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Symmetric Duality for Mathematical Programming in Complex Spaces with Higher-Order F-univexity

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Abstract: In this study, we established appropriate duality results for a pair of Wolfe and Mond-Weir type symmetric dual for nonlinear programming problems in complex spaces under higher order F-univexity, F-unicavity/F-pseudounivexity, F-pseudounicavity. Results of this paper are real extension of previous literature.

Key words: Duality theorem, higher-order (phi, ρ)-(pseudo/quasi)-convexity, higher-order duality, Mathematical Subject Classification (2000), multiobjective programming, 90C29, 90C46

INTRODUCTION

Bector *et al.* (1994) introduced the concept of univexity. Mishra (1998) and Mishra and Reuda (2001, 2005) have applied and generalized these functions. Mathematical programming in complex spaces originated from Levinson's discussion of linear problems (Levinson, 1966). Mishra *et al.* (2003) have generalized this upto first order, second order with symmetric duality. Further Mishra and Rueda (2003) have given second order generalized invexity and it's duality theorems in multiobjective programming problems. For more details reader may consult (Ferraro, 1992; Lai, 2000; Liu, 1997; Liu *et al.*, 1997). Symmetric duality in real mathematical programming was introduced by Dorn (1961), who defined a program and its dual to be symmetric if the dual of the dual is original problem. Later, Mond and Weir (1981) presented a pair of symmetric dual nonlinear programs which allows the weakening of the convexity-concavity condition. For more work on symmetric duality in real spaces (Chandra, 1998; Gulati *et al.*, 1997; Mishra, 2000a, b). Mangasarian (1975) considered a nonlinear program and discussed second order duality under certain inequalities. Mond (1974) assumed simple inequalities respectively and indicated a possible computational advantage of the second order dual over the first order dual. Bector and Chandra (1987) defined the functions satisfying these inequalities (Mond, 1974) to be bonvex/boncave. Mishra (2000a, b) obtained second order duality results for a pair of Wolfe and Mon-Weir type second order symmetric dual nonlinear programming problems in real spaces under second order F-convexity, F-concavity and second order F-convexity is an extension of F-convex functions introduced by Hanson and Mond (1982). Mishra (2000a, b) formulated a pair of multiobjective second order symmetric dual programs for arbitrary cones in real space. The model considered in (Mishra, 2000a, b) unifies the Wolfe and the Mond-Weir type second order vector symmetric dual models. Gupta (1983) formulated a second order nonlinear symmetric dual program on the pattern of second order dual formulation as given by Mangasarian (1975) for the real case, but the constraints in the formulation (Gupta, 1983) is linear. Lai (2000) extended the concept of F-convex functions to the complex case and established sufficient optimality and duality theorems for a pair of nondifferentiable fractional complex programs.

In this study, we defined F-univex in the complex space in two variables . Hence extend the concepts of F-pseudounivex, F-pseudounicave functions and study symmetric duality under the aforesaid assumptions for Wolfe and Mond-Weir type models with Higher in complex spaces.

MATERIALS AND METHODS

Mishra *et al.* (2003) to compare vectors, we will distinguish between $\leq$ and $\leqq$ or $\geq$ and $\geqq$, specifically, in complex space. Let For C^n denotes n-dimensional complex spaces. For z$\hat{I}$$C^n$, let the real vectors Re (z) and Im (z) denote the real and imaginary parts, respectively, and let $\bar{z}$ = Re(z) – iIm(z) be the conjugate of z . Given a matrix A= $\left[a_{ij}\right] \in C^{n\times n}$, where $C^{n\times n}$ is the collection of m×n complex matrices, let $\bar{A} = \left[\bar{a}_{ij}\right] \in C^{n\times n}$ denote its conjugate matrix, let $A^H = \left[\bar{a}_{ij}\right]$ denote its

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conjugate transpose. The inner product of $x,y \in C^n$ is $(x,y) = y^H x$ .Let R_+ denote the half line $[0, \infty[$.

$$z \in C^n, v \in C^m, \mathrm{Re}(z) \leq \mathrm{Re}(v) \Leftrightarrow \mathrm{Re}(z_i) \leq \mathrm{Re}(v_i), \textit{ for all } I = 1,2,.,.,.,n, \mathrm{Re}(z) \neq \mathrm{Re}(v)$$

$$z \in C^n, v \in C^m, \mathrm{Re}(z) \leq \mathrm{Re}(v) \Leftrightarrow \mathrm{Re}(z_i) \leq \mathrm{Re}(v_i), \textit{ for all } I = 1,2,.,.,.,n, \mathrm{Re}(z) \neq \mathrm{Re}(v)$$

Similar notations are applied to distinguish between $\geq$ and $\geqq$. For a complex function $f: C^n \times C^n \times C^n \times C^n \to C^k$ analytic with respect to $\zeta = (w^1, w^2)$, $z \in C^n$, define gradients by:

$$\nabla_z f_i(v, \bar{v}, \zeta) = \left[\frac{\partial f}{\partial w_i^1}(v, \bar{v}, \zeta)\right] \text{ i=1,2,.,.,.,.,n . } \nabla_{\bar{z}} f_i(v, \bar{v}, \zeta) = \left[\frac{\partial f}{\partial w_i^2}(v, \bar{v}, \zeta)\right] \text{ i=1,2,.,.,.,.,n .}$$

In order to define generalized F-convexity, we follow the lines adopted in Mishra *et al.* (2003). In which $F: C^n \times C^n \times C^m \to R$ be sublinear on the third variable . Then we can generalized F-univexity for analytic functions and and ψ is increasing function and satisfying:

$$\text{(i)} \quad \psi(a) \geq 0 \Rightarrow a \geq 0 \qquad \text{(ii)} \quad a \geq b \Rightarrow \psi(a) \geq \psi(b)$$

Definition: The real part Re f of an analytic function $f: C^n \times C^n \times C^n \times C^n \to C$ is said to be higher order F-univex at $(z_0, \bar{z}_0)$ with respect to R_+, and $b: C^m \times C^n \times C^m \times C^n \times C^m \times C^n \times C^n \times C^n \to C_+$ if for any $z \in C^n$ for fixed $(w, \bar{w}) \in C^{2n}$,

$$\mathrm{Re}\Big[b(z, \bar{z}, z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})\Big[\psi\Big(f(z, \bar{z}, w, \bar{w}) - f(z_0, \bar{z}_0, w, \bar{w})$$
$$- h(z_0, \bar{z}_0, w, \bar{w}, p, \bar{p}) + P^T \overline{\nabla_p h(z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})} p + p^H \nabla_p h(z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})\Big)\Big]\Big] \textit{ for}$$
$$\geq F\Big(z_0, z_0; \Big(\overline{\nabla_p h(z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})} p + \nabla_p h(z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})\Big) p\Big)$$

some arbitrary sublinear functional F.

Definition: The real part Re f of an analytic function $f: C^n \times C^n \times C^n \times C^n \to C$ is said to be higher order F-unicave at $(w_0, \bar{w}_0)$ with respect to R_+ and $b: C^n \times C^n \times C^n \times C^n \times C^n \times C^n \times C^n \times C^n \to C_+$ for fixed $(z, \bar{z}) \in C^n \times C^n$, if:

$$\mathrm{Re}\Big[b(z, \bar{z}, z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})\Big[\psi\Big(f(z, \bar{z}, w_0, \bar{w}_0) - f(z, \bar{z}, w, \bar{w})$$
$$- h(z, \bar{z}, w, \bar{w}, p, \bar{p}) + P^T \overline{\nabla_p h(z, \bar{z}, w, \bar{w}, p, \bar{p})} p + p^H \nabla_p h(z, \bar{z}, w, \bar{w}, p, \bar{p})\Big)\Big]\Big] \textit{ for}$$
$$\geq F\Big(w, w_0; -\Big(\overline{\nabla_p h(z, \bar{z}, w, \bar{w}, p, \bar{p})} p + \nabla_p h(z, \bar{z}, w, \bar{w}, p, \bar{p}) p\Big)\Big)$$

some arbitrary sublinear functional F.

Definition: The real part Re f of an analytic function $f: C^n \times C^n \times C^n \times C^n \rightarrow C$ is said to be higher order F-pseudounivex at $(z_0, \bar{z}_0)$ with respect to R_+ and $b: C^n \times C^n \times C^n \times C^n \times C^n \times C^n \times C^n \times C^n \rightarrow C_+$ for fixed $(w, \bar{w})$, if:

$$F\left(z, z_0; -\left(\overline{\nabla_p h(z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})}p + \nabla_p h(z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})p\right)\right) \geq 0$$
$$\Rightarrow \mathrm{Re}\Big[b(z, \bar{z}, z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})\Big[\psi\Big\{f(z, \bar{z}, w, \bar{w}) - f(z_0, \bar{z}_0, w, \bar{w}) - h(z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})$$
$$+ p^T\overline{\nabla_p h(z, \bar{z}, w, \bar{w}, p, \bar{p})}p + p^H\nabla_{\bar{p}} h(z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})\Big\}\Big]\Big] \geq 0$$

for all $z \in C^n$ and for some arbitrary sublinear functional F.

Definition: The real part Reof an analytic function $f: C^n \times C^n \times C^m \times C^m \rightarrow C$ is said to be higher order F-pseudounicave at $(w_0, \bar{w}_0)$ with respect to R_+ and $b: C^n \times C^n \times C^n \times C^n \times C^n \times C^n \times C^n \times C^n \rightarrow C_+$ for fixed $(z, \bar{z}) \in C^n \times C^n$, if:

$$F\left(w, w_0; -\left(\overline{\nabla_p h(z, \bar{z}, w, \bar{w}, p, \bar{p})}p - \nabla_p h(z, \bar{z}, w, \bar{w}, p, \bar{p})p\right)\right) \geq 0$$
$$\Rightarrow b(z, \bar{z}, z_0, \bar{z}_0, w, \bar{w}, p, \bar{p})\Big[\psi\Big\{f(z, \bar{z}, w_0, \bar{w}_0) - f(z, \bar{z}, w, \bar{w}) - h(z, \bar{z}, w, \bar{w}, p, \bar{p})$$
$$+ p^T\overline{\nabla_p h(z, \bar{z}, w, \bar{w}, p, \bar{p})}p + p^H\nabla_{\bar{p}} h(z, \bar{z}, w, \bar{w}, p, \bar{p})p\Big\}\Big]\Big] \geq 0$$

for all $w \in C^n$ some arbitrary sublinear functional F.

Lemma:

(i) If $\mathrm{Re} f(.,., w, \bar{w})$ is F-univex and $\mathrm{Re}\, g(.,., w, \bar{w})$ is F-unicave, then $\mathrm{Re}\, p(.,., w, \bar{w})$ is F-pseudounivex.

(ii) If $\mathrm{Re} f(.,., w, \bar{w})$ is F-unicave and $\mathrm{Re}\, g(.,., w, \bar{w})$ is F-univex, then $\mathrm{Re}\, p(.,., w, \bar{w})$ is F-pseudounicave.

RESULTS AND DISCUSSION

Higher order Wolfe type symmetric duality: We consider the following higher-order Mond-Weir type pair and prove a weak duality theorem.

Primal (HMP):

$$\mathit{Minimize}\ \pi(z, \bar{z}, w, \bar{w}, p, \bar{p}) =$$
$$\mathrm{Re}\Big[\psi\Big\{f(z, \bar{z}, w, \bar{w}) + h(z, \bar{z}, w, \bar{w}, p, \bar{p}) - p^T\overline{\nabla_p h(z, \bar{z}, w, \bar{w}, p, \bar{p})}p$$
$$- p^H\nabla_{\bar{p}} h(z, \bar{z}, w, \bar{w}, p, \bar{p})p\Big\}\Big]$$

Subject to $\mathrm{Re}\left[\left(\overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p+\nabla_{\bar{p}}h(z,\bar{z},w,\bar{w},p,\bar{p})p\right]\le 0 \quad z\ge 0;$

Dual (HMD):

$$Maximize\ \vartheta(u,\bar{u},v,\bar{v},p_1,\bar{p}_1) =$$
$$\mathrm{Re}\left[\psi\left\{f(u,\bar{u},v,\bar{v})+h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)-p^T\overline{\nabla_{p_1}h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)}\right.\right.$$
$$\left.\left.-p_1^H\nabla_{\bar{p}_1}h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)\right\}\right]$$

Subject to $\mathrm{Re}\left[\left(\overline{\nabla_{p_1}h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)}p+\nabla_{\bar{p}_1}h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)p\right]\ge 0 \quad v\ge 0;$

Theorem: Let $\mathrm{Re}f[.,.,w,\bar{w}]$ be higher-order F_0-univex at $(u,\bar{u})$ and $\mathrm{Re}f[z,\bar{z},.,.]$ be higher-order F_1-unicave at $(w,\bar{w})$ and for all $(z,\bar{z},w,\bar{w})$ feasible for (HMP) and all $(u,\bar{u},v,\bar{v})$ feasible for (HMD).

(I) $F_0(z,u;\xi_1+\xi_2)+\mathrm{Re}\left[u^T\xi_1+u^H\xi_2\right]\ge 0,\ \xi_1,\xi_1\in C^n$

(II) $F_1(v,w,\eta_1+\eta_2)+\mathrm{Re}\left[w^T\eta_1+w^H\eta_2\right]\ge 0,\ \eta_1,\eta_2\in C^n$

Then $\inf(HMP)\ge Sup(HMD)$.

Proof: Let $b: C^n\times C^n\times C^n\times C^n\times C^n\times C^n\times C^n\times C^n\times C^n\times C^n\times C^n\times C^n\to C_+$

$\xi_1=\overline{\nabla_{p_1}h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)},\ \xi_2=\nabla_{\bar{p}_1}h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)p_1$

Then $F_0(z,u;\xi_1+\xi_2)\ge \mathrm{Re}\left[-u^T\xi_1-u^H\xi_2\right]\ge 0$

Which by the higher-order F_0-univexity of $\mathrm{Re}f[.,.,w,\bar{w}]$ at $(u,\bar{u})$ yields,

$$\mathrm{Re}\left[b(z,\bar{z},w,\bar{w},u,\bar{u},p,\bar{p},p_1,\bar{p}_1)\left(\psi\left\{f(z,\bar{z},v,\bar{v})-f(u,\bar{u},v,\bar{v})-h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)\right.\right.\right.$$
$$\left.\left.\left.+P_1^T\overline{\nabla_{p_1}h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)}p_1+p_1^H\nabla_{\bar{p}_1}h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)p_1\right\}\right\}\right]\ge F_0(z,u;\xi_1+\xi_2) \tag{1}$$

Let

$$\eta_1=-\overline{\nabla_{\bar{p}}h(z,\bar{z},w,\bar{w},p,\bar{p})}p,\quad \eta_2=-\overline{\nabla_{\bar{p}}h(z,\bar{z},w,\bar{w},p,\bar{p})}p$$
$$F_1(v,w;\eta_1+\eta_2)\ge\mathrm{Re}\left[-w^T\eta_1-w^H\eta_2\right]\ge 0$$

which by the higher-order F_1-unicavity of $\mathrm{Re}f[z,\bar{z},.,.]$ at $(w,\bar{w})$ gives:

$$\operatorname{Re}\Big[b(z,\bar{z},w,\bar{w},u,\bar{u},p,\bar{p},p_1,\bar{p}_1)\Big\{\psi\Big(f(z,\bar{z},w,\bar{w})-f(z,\bar{z},v,\bar{v})-h(z,\bar{z},w,\bar{w},p,\bar{p}) + P^T\overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p + P^H\nabla_{\bar{p}}h(z,\bar{z},w,\bar{w},p,\bar{p})p\Big)\Big\}\Big] \geq F_0(z,u;\eta_1+\eta_2) \tag{2}$$

combining (1) and (2) , and using property of function ψ and *b*:

$$\begin{aligned}&\operatorname{Re}\Big[b(z,\bar{z},w,\bar{w},u,\bar{u},v,\bar{v},p,\bar{p},p_1,\bar{p}_1)\Big\{f(z,\bar{z},w,\bar{w})-h(z,\bar{z},w,\bar{w},p,\bar{p}) \\ &+ P^T\overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p + P^H\nabla_{\bar{p}}h(z,\bar{z},w,\bar{w},p,\bar{p})p\Big\}\Big] - \\ &\operatorname{Re}\Big[b(z,\bar{z},w,\bar{w},u,\bar{u},v,\bar{v},p,\bar{p},p_1,\bar{p}_1)\Big\{f(u,\bar{u},v,\bar{v})-h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1) \\ &+ P_1^T\overline{\nabla_{p_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)}p_1 + P_1^H\nabla_{\bar{p}_1}h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)p_1\Big\}\Big] \\ &\geq F_0(z,u;\xi_1+\xi_2)+F_0(v,w;\eta_1+\eta_2)\end{aligned} \tag{3}$$

Using the hypothesis of the theorem,

$$F_0(z,u;\xi_1+\xi_2) \geq \operatorname{Re}\left[-u^T\xi_1-u^H\xi_2\right], \text{ and } F_1(v,w;\eta_1+\eta_2) \geq \operatorname{Re}\left[-w^T\eta_1-w^H\eta_2\right]$$

Thereafter, using property of function ψ and *b*, we get:

$$\begin{aligned}&\operatorname{Re}\Big[b(z,\bar{z},w,\bar{w},u,\bar{u},v,\bar{v},p,\bar{p},p_1,\bar{p}_1)\Big\{\psi\Big(f(z,\bar{z},w,\bar{w})-h(z,\bar{z},w,\bar{w},p,\bar{p}) \\ &+ P^T\overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p + P^H\nabla_{\bar{p}}h(z,\bar{z},w,\bar{w},p,\bar{p})p\Big)\Big\}\Big] \geq \\ &\operatorname{Re}\Big[b(z,\bar{z},w,\bar{w},u,\bar{u},v,\bar{v},p,\bar{p},p_1,\bar{p}_1)\Big\{\psi\Big(f(u,\bar{u},v,\bar{v})-h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1) \\ &+ P_1^T\overline{\nabla_{p_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)}p_1 + P_1^H\nabla_{\bar{p}_1}h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)p_1\Big)\Big\}\Big]\end{aligned}$$

The higher order strong duality theorem can be developed on the lines of Mishra *et al*. (2003) in the view of the above theorem.

Mond-Weir type symmetric duality in higher-order: We consider the following higher-order Mond-Weir type pair and prove a weak duality theorem.

Primal (HMP):

$$\begin{aligned}&\textit{Minimize}\ \operatorname{Re}\Big[\psi\Big(f(z,\bar{z},w,\bar{w})+h(z,\bar{z},w,\bar{w},p,\bar{p})-P^T\overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p \\ &+ P^H\nabla_{\bar{p}}h(z,\bar{z},w,\bar{w},p,\bar{p})p\Big)\Big]\end{aligned}$$

Subject to:

$$\operatorname{Re}\left[\overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p + \nabla_{\bar{p}} h(z,\bar{z},w,\bar{w},p,\bar{p})p\right] \le 0$$

$$\operatorname{Re}\left[w^T \overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p + w^H \nabla_{\bar{p}} h(z,\bar{z},w,\bar{w},p,\bar{p})p\right] \ge 0, \quad z \ge 0;$$

Dual (HMD):

$$\text{Maximize } \operatorname{Re}\Big[\sigma\Big(f(u,\bar{u},v,\bar{v}) + h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)$$
$$- p_1^T \overline{\nabla_{p_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)}p_1 + p_1^H \nabla_{\bar{p}_1} h(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)p_1\Big)\Big]$$

Subject to:

$$\operatorname{Re}\left[\overline{\nabla_{p_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)}p_1 + \nabla_{\bar{p}} h(u,\bar{u},v,\bar{v},p,\bar{p})p\right] \ge 0$$

$$\operatorname{Re}\left[u^T \overline{\nabla_{p_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)}p_1 + u^H \nabla_{\bar{p}_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)p_1\right] \le 0, \quad v \ge 0;$$

Theorem: Let $\operatorname{Re} f[.,.,w,\bar{w}]$ be higher-order F_0-pseudounivex at $(u,\bar{u})$ and $\operatorname{Re} f[z,\bar{z},.,.]$ be higher-order F_1-pseudounicave at $(w,\bar{w})$ and for all $(z,\bar{z},w,\bar{w})$ feasible for (HMP) and all $(u,\bar{u},v,\bar{v})$ feasible for (HMD) .

(I) $F_0(z,u;\xi_1+\xi_2) + \operatorname{Re}\left[u^T\xi_1 + u^H\xi_2\right] \ge 0, \quad \xi_1,\xi_2 \in C^n$

(II) $F_0(z,u;\eta_1+\eta_2) + \operatorname{Re}\left[u^T\eta_1 + u^H\eta_2\right] \ge 0, \quad \eta_1,\eta_2 \in C^n$

Then $\inf(HMP) \ge Sup(HMD)$

Proof:
Let $b: C^n \times C^n \times C^m \times C^m \times C^n \times C^n \times C^n \times C^n \times C^n \times C^m \times C^n \times C^m \to C_+$

$$\xi_1 = \overline{\nabla_{p_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)},\ \xi_2 = \nabla_{\bar{p}_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)p_1$$

Then $F_0(z,u;\xi_1+\xi_2) \ge \operatorname{Re}\left[-u^T\xi_1 - u^H\xi_2\right] \ge 0$

which by the higher-order F_0-pseudounivexity of $\operatorname{Re} f[.,.,w,\bar{w}]$ at $(u,\bar{u})$ yields,

$$\operatorname{Re}\Big[b(z,\bar{z},w,\bar{w},u,\bar{u},v,\bar{v},p,\bar{p},p_1,\bar{p}_1)\Big(\psi\Big(f(z,\bar{z},v,\bar{v}) - f(u,\bar{u},v,\bar{v}) - h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)$$
$$+ p_1^T \overline{\nabla_{p_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)}p_1 + p_1^H \nabla_{p_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)p_1\Big)\Big)\Big] \ge 0 \tag{4}$$

Let

$$\eta_1 = -\overline{\nabla_{\bar{p}} h(z,\bar{z},w,\bar{w},p,\bar{p})}p, \quad \eta_2 = -\overline{\nabla_{\bar{p}} h(z,\bar{z},w,\bar{w},p,\bar{p})}p$$
$$F_1(v,w;\eta_1+\eta_2) \geq \mathrm{Re}\left[-w^T\eta_1 - w^H\eta_2\right] \geq 0 \quad ,$$

which by the higher-order F_1-pseudounicavity of $\mathrm{Re} f[z,\bar{z},.,.]$ at $(w,\bar{w})$ gives:

$$\mathrm{Re}\Big[b(z,\bar{z},w,\bar{w},u,\bar{u},v,\bar{v},p,\bar{p},p_1,\bar{p}_1)\Big\{\psi\Big\{f(z,\bar{z},w,\bar{w}) - f(z,\bar{z},v,\bar{v}) - h(z,\bar{z},w,\bar{w},p,\bar{p}) + P^T\overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p + P^H\nabla_{\bar{p}} h(z,\bar{z},w,\bar{w},p,\bar{p})p\Big\}\Big\}\Big] \geq 0 \tag{5}$$

combining (1) and (2) , and using property of function ψ and *b*:

$$\mathrm{Re}\Big[b(z,\bar{z},w,\bar{w},u,\bar{u},v,\bar{v},p,\bar{p},p_1,\bar{p}_1)\Big\{f(z,\bar{z},w,\bar{w}) - h(z,\bar{z},w,\bar{w},p,\bar{p}) - P^T\overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p - P^H\nabla_{\bar{p}} h(z,\bar{z},w,\bar{w},p,\bar{p})p\Big\}\Big] \geq$$
$$\mathrm{Re}\Big[b(z,\bar{z},w,\bar{w},u,\bar{u},v,\bar{v},p,\bar{p},p_1,\bar{p}_1)\Big\{f(u,\bar{u},v,\bar{v}) - h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1) + P_1^T\overline{\nabla_{p_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)}p_1 + P_1^H\nabla_{\bar{p}_1} h(u,\bar{u},v,\bar{v},p_1,\bar{p}_1)p_1\Big\}\Big]$$

That i.e., inf(HMP) $\geq$ sup (HMD).

The higher order strong duality theorem can be developed on the lines of Mishra *et al.* (2003) in the view of the above theorem.

Higher order dual fractional programming: In this section, we extend above to the higher order complex fractional symmetric dual pair (HFP) and (HFD) as follows:

Primal (HFP):

$$\mathit{Minimize}\ \frac{\mathrm{Re}\Big[\psi\Big\{f(z,\bar{z},w,\bar{w}) + h(z,\bar{z},w,\bar{w},p,\bar{p}) - P^T\overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p + P^H\nabla_{\bar{p}} h(z,\bar{z},w,\bar{w},p,\bar{p})p\Big\}\Big]}{\mathrm{Re}\Big[\psi\Big\{g(z,\bar{z},w,\bar{w}) + h(z,\bar{z},w,\bar{w},p_1,\bar{p}_1) - P_1^T\overline{\nabla_{p_1} h(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)}p_1 + P_1^H\nabla_{\bar{p}_1} h(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)p_1\Big\}\Big]}$$

Subject to:

$$\mathrm{Re}\Big[G(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)\Big\{\overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p + \nabla_{\bar{p}} h(z,\bar{z},w,\bar{w},p,\bar{p})p\Big\} - H(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)\Big\{\overline{\nabla_{p_1} h(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)}p_1 + \nabla_{\bar{p}_1} h(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)p_1\Big\}\Big] \leq 0$$

$$\mathrm{Re}\Big[w^T \Big\{ G(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)\overline{\nabla_p h(z,\bar{z},w,\bar{w},p,\bar{p})}p$$
$$- H(z,\bar{z},w,\bar{w},p,\bar{p})\overline{\nabla_{p_1} h(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)}p_1 \Big\}$$
$$w^H \Big\{ G(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)\nabla_{\bar{p}} h(z,\bar{z},w,\bar{w},p,\bar{p})p$$
$$- H(z,\bar{z},w,\bar{w},p,\bar{p})\nabla_{\bar{p}_1} h(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)p_1 \Big] \geq 0, \quad z \geq 0;$$

where,

$$G(z,\bar{z},w,\bar{w},p_1,\bar{p}_1) = \mathrm{Re}\Big[\psi\Big\{ g(z,\bar{z},w,\bar{w}) + h(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)$$
$$- p_1^T \overline{\nabla_{p_1}(z,\bar{z},w,\bar{w},p_1,\bar{p}_1)} - p_1^H \nabla_{p_1}(z,\bar{z},w,\bar{w},p_1,\bar{p}_1) \Big\} \Big]$$

and

$$H(z,\bar{z},w,\bar{w},p,\bar{p}) = \mathrm{Re}\Big[\psi\Big\{ f(z,\bar{z},w,\bar{w}) + h(z,\bar{z},w,\bar{w},p,\bar{p})$$
$$- p^T \overline{\nabla_p(z,\bar{z},w,\bar{w},p,\bar{p})} - p^H \nabla_p(z,\bar{z},w,\bar{w},p,\bar{p}) \Big\} \Big]$$

Dual (HMD):

$$\mathit{Maximize}\ \frac{\begin{array}{l}\mathrm{Re}\Big[\sigma\Big\{ f(u,\bar{u},v,\bar{v}) + h(u,\bar{u},v,\bar{v},p_2,\bar{p}_2) \\ - P_2^T \overline{\nabla_{p_2} h(u,\bar{u},v,\bar{v},p_2,\bar{p}_2)} + P_2^H \nabla_{\bar{p}_2} h(z,\bar{z},w,\bar{w},p_2,\bar{p}_2) \Big\} \Big]\end{array}}{\begin{array}{l}\mathrm{Re}\Big[\sigma\Big\{ f(u,\bar{u},v,\bar{v}) + h(u,\bar{u},v,\bar{v},p_3,\bar{p}_3) \\ - P_3^T \overline{\nabla_{p_3} h(u,\bar{u},v,\bar{v},p_3,\bar{p}_3)} + P_3^H \nabla_{\bar{p}_3} h(z,\bar{z},w,\bar{w},p_3,\bar{p}_3) \Big\} \Big]\end{array}}$$

$$\mathrm{Re}\Big[u^T \Big\{ G(u,\bar{u},v,\bar{v},p_3,\bar{p}_3)\overline{\nabla_{p_3} h(u,\bar{u},v,\bar{v},p_2,\bar{p}_2)}p_2 \Big\}$$
$$- H(u,\bar{u},v,\bar{v},p_2,\bar{p}_2)\overline{\nabla_{p_1} h(u,\bar{u},v,\bar{v},p_3,\bar{p}_3)}$$
$$+ u^G \Big\{ G(u,\bar{u},v,\bar{v},p_3,\bar{p}_3)\nabla_{\bar{p}_2} h(u,\bar{u},v,\bar{v},p_2,\bar{p}_2)p_2$$
$$- H(u,\bar{u},v,\bar{v},p_2,\bar{p}_2)\nabla_{\bar{p}_3} h(u,\bar{u},v,\bar{v},p_3,\bar{p}_3)p_3 \Big] \leq 0, \quad v \geq 0;$$

where,

$$G(u,\bar{u},v,\bar{v},p_2,\bar{p}_2) = \mathrm{Re}\Big[\psi\Big\{ g(u,\bar{u},v,\bar{v}) + h(u,\bar{u},v,\bar{v},p_2,\bar{p}_2)$$
$$- p_2^T \overline{\nabla_{p_2}(u,\bar{u},v,\bar{v},p_2,\bar{p}_2)} - p_2^H \nabla_{p_2}(u,\bar{u},v,\bar{v},p_2,\bar{p}_2) \Big\} \Big]$$

and

$$H(u,\bar{u},v,\bar{v},p_3,\bar{p}_3) = \mathrm{Re}\Big[\psi\Big\{f(u,\bar{u},v,\bar{v}) + h(u,\bar{u},v,\bar{v},p_3,\bar{p}_3) - p_3^T\overline{\nabla_{p_3}(u,\bar{u},v,\bar{v},p_3,\bar{p}_3)} - p_3^H\nabla_{p_3}(u,\bar{u},v,\bar{v},p_3,\bar{p}_3)\Big\}\Big]$$

It is assumed that Re G>0 and Re H>0 throughout the feasible regions defined by the primal (HFP) and the dual problem (HFD). Lemma above can be extended to the higher order and also next theorem on the line of Mishra *et al.* (2003).

CONCLUSION

- If we take b = 1 and $\psi(f) = f, h = \nabla f$, $h = p\nabla f + \frac{1}{2}p\nabla^2 f p$ then this is an earlier work by Mishra *et al.* (2003) On the same condition if it is single objective and f is real and differential then this is an earlier work Mishra (2000a, b).
- If we take b = 1 and $\psi(f) = f$, the function f to be real and $F_0 = F_1$ differentiable and, then this is an earlier work by Gulati and Ahmad (1997).

As Lai (2000), complex programming problems are applied to various fields of electrical engineering. For details the reader may refer to Stancu-Minasian (1997) and Denoho (1981).

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