\lambda K+2k\right)\int_0^{t\wedge v_n}E\left(\left|\gamma(s)-\overline{\gamma}(s)\right|^2\right)ds. \tag{47}$$

Hence,

$$E\left[\sup_{s\in[0,T]}\left|K_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\right]$$
$$\leq3\left(A^2F_1^2\eta^2+9+\lambda+9\overline{\alpha}\right)$$
$$\times\int_0^{t\wedge v_n}E\sup_{u\in[0,s]}\left|K_{u\wedge v_n}-Q_{u\wedge v_n}\right|^2ds$$
$$+12h^{1\wedge2\gamma}T\left[\left(\overline{\alpha}+K_d\right)\left(C_1(d)+C_3(d)\right)\right.$$
$$\left.+K_d\left(C_2(d)+C_4(d)\right)\right]$$
$$+6\left(k+\lambda K\right)Ch^{1-2/p}$$
$$+\left[6\left(3k+2\lambda K_d+K_d\right)T+1\right]\left(C_1(d)+C_2(d)\right)h^{1\wedge2\gamma}$$
$$+12\left(3k+2\lambda K_d+2K_d\right)$$
$$\times\int_0^{t\wedge v_n}E\left[\sup_{0\leq u\leq s}\left|K_{u\wedge v_n}-Q_{u\wedge v_n}\right|^2\right]ds$$

$$\leq C_5(d)\, h^{(1-2/p)\wedge 2\gamma} + C_6(d) \int_0^{t\wedge v_n} E\left[\sup_{0\leq u\leq s} \left|K_{u\wedge v_n} - Q_{u\wedge v_n}\right|^2\right] ds. \tag{48}$$

Applying Gronwall's inequality, we obtain a bound of the form

$$E\left[\sup_{s\in[0,T]} \left|K_{s\wedge v_n} - Q_{s\wedge v_n}\right|^2\right] \leq C_d h^{(1-2/p)\wedge 2\gamma}, \tag{49}$$

where $C_d = C_5(d) e^{C_6(d)T}$. □

Theorem 16. *Under Assumptions 2–7 and let* $0 < \theta < 1$, *then*

$$E\left[\sup_{0\leq t\leq T} \left|Q_t - K_t\right|^2\right] \leq C_d' h^{(1-2/p)\wedge 2\gamma}. \tag{50}$$

Proof. Let $Z_t = Q_t - K_t$, it is easy to see that

$$\begin{aligned} &E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^2\right] \\ &= E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^2 1_{\{\sigma_n > T, \rho_n > T\}}\right] + E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^2 1_{\{\sigma_n\leq T \text{ or } \rho_n\leq T\}}\right] \\ &= E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^2 1_{\{v_n > T\}}\right] + E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^2 1_{\{\sigma_n\leq T \text{ or } \rho_n\leq T\}}\right] \\ &\leq E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^2 1_{\{t\wedge v_n\}}\right] + E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^2 1_{\{\sigma_n\leq T \text{ or } \rho_n\leq T\}}\right]. \end{aligned} \tag{51}$$

By Young's inequality $xy \leq x^p/p + y^q/q$, $1/p + 1/q = 1$, for any a, b, p, q, and $\delta > 0$, we have

$$ab \leq a\delta^{1/p} \frac{b}{\delta^{1/p}} \leq \frac{\left(a\delta^{1/p}\right)^p}{p} + \frac{b^q}{q\delta^{q/p}} = \frac{a^p\delta}{p} + \frac{b^q}{q\delta^{q/p}}. \tag{52}$$

Let $p = 2$, $\delta = h > 0$; we have

$$\begin{aligned} &E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^2 1_{\{\sigma_n\leq T \text{ or } \rho_n\leq T\}}\right] \\ &\leq \frac{h}{2} E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^4\right] + \frac{1}{2h} P\left\{\sigma_n \leq T \text{ or } \rho_n \leq T\right\}. \end{aligned} \tag{53}$$

Note

$$\begin{aligned} E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^4\right] &\leq 8\left(E\left[\sup_{0\leq t\leq T} \left|K_t\right|^4\right] + E\left[\sup_{0\leq t\leq T} \left|Q_t\right|^4\right]\right) \\ &\leq 8\left(C_1 + C_3\right), \\ P\left\{\sigma_n \leq T\right\} &= E\left[1_{\sigma_n\leq T} \frac{\left|K_{\sigma_n}\right|^4}{n^4}\right] \leq \frac{C_1}{n^4}, \\ P\left\{\rho_n \leq T\right\} &= E\left[1_{\rho_n\leq T} \frac{\left|Q_{\rho_n}\right|^4}{n^4}\right] \leq \frac{C_3}{n^4}. \end{aligned} \tag{54}$$

Then

$$\begin{aligned} &E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^2 1_{\{\sigma_n\leq T \text{ or } \rho_n\leq T\}}\right] \\ &\leq 4h\left(C_1 + C_3\right) + \frac{1}{2hn^4}\left(C_1 + C_3\right). \end{aligned} \tag{55}$$

By Theorem 15, (51) becomes

$$\begin{aligned} &E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^2\right] \\ &\leq C_d h^{(1-2/p)\wedge 2\gamma} + 4h\left(C_1 + C_3\right) + \frac{1}{2hn^4}\left(C_1 + C_3\right). \end{aligned} \tag{56}$$

Let $n \geq (2h^2)^{-1/4}$; then

$$\begin{aligned} E\left[\sup_{0\leq t\leq T} \left|Z_t\right|^2\right] &\leq C_d h^{(1-2/p)\wedge 2\gamma} + 5h\left(C_1 + C_3\right) \\ &= C_d' h^{(1-2/p)\wedge 2\gamma}. \end{aligned} \tag{57}$$

The proof is completed. □

Theorem 17. *Under Assumptions 2–7 and there is a constant* C *such that* $E[|\gamma_i|^p] \leq C$ *for some* $p > 2$, *the numerical approximate solution (7) will converge to the exact solution to (1) in the sense*

$$\lim_{h\to 0} E\left[\sup_{0\leq t\leq T} \left|Q_t - K_t\right|^2\right] = 0. \tag{58}$$

Proof. The proof is easily deduced from Theorem 16. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This paper was partially supported by The National Natural Science Foundation (61374040), Key Discipline of Shanghai (S30501), Scientific Innovation Program (13ZZ115), and Graduate Innovation Program of Shanghai (54-13-302-102, JWCXSL1301).

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